Al-Karajī (or Al-Karkh | Encyclopedia.com

(*fl*. Baghdad, end of tenth century/beginning of eleventh), *mathematics*.

Virtually nothing is known of al-Karajī's life; even his name is not certain. Since the translations by Woepcke and Hochheim he has been called al-Karkhī, a name adopted by historians histroians of mathematics.[1] In 1933, however, Giorgio Levi della Vida rejected this name for that of al-Karajī.[2] This debate would have been pointlessif certain authors had not attempted to use the name of this mathematician to deduce his origins: Karkh, a suburb of Baghdad, or Karaj, an Iranian city. In the epresent state of our knowledge della Vida's argument is plausible but not decisive. On the basis of the manuscripts consulted it is far from easy to decide in favor of either name.[3] Turning to the "commentators"does not take us any further.[4] For example, the *al-Bāhir fī'l jabr* of al-Samawʾal cites the name al-Karajī, as indicated in MS Aya Sofya 2718. On this basis some authors have sought to derive a definitive argument in favor of this name.[5] On the other hand, another hitherto unknown manuscript of the same text (East Efendi 3155) gives the name al-Karkhī.[6] Because the use of the name al-Karajī is beginning to predominate—for no clear reasons—and because we do not wish to add to the alreadyy great' confusion in the designation of Arab authors, we shall use the name al-Karajī—refraining from any speculation designed to infer our subject's orgins from this name. It is sufficient to know that he lived and produced the bulk of his work in Baghdad at the end of the tenth centry and the beginning of the eleventh and that he probably left that city for the "mountain countries,"[7] where he appears to have ceased writing mathematical works in order to devote himself to composing works on engineering, as indicated by his book on the drilling of wells.

Al-Karajī's work holds an especially important place in the history of mathematics. Woepcke remarked that it "offers first the most complete or rather the only theory of algebraic calculus among the Arabs known to us up to the present time."[8] It is true that al-Karajī employed an approach entirely new in the tradition of the Arab algebraists—al-Khwārizmī, Ibn al-Fatḥ, Abū Kāmil—commencing with an exposition of the theory of algebraic calculus.[9] The more or less explicit aim of this exposition was to find means of realizing the autonomy and specificity of algebra, so as to be in a position to reject, in particular, the geometric representation of algebraic operations. What was actually at stake was a new beginning of algebra by means of the systematic application of the operations of arithmetic to the interval $[0, \infty]$. This arithmetization of algebra was based both on algebra, as conceived by al-Khwārizmī and developed by Abū Kāmil and many others, and on the translation of the *Arithmetica* of Diophantus, commented on and developed by such Arab mathematicians as Abu'l Wafāʾ al-Būzjānī.[10] In brief, the discovery and reading of the arithmetical work of Diophantus, in the light of the algebraic conceptions and methods of al-Khwārizmī and other Arab algebraists, made possible a new departure in algebra by al-Karajī, the author of the first account of the algebra of polynomials.

In his treatise on algebra, *al-Fakhrī*, al-Karajī first presented a systematic study of algebraic exponents, then turned to the application of arithmetical operations to algebraic terms and expressions, and concluded with a first account of the algebra of polynomials. He studied[11] the two sequnces, $x, x^2,\ldots, x^9,\ldots, 1/x, 1/x^2,\ldots, 1/x^9,\ldots$ and, successively, formulated the following rules:

(1) (2)

In order to appreciate the importance of this study, it is necessary to see how al-Karajī's more or less immediate successors exploited it. For example, al-Samawʾal[12] was able, on the basis of al-Karajī's work, to utilize the isomorphism of what would now be called the groups (Z, +) and $([x^n; n \in Z], +)$ in order to give for the first time, in all its generality, the rule equivalent to $x^m x^n = x^{m+n}$, where $m, n \in Z$.

In applying arithmetical operations to algebraic terms and expressions, al-Karajī first considered the application of these rules to monomials before taking up "composed quantities," or polynomials. For multiplication he thus demonstrated the following rules: (1) $(a/b) \cdot c=ac/b$ and (2) $a/b \cdot c/d=ac/bd$, where $a, b, c,$ and d are monomials. He then treated the multiplication of polynomials, for which he gave the general rule. He proceeded in the same manner and with the same concern for the symmetry of the operations of addition and subtraction. Yet this algebra of polynomials was uneven. In division and the extraction of roots al-Karajī did not achieve the generality already attained for the other operations. Hence he considered only the division of one monomial by another and of a polynomial by a monomial. Nevertheless, these results permitted his successors—notably al-Samawʾal—to study, for the first time to our knowledge, divisibility in the ring $[Q(x) + Q(1/x)]$ and the approximation of whole fractions by elements of the same ring.[13] As for the extraction of the square root of a polynomial, al-Karajī succeeded in giving a general method—but it is valid only for positive coefficients. This method allowed al-Samawʾal to solve the problem for a polynomial with rational coefficients or, more precisely, to determine the root of a square element of the ring $[Q(x) + Q(1/x)]$.[14] Al-Karajī's method consisted in giving first the development of $(x_1 + x_2 + x_3)^2$—where x_1, x_2, and x_3 are monomials—for which he proposed the canonical form

This last expression is itself, in this case, a polynomial ordered according to decreasing powers. Al-Karajī then posed the inverse problem: finding the root of a five-term polynomial. He therefore considered this polynomial to be of the canonical form and proposed two methods. The first consisted in taking the sum of the roots of two extreme terms—if these exist—and the quotient of either the second term divided by twice the root of the first or of the fourth term divided by twice the root of the last.[15] The second method consisted in subtracting from the third term twice the product of the root of the root of the first term times the root of the last term, then the root of the reminder from the subtraction is added to the roots of the extreme terms. Great care must be exercised here. This form is not restricted to the particular example, and al-Karajī's method, as can be seen in *al-Badī*', is general.[16]

Again with a view to extending algebraic computation al-Karajī pursued the examination of the application of arithmetical operations to irrational terms and expressions.

"How multiplication, division, additon, subtraction, and the extraction of roots may be used [on irrational algebraic quantities]." [17] This was the problem posed by al-Karajī and used by al-Samaw'al as the title of the penultimate chapter of his work on the use of arithmetical tools on irrational quantites. The problem marked an important stage in al-Karajï's whole project and therefore also in the extension of the algebraic calculus . Just as he had explicitly and systematically applied the operations of elementary arithmetic to rational quantities, al-Karajī. in order to achieve his objectives, wished to extend this application to irrational quantities in order to show that they still retained their properties. This project, while conceived as purely theoretical, led to a greatly increased knowledge of the algebraic structure of real numbers. Clear progress indeed, but to make it possible it was necessary to risk a setback—a risk at which some today would be scandalized—in that it did not base the operation on the firm ground of the theory of real numbers. The arithmetician–algebrasists were only interested in what we might call the algebra of R and did not attempt to construct the field of real numbers. Here progress was made in another algebraci field, that of geometrical algebra, later revived by al-Hayyām and Šaraf al-Dīn al-Tusī.[18] In the tradition of this algebra al-Karajī and al-Samaw'al could extend their algebraical operations to irrational quantities without questioning the reasons for their success or justifying the extension. Because an unfortunate lack of any such justification gave the sense of a setback al-Karajī simultaneously adopted the definitions of books VII and X of the *Elements*. While he borrowed from book VII the definition of number as "a whole composed of unities" and of unity—not yet a number—as that which "qualifies by an existing whole," it is in conformity with book X that he defined the concepts of incommensurability and irrationality. For Euclid, however, as for his commenta'ors, these concepts apply only to geometrical objects or, in the expression of Pappus, they "are a property which is essentially geometrical." [19] "Neither incommensurability nor irrationality," he continued, "can exist for numbers. Numbers are rational and commensurable." [20]

Since al-Karajī explicitly used the Euclidean definitions as a point of departure, it would have been useful if he could have justified his use of them on incommensurable and irrational quantities. His works may be searched in vain for such an explanation. The only justification to be found is extrinsic and indirect and is based on his conception of algebra. Since algebra is concerned with both segments and numbers, the operations of algebra can be applied to any object, be it geometrical or arithmetical. Irrationals as well as rationals may be the solution of the unknown in algebraic operations precisely because they are concerned with both numbers and geometrical magnitudes. The absence of any intrinsic explanation seems to indicate that the extension of algebraic calculation—and therefore of algebra—needed for its development to forget the problems relative to the construction of R and to surmount any potential obstacle, in order to concentrate on the algebraci structure. An unjustified leap, indeed, but a fortunate one for the development of algebra. This is the exact meaning of al-Karajī when he writes, without transition immediately after referring to the definitions of Euclic, "I show you how these quantities [incommensurables, irrationals] are transposed into numbers." [21].

One of the consequences of this project, and not the least important, is the reinterpretation of book X of the *Elements*.[22] This had until then been considered by most mathematicians, even by one so important as Ibn al-Haytham, as merely a geometry book. For al-Karajï its concepts concerned magnitudes in general, both numerical and geometric, and by algebra he classified the theory in this book in what was later to be known as the theory of numbers. To extend the concepts of book X of the *Elements* to all algebraic quantities al-Karjī began by increasing their number. "I say that the monomials are infinit: the first is absolutely rational, five for example, the second is potentially rational, as the root of ten, the third is defined by reference to its cube as the *côté* of twenty, the fourth is the *médiale* defined by reference to the square of its square, the fifth is the square of the quardrato-cube, then the *côté* to the cubo-cube and so on to infinity."[23] In the same way binomials can also be split infinitely. In this field, as in so many others, al-Samaw'al is continuing the work of al-Karajī. At the same time one contribution belongs to him alone and that is his generalization of the division of a polynomial with irrational terms.[24] He thus developed the calculus of radicals introduced by his predecessors. At the beginning of *al-Badī';*[25] is a statement—for the monomials *x1, x2* and the strictly positive natural integers *m, n*—of the rule that make it possible to calculate the following:

Al-Karajī next discussed the same operations carried out on polynomials and gave, among others, rules that allow calculation of expressions such as

In addition he attempted, unsuccessfully, to calculate

In the same spirit al-Karajī took up binomial developments. In *al-Fakhrī*[26] he gives the development of $(a + b)^3$, and in *al-Badī';*[27] he presents those of $(a+b)^3$, and $(a+b)^4$. In a long text of al-Karajī reported by al-Samaw'al are the table of binomial coefficients, its formation law and the expansion for integer *n*.[28]

To demonstrate the preceding proposition as well as the proposition $(ab)^n = a^n b^n$, where a and b are commutative and for all $n \in N$, al-Samawʾal uses a slightly old-fashioned from of mathematical induction. Before proceeding to demonstrate the two propositions he shows that multiplication is commutative and associative—$(ab)(cd) = (ac)(bd)$—and recalls the distributivity of multiplication with respect to addition —$(a+b)\lambda = a\lambda + b\lambda$. He then uses the expansion of $(a+b)^{n-1}$ to prove the identity for $(a+b)^n$ and that of $(ab)^{n-1}$ to far as we know, that we find a proof that can be considered the beginning of mathematical induction.

Turning to the theory of numbers, al-Karajī pursued further the task of extending algebra computation. He demonstrated the following theorems:[29]

Actually al-Karajī did not demonstrate this theorem; he only gave the equivalent form

The algebraic demonstration appeared for the first time in al-Samawʾal:[30]

For al-Karajī, the "determination of unknowns starting from known premises" is the proper task of algebra.[31] The aim of algebra is to show how unknown quantities are determined by known quantities through the transformation of the given equations. This is obviously an analytic task, and algebraic equations. One can thus understand the extension of algebraic computation and why al-Karajī's followers[32] did not hesitate to join algebra to analysis and, to a certain extent, to oppose it to geometry, thus affirming its autonomy and its independence. Since al-Khwārizmī the unity of the algebraic object was no longer founded in the unity of mathematical entities but in that of operation. It was a question, on the one hand, of the operations necessary to reduce an arbitrary problem to one form of equation—or, more precisely, to one of the canonical types stated by al-Khwārizmī—and, on the other hand. of the operations necessary to give particular solutions, that is, the "canons." In the same fashion al-Karajī took up the six canonical equations[33]— $ax=b$, $ax^2=bx$, $ax^2=b$, $ax^2+bx=c$, $ax^2+c=bx$, $bx+c=ax^2$— in order to solve equations of higher degree: $ax^{2n} + bx^n=c$, $ax^{2n} + c = bx^n$, $bx^n + c = ax^{2n}$, $ax^{2n+m} = bx^{n+m} + cx^m$.

Next, following Abū Kāmil in particular, al-Karajī studied systems of linear equations [34] and solved, for example, the system $x/2 + w = s/2$, $2\,y/3 + w = s/3$, $5\,z/6 + w = s/6$, where $s = x + y + z$ and $w = 1/3\ (x/2 + y/3 + z/6)$.

The translation of the first five books of Diophantu's *Arithmetica* revealed to al-Karajī the importance of at least two fields. Yet, unlike Diophantus, he wished to elaborate the theoretical aspect of the fields under consideration. Therefore al-Karajī benefited from both a conception of algebra renewed by al-Khwārizmī and a more developed theory of algebraic computation, and he was able, through his reading of Diophantus, to state in a general form propositions still implicit in Diophantus and to add to them others not initially foreseen. In *al-Fakhrī*, as in *al-Badī';*, by indeterminate analysis (*istiqrāʾ*)[35] al-Karajī meant "to put forward a composite quantity [that is, a polynomial or algebraic expression] formed from one, two, or three successive terms, understood as a square but the formulation of which is nonsquare and the root of which one wishes to extract."[36] By the solution in q of a polynomial with rational coefficients al-Karajī proposed to find the values of x in q such that p (x) will be the square of a rational number. In order to solve in this sense, for example, $A\ (x) = ax^{2n} + bx^{2n-1}$, where $n = 1, 2, 3, . . .$ divide by x^{2n-2} to arrive at the form $ax^2,+ b$, which should be set equal to a square polynomial of which the monomial of maximum degree is ax^2, such that the equation has a rational root.

Al-Karajī noted that problems of this type have an infinite number of solutions and proposed to solve many of them, some of which were borrowed from Diophantus while others were of his own devising. An exhaustive enumeration of these problems cannot be given here. We shall present only the principal types of algebraic expressions of polynomials that can be set equal to a square.[37]

1. Equations in one unknown:

$ax^n = u^2$

$ax + bx = u^2$ and in general $ax^{2n} + bx^{2n+1} = u^2$

$ax^2 + b = u^2$ and in general $ax^{2n} + bx^{2n-2} = u^2$

$ax^2 + bx + c = u^2$ and in general $ax^{2n} + bx^{2n-1} + cx^{2n-2} = u^2$

$ax^3 + bx^2 = u^2$ and in general $ax^{2n+1} + bx^{2n} =u^2$ for $n = 1,2,3,. . ..$

2. equations in two unknowns:

$x^2 + y^2 = u$, $x^3 + y^3 = u^2$, $(x^2)^{2m} + (y^3)^{2m+1} = u^2$ $(x^{2m+1})^{2m+1}-(y^{2m})^{2m} = u^2$.

3. Equation in three unknowns:

$x^2 + y^2 + z^2 + (x+ y + z) = u^2$.

4. Two equations in one unknown:

5. Two equations in two unknowns:

6. Two equations in three unknowns:

7. Three equations in two unknowns:

8. Three equations in three unknowns:

In al-Karajī᾽s work there are other variations on the number of equations and of unknowns, as well as a study of algebraic expressions and or polynomials that may be set equal to a cube. From a comparison of the problems solved by al-Karajī and those of Diophantus it was found that "more than a third of the problems of the first book of Diophantus, the problems of the second book starting with the eighth, and virtually all the problems of the third book were included by al-Karajī in his collection,"[38] It should be noted that al-Karajī added new problems.

Two sorts of preoccupations become evident in al-Karajī᾽s solutions: to find methods of ever greater generality and to increase the number of cases in which the conditions of the solution should be examined. Hence, for the equation $ax^2 + bx + c = u^2$—although he supposed that its solution requires that a and c be positive squares—he considered the various possibilities: a is a square, b is a square, neither a nor b is a square in $ax^2 + b = u^2$ but $— b/a$ is a square. In addition he showed that $\pm(bx — c) — x^2 = u^2$ has no rational solution unless $b^2/4 \pm c$ is the sum of two squares.[39] Another example is that of the solution of the system $ax + b = u^2$ and $ax + c = v^2$ where he set up $b — c = a \cdot (b — c)/\alpha$ and took $ax + b = (a + [b — c]/a)^2/4$.

The same preoccupation appears in his solution of the system $x^2 + y = u^2$ and $y^2 + x = v^2$, where he sought first to transform $x = at$ and $y = bt$, $a > b$, in order to posit $(a — b)t = \lambda\, a^2 + t^2 + bt = u$; $b^2t^2 + at = v$, and to solve the problem by means of the demonstrated identity

This concern with generality is also evident in the following two examples: (1) $x^3 + y^3 = u^2$, where be set $y = mx$ and $u = nx$, with $n, m \in q$ and derived $x = n^2/1 + m^2$—a method applicable to more general rational problems of the form $ax^n + by^n = cu^{n-1}$— and $x^3 + ax^2 = u^2$; $x^3 — bx^2 = v^2$, where a and b are integers; he set $u = mx$, $v = nx \Rightarrow x = m^2 — a = n^2 + b$, from which he showed that the condition that m and n should fulfill is $m^2 — n^2 = a + b$. He set $m = n + t$ and obtained $2\,nt + t^2 = a + b \Rightarrow n = a + b — t^2/2\,t$

A great many other examples could be cited to illustrate al-Karaji's incontestable concern with generality and with the study of solutions, as well as a considerable number of other mathematical investigations and results. His most important work, however, remains this new start he gave to algebra, an arithmetization elicited by the discovery of Diophantus by a mathematician already familiar with the algebra of al-Khwārizmī. This new impetus was understood perfectly and extended by al-Karajī's direct successors, notably al-Samaw᾽al. It is this tradition, as all the evidence indicates, of which Leonardo Fibonacci had some knowledge, as perhaps did Levi ben Gerson.[40]

NOTES

1. F. Woepcke, *Extrait du Fahri, traité d'algébre* (Paris, 1853); A. Hochhiem, *Al-Kāfīl fīl Ḥisāb*, 3 pts. (Halle, 1877–1880).

2. G. Levi della Vida, "Appunti e quesiti di storia letteraria araba, IV," in *Rivista degli studi orientali*, **14** (1933), 264 ff.

3. No claim for completeness is made for this table, because of the dispersion of the Arabic MSS and their insufficient classification.

Title	al-Karkhī	al-Karajī
alFakhrī	BN Paris 2495 East Efendi Istanbul 3157 Cairo Nat. Lib., 21	Köprülü Istanbul 950
al-Kāfī	Gotha 1474	Topkapi Sarayī, Istanbul A. 3135
	Alaxandria 1030	
		Damat, Istanbul no. 855
		Sbath Cairo 111
al-Badī';		Barberini Rome 36, 1
ilal-ḥisā al-jabr	Hūsner pasha, Istanbul 257	Bodleian Library I, 968, 3

Title	al-Karkhī	al-Karajī
Inbat al-miyāh al-khafiyyat	Publ. Hyderabad, 1945, on the basis of the MSS. of the library of Aya Sofya and of the library of Bankipore.	

4. One encounters the same difficulties when one considers the MSS of the later Arab commentators and scholars. Thus in the commentaries of al-Shahrazūrī (Damat 855) and of Ibn al-Shaqqāq (Topkapi Sarayī A. 3135), both of which refer to *al-Kāfi*, one finds the name al-Karajī, whereas in MS Alexandria 1030 one finds al-Kharkhī.

5. See A. Anbouba, *L'algèbre al-Badī'; d'al-Karajī* (Beirut, 1964), p. 11; this work has an introduction in French.

6. This MS was classified as anonymous until the present author identified it as being the *al-Bāhir* of al-Samawʾal. See R. Rashed, "L'arithmétisation de l'algébre au 1 léme siècle," In *Actes du Congrè de l'histoire des sciences* (Moscow, in press); and R. Rashed and S. Ahmed, *L'algèbre al-Bāhir d'al-Samawʾal* (Damascus, 1972).

7. In Arab dictionaries the" mountain countries" include the cities located between "Ādharbayjān, Arab Iraq, Khourestan, Persia, and the land of Deīlem (a land bordering the Caspian Sea)."

8. Woepcke, *op. cit.*, p. 4.

9. See R. Rashed, "Algèbre et linguistique: L'analyse combinatoire dans la science arabe," in R. Cohen, ed., *Boston Studies in the Philosophy of Science*, X (Dordrecht).

10. See M. I. Medovoi, "Mā yaḥtāj ilayh al-Kuttaāb waʾl-';ummāl min sinā';at al-ḥisab," in *Istoriko-mathematicheskie issledovaniya***13** (1960), pp. 253–324.

11. *Al-Fakhrī* see Woepcke, *op. Cit.*, p. 48.

12. See al-Samaʾwal, *op. cit.*, pp. 20 ff. of the Arabic text.

13. *Ibid.*

14. *Ibid.*, p. 60 of the Arabic text.

15. For example, for the first method, to find the root of $x^6 + 4x^5 + (4x^4 + 6x^3) + 12x^2 + 9$; one takes the roots of x^3 and of 9; one then divides $4x^5$ by x^3 or $12x^2$ by 3; in both cases one obtains $4x^2$. The root sought is thus $(x^3 + 2x^2 + 3)$. For the second method, take $x^8 + 2x^6 + 11x^4 + 10x^2 + 25$. one finds the roots of x^8 and of 25; x^4 and 5, then subtracts as indicated to obtain x^4, the root of which is x^2. The root sought is thus $(x^4 + x^2 + 5)$. See *al-Fakhrī*, p. 55; and al-Badi';, p. 50 of the Arabic text.

16. Al-Samawal, *op. cit.*

17. *Al-Badi';*, p. 31 of the Arabic text.

18. See Šaraf al-Dī al Tusī, MSS India office 80th 767 (I.O. 461) and the important work on decimal numbers.

19. See *The Commentary of Pappus on Book X of Euclid's Elements*, W. Thomson, ed. (Cambridge, Mass., 1930), p. 193.

20. *Ibid.*

21. Al-Karajī, *op. cit.*, p. 29 of the Arabic text.

22. For Euclid, book X, see Van der Waerden, *Erwachende Wissenschaft* (Basel-Stuttgart, 1956). J. Vuillemin, *La philosophie de l'algèbre* (Paris, 1962), and P. Dedron and J. Itard, *Mathéematiques et mathématisation* (Paris, 1959).

23. *Al-Badi';, p.* of the Arabic text.

24. See the introduction to the present author's edition of al-Bāhir, cited above (note 7).

25. See Anbouba, *op. cit.*,pp. 32 ff. of the Arabic text and *pp.* 36 ff. of the French intro.

26. See *al-Fakhrī,* in Woepcke, *op. cit., p.* 58.

27. See *al-Badi';*, in Anbouba, *op. cit., p* 33 of the Arabic text.

28. See the chapter on numerical principles in al-Samawʾal, *op. cit.*

29. See *al-Fakhrī*, in Woepcke, *op. cit., pp.* 59 ff.

30. See al-Samawʾal, *op. cit., pp.* 71 ff. of the Arabic text.

31. see *al-Fakhrī*, in Woepcke, *op. cit., p.* 63, with the trans. improved by comparison with MSS of the Bibliothàeque Nationale, Paris.

32. See *al-Samaw‘;al*, *op . cit.*, pp. 71 ff. of the Arabic text.

33. See *al-Fakhri*, in Woepcke, *op.cit., pp.* 64 ff.

34. *lbid.*, *pp.* 90–100.

35. *lbid.*, *p.* 72; *Al-Badi‘;*, in Anbouba, *op.cit., p.* 62 of the Arabic trext.

36. *Al-Fakhrī*, with trans, improved by comparison with the MSS of the Bibliothèeque NAtionale.

37. See *al-Fakhrī* and *al-Badi‘;*.

38. See *al-Fakhrī, op. cit.*, p. 21.

39. *lbid.*, *p.* 8.

40. See the compariosn made by Woepcke, *op.cit.*; and G. Sarton, *Introduction to the History of Science* (1300–1500), *p*.596.

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I. Original Works. In addition to the works cited in note 3, all of which have been published except ‘;*ĭlal ḥisāb al-jabr*, the Arabic bibliographies and al-Karajī himself mention other texts that seem to have been lost. Those mentioned in the bibliographies are *Kitāb al ‘;uqūd wa’l abniyah* (“Of Vaults and Buildings“) and *Al-madkhal fī ’ilm al-nujūm* (“Introduction to Astronomy”). Cited by Karajī in *al-Fakhrī* are *Kitāb nawādir al-ashkāl* (“On Unusual Problems”) and *Kitāb al dūr wa’l wiṣāyāa (“On Houses and Wills”);* and in *al-Badi‘;*, “In Inderminate Analysis” and *Kitāb fi’l-hisāb al-hindi* (“On Indian Computation”). finally, al-Samawʾal mentions a book by al-Karaji from which he has extracted his text on binomial coefficients and expansion.

II. Secondary Literature. Besides the works cited in the notes, see Amir Moez, “Comparison of the Methods of Ibn Ezra and Karhī,” in *Scripta mathematica* **23** (1957); and L. E. Dickson, *History of the Theory of Numbers* (New york, 1952).

See also R. Rashed, “L’induction mathématique-al-Karaji et As-Samaw’al,” in *Archive for History of Exact Sciences*, **1** (1972), 1–21.

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