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(*b*. Berlin, Germany, 10 February 1901;d. Boston, Massachusetts, 17 April 1977)

mathematics.

Brauer was the youngest of three children of Max Brauer, an influential and wealthy businessman in the wholesale leather trade, and his wife, Lilly Caroline Jacob. He attended the Kaiser-Friedrich-Schule in Berlin-Charlottenburg and had an interest in science and mathematics as a young boy, an interest that owed much to the influence of his gifted brother Alfred, who was seven years older.

In February 1919 Brauer enrolled in the Technische Hochschule in Berlin but soon realized that, in his own words, his interests were “more theoretical than practical.” He transferred to the University of Berlin after one term. He took his Ph.D. there in 1925, under the guidance of the algebraist Issai Schur.

On 17 September 1925 Brauer married Ilse Karger, a fellow student and the daughter of a Berlin physician. They had two sons, George Ulrich and Fred Günther, both of whom became research mathematicians.

Brauer’s first academic post was at the University of Königsberg (now Kaliningrad), where he remained until dismissed by Hitler’s 1933 decree banning Jews from university teaching. He spent 1933 and 1934 at the University of Kentucky at Lexington, and 1934 and 1935 at the Institute for Advanced Study at Princeton, where he was assistant to Hermann Weyl. After this Brauer held professorships at the University of Toronto (1935–1948), the University of Michigan at Ann Arbor (1948–1952), and Harvard (1952–1971). He lived near Boston for the rest of his life. Weakened by aplastic anemia, he died of a generalized infection.

Brauer was an elected member of the Royal Society of Canada, the American Acadenn of Arts and Sciences, the National Academy of Sciences, the London Mathematical Society, the Akademie der Wissenschaiten (Göttingen), and the American Philosophical Society. He was president of the Canadian Mathematical Congress (1957–1958) and of the American Mathematical Society (1959–1960). He received the Guggenheim Memorial Fellowship (1941–1942), the Cole Prize of the American Mathematical Society (1949), and ihe National Medal for Scientific Merit (1971).

Brauer was one of the most influential algebraists of the twentieth century. He built on the foundations of the representation theory of groups that were laid by Georg Frobenius, William Burnside, and Issai Schur in the years 1895–1910; and over his long career he brought the representation theory of finite groups, in particular, to a remarkable depth and sophistication. His first important research, however, was concerned with the representations of a continuous (topological) group.

By a representation of a given linear group Γ is meant a homomorphism $H:\Gamma\rightarrow GL(N, C)$, whereby each element s of Γ is represented by a nonsingular complex matrix or linear transformation H(s) of some finite degree N. If Γ is a topological group, it is assumed that H is continuous, that is, that each matrix coefficient of H (s) is a continuous function of s. Among topological groups the most important are the classical linear groups, such as the group O(n) of all real orthogonal transformations of n variables, or its subgroup SO(n) (often called the rotation group) consisting of the orthogonal transformations of determinant I.

In 1897 Adolf Hurwitz introduced a new and fundamental idea into the study of such groups, that of an invariant integral. He defined such an integral for SO(n) and used it to calculate polynomial invariants for this group. Schur realized that his own treatment of Frobenius’ character theory of a finite group I could be extended to a continuous linear group I on which an invariant integral could be defined. In a series of papers published in 1924, he used Hurwitz’s integral to find the irreducible characters of O(n). Brauer was attending Schur’s seminar at this time, and Schur suggested to him that it might be possible to fuul a purely algebraic treatment of this work, that is, one that did not rely on the analytic notion of an integral. Brauer found such a treatment, and with it calculated the irreducible characters of the groups O(n) and SO(n); this became his dissertation, for which he was awarded the Ph.D. summa cum laude in 1926.

While Brauer was writing his dissertation, Hermann Weyl was working on his papers on the representations of semisimple Lie groups (these include the classical linear groups). This work of Weyl’s has claim to he the finest single mathematical achievement of the twentieth century. It is based on Schur’s methods with the invariant integral and Élie Cartan’s construction of representations of a semisimple Lie group I by means of representations of its Lie algebra g: Schur’s and Brauer’s results on $O(n)$ and $SO(n)$ come out as special cases. Weyl’s results and methods have been the starting point of a huge amount of research in pure mathematics and in quantum physics. By contrast Brauer’s algebraic treatment for the orthogonal groups is little known—it uses difficult and (now) unfashionable techniques from the theories of determinants and invariants, and was published only as his Ph.D. dissertation.

Brauer greatly admired Weyl and during his year as Weyl's assistant at the Institute for Advanced Study, he briefly returned to the classical linear groups. From this period date a joint paper with Weyl on spinors and a paper in which Brauer calculates, by purely algebraic means, the Poineаré polynomials of the classical groups (unitary, symplectic, and orthogonal). Brauer's last paper on continuous groups, published in 1937, hints at a general representation theory for continuous groups, strictly algebraic in nature and based on invariant theory. A promised sequel never appeared.

Much of Brauer's work while he was in Königsberg (1925–1933) was concerned with simple algebras and rooted in Schur's theory of splitting fields. Suppose k is a given field, K an algebraically closed field that contains k, and that $H{:}\Gamma\rightarrow GL(f, K)$ is an irreducible representation of some group Γ. Then each elements of Γ is represented by a nonsingular f x f matrix $H(s)$ whose coefficients lie in K; the condition that H be irreducible means that the set of all K-linear combinations of the H(s) s ∊ Γ, is the full matrix algebra K_f of all f x f matrices over K. We shall assume that the trace X(s) of each matrix H(s) lies in the ground field k. Then a field L $(k \subseteq L \subseteq K)$ is a splitting field for H (or for its character X) if L is a finite extension of k and there exists a matrix $P \in GL\ (f, K)$ such that all the coefficients of all the matrices $P^{-1}H(s)\ P,\ s{\in}\Gamma$ lie in L. The least degree $m_g\ (H) = m_g\ (X)$ over k, among all such splitting fields L, is called the Schur index of H or X over k; Schur had initiated the study of splitting fields (in the case where k is an algebraic number field and k = ℂ) in the early 1900's, and had proved a number of facts about the Schur index.

Brauer and Emmy Noether (who was then at Göttingen) showed, in a paper published in 1927, how the splitting fields H are determined by the algebra A of all k-linear combinations of the matrices $H(s),\ s \in I'$. A is a finite dimensional simple algebra over k that is central; its center consists only of the scalar multiples of the identity. A given field L of finite degree $(L{:}k)$ is a splitting field for H if and only if $L \otimes_b A$ is isomorphic to the full matrix algebra L_f over L. This last condition depends only on the algebra A, and a field L that satisfies it is called a splitting field for A. Brauer and Noether's main result (proved under certain restrictions on the ground field k, which were later shown by Noether to be unnecessary) was that the splitting fields of a given central, simple k-algebra A are (up to isomorphism) the same as the maximal subfields of the algebras B in the same algebra class as A. The algebra class [A] of A is defined as follows. By Joseph Wedderburn's structure theorem (1907). A is isomorphic to the algebra D_i of all t x t matrices over a certain central division algebra D and for a certain positive integer t: the class [A] then consists of all central simple algebras over k that are isomorphic to D_s, for any positive integer s. Theset of all such algebra classes, with given ground field k, forms a group $B(k)$, now known as the Brauer group of k; the product of classes $[A],\ [B]$ is defined to be the class $[A \otimes_k B]$, and the identity clement of $B(k)$ is the class [k]. It has turned out that $B(k)$ is a fundamentally important invariant of the field k. Brauer studied it with the help of a technique of factor sets, and from this beginning has grown the theory of Galois cohomology and its many uses in number theory. In another direction. M. Auslander and O. Goldman showed in 1960 how to define $B(R)$ for an arbitrary commutative ring R, thereby beginning a new chapter in commutative algebra.

Brauer's work with Emmy Noether brought him into contact not only with this influential algebraist and her school in Göttingen but also with the famous conjecture of L. E. Dickson that every central simple algebra A over an algebraic number field k contains a maximal subfield L that is a Galois extension of k with cyclic Galois group. This conjecture was proved in 1931 in a joint paper by Brauer. Noether, and H. Hasse that is the culmination of a long development in the theory of algebras. Brauer's association with this work secured his reputation as one of the best mathematicians of the rising generation in Germany.

Brauer was abruptly dismissed in 1933, along with all other Jewish university teachers in Germany. The disadvantages and disruptions of a forced emigration at the age of thirty-two were offset to some extent by the new contacts Bruer made, not only with American mathematicians but also with other German scientists who found refuge in America in the 1930's Theyear 1934–1935, when Brauer was Weyl's assistant at the Institute for Advanced Study, saw an extraordinary gathering of mathematicians and physicists of the first rank: J. W. Alexander, Albert Einstein John von Neumann, Oswald Veblen, and Weyl were permanent professors at the Institute; and the mathematics faculty at Princeton University included Salomon Bochner. S. Lefschetz, and Joseph H. M. Wedderburn. Among the visiting members of the Institute that year were, besides Brauer, W. Magnus, C. L. Siegel, and Oscar Zariski.

In 1935 Brauer took up a post as assistant professor at the University of Toronto, where he remained until 1948. becoming in due course associate and then full professor. Here he developed his modular representation theory of finite groups, which will probably continue to be regarded as his most original and characteristic contribution to mathematics.

Representation theory began with Frobenius paper "Über Gruppencharaktere," published in 1896. In Frobenius' theorem the irreducible characters of a group were defined in terms of what can now be described as the representations of the center of the group algebra ℂG (ℂG is the linear algebra over the complex field ℂ having the elements of G as basis). Schur (who had been Frobenius' pupil) reformulated Frobenius' character theory in a beautiful paper published in 1905 that became the basis of all subsequent expositions of the subject. Schur defined the character X_H of an arbitrary representation $H{:}G\rightarrow GL(N, \mathbb{C})$ of G to be the complexvalued function X_H:Gℂ given by X_H (s), = trace H(s), $s \in G$. If H is an irreducible representation, then X_H is said to be a simple, or irreducible, character. Schur rederived Frobenius' orthogonality relations for the irreducible characters, from which follows the fundamental theorem: Two representations H, H' of the group G are equivalent if, and only if, their characters are equal, $X_H = X_H$.

It was recognized very soon that Frobeniusk arguments do not hold when ℂ is replaced by an algebraically closed field k of finite characteristic p, although L. E, Dickson proved in 1902 that the orthogonality relations for the characters are still true, provided p does not divide the order |G| of G. In two later papers Dickson considered the case where p divides |G|. In this case the group algebra kG is not semisimple. A representation $F{:}G\to GL(n, k)$ is no longer determined up to equivalence by the natural character X_F = trace F. After Dickson's papers were published in 1907, little more was done on representations over fields of finite characteristic until the mid 1930's, when Brauer laid the foundations of his modular representation theory in three fundamental papers, the last two of which were written with C. Nesbitt, who took his Ph.D. at Toronto under Brauer's supervision.

Let G_0 denote the set of all p'-elements (or p-regular elements) of G; these are the elements whose order is prime to p. Let |G| = p^0m, where a ≥ 0 and p does not divide m. Then each element of G_0 satisfies the equation $g^m=1$, so if $F{:}G\to GL(n, k)$ is a representation, the eigenvalues $\alpha_1, \ldots\alpha_n$ of F[g] are mth roots of unit in the field k. Theset $Um(k)$ of all mth roots of unit in k forms a cyclic group of order m. the group operation being multiplication in k. But the multiplicative group U_m (ℂ) of all complex mth roots of unity is also cyclic of order m. Therefore one can find a multiplicative isomorphism c:U_m $(k)\to U_m$ (ℂ); in general this can be done in many ways. Choose any such isomorphism c. Brauer defines the modular character (later known as the Brauer character) of the representation F to be the function $\beta_F{:}G_0\to$ ℂ given by

This gives a complex valued function in place of the k-valued natural character X_f, for which

by the daring—almost impudent—device of complexifying the eigenvalues $\alpha_1, \ldots, \alpha_n$ That definition (l), and not the natural definition (2), is the correct basis for a modular character theory soon becomes clear. If $F1\ldots F1$ is a full set of irreducible representations of G over k, then their Brauer characters $\beta_1, \ldots, \beta_i$ are linearly independent functions on G_0. For any representation $F{:}G \to GL(n, k)$ one has $\beta_F = \Sigma n_i\,(F)\beta_i$ where n(F) is the multiplicity with which F_i appears as a composition factor in F. Brauer characters (like natural characters) are class functions, that is. $\beta_F\,(g) = \beta_F\,(g')$ whenever g, g´ belong to the same conjugacy class of G. Frobenius had shown that the number of irreducible ordinary characters of G is equal to the number of conjugacy classes of G; Brauer showed that the number l of irreducible modular characters is equal to the number of p-regular classes, that is, to the number of conjugaey classes lying in G_0.

From the beginning, Brauer saw the modular character theory—which is based on representations of G over a field k of finite characteristic p—as a source of information on the ordinary characters— which is based on representations on ℂ or some other field of characteristic zero. If $X_1\ldots X_0$ are the irreducible ordinary characters of G. and $\beta_1\ldots, \beta_1$ are the irreducible modular characters, there exist nonnegative integers d_ai such that the equations

hold for all elements g in G_0. To explain these equations, we need some technical preliminaries. Let L be a field of characteristic zero that is a splitting field for G; this means that for each σ = 1, …, s, the character X_σ can be obtained from a matrix representation X_σ such that all the coefficients of the matrices X_σ lie in L. This is to say that L is a splitting field for all the X_σ in the sense we used earlier. We assume that L has a subring R that is a principal ideal domain, such that L is the field of quotients of R: moreover, R should have aprime ideal v containing the integer p. We identify $\bar{R} = R/p$ (which is a field of characteristic p) with a subfield of the field k. The matrix representations X_σ can be chosen so that the matrix coefficients of X_σ (g) all lie in R.

Taking these mod p, we get a modular representation of G, that is, a representation over the field k. The Brauer character of $\bar{X\sigma}$ is (suitably identifying a part of L with a part of ℂ, so that the character values $X_\sigma[g]$ can be regarded as elements of ℂ) identical with the restriction of X_σ to G_0. Equations (3) therefore say that $d_{\sigma i}$ is the multiplicity of F_i as composition factor of $\bar{X\sigma}$: Brauer called the d_i decomposition numbers; they record the decomposition that the ordinary irreducible representations X_σ of G undergo when they are reduced mod v. But these numbers have another, quite separate interpretation. Brauer found that the modular irreducibles $F_1, \ldots, F_l$ are in natural correspondence with the indecomposable direct summands $U_1, \ldots, U_l$ of the regular representation of the group algebra $A = kG$. (He showed, in fact, in joint work with Nesbitt, that this holds for any finite-dimensional k-algebra A. U_l is what is now called the projective cover of F_l. Brauer's ideas from this period—the late 1930's—pervade much modern research on algebras and their representations.) If splitting field L is taken to be complete with respect to a suitable discrete valuation, with R as the ring of valuation integers, then each U_l can be lifted to a representation U_1 of G over R, that is, to a representation in characteristic zero, which therefore has an ordinary character η_i, say.

There hold then the remarkable equations

that are in some sense dual to (3), From (3) and (4) one deduces equations

where the c_{ij} are the Cartan invariants of the algebra $A = kG{:}\ c_{ij}$ a may be defined as the multiplicity (as composition factor) of F_i *in* U_i Cartan invariants exist for any algebra A, but equations (5) show that for a group algebra $A = kG$, the l *x* l matrix $C = (c_{ij})$ has special properties: it is symmetric and positive definite. Brauer also discovered the deep theorem that det C is a power of the characteristic p of k. This was published in the first of a remarkable series of papers appealing in 1941 and 1942. In these Brauer introduced new and sophisticated methods for the study of group characters, and began to give applications of his theory to the structure theory of finite groups.

Fundamental to this work was the idea of a block. Blocks are most easily defined by taking a decomposition 1 = e_1 +... +e_1 1 as sum of orthogonal primitive idempotents of e_τ, of the center Z(kG) of kG. This can be lifted to a corresponding decomposition I = $\hat{e}_1$ +... + ê in Z(RG). An ordinary (or modular) irreducible character ψ of G is said to belong to the block B_i of G if $\hat{e}_i$ (or e_i) is represented by the identity matrix in a representation corresponding to ψ. In this way both sets {X_i, ..., X_s} and {β_1, ..., β_1 fy} are partitioned among the t blocks B_1, ..., B_1 of G. Block theory aims to give information about the ordinary characters of a given block B_1 of G, in terms of information available for a block b in some *p*-local subgroup *H* of *G* (*H* is usually the normalizer or centralizer in *G* of some p-subgroup of *G*)

In the most favorable cases, Brauer's methods show that a part of the character tabic for H is almost identical with a part of the character table for G: this is now much used in the computation of character tables for known finite groups. Brauer saw in his theory a potential tool for studying finite simple groups. (We shall return to this later.)

The main facts about blocks of a group *G* and their relation to blocks of subgroups *H* of *G* as well as the important refinement to equations (3) involving coefficients called the generalized decomposition numbers, had been published (sometimes without proofs, which appeared years later) by 1947. In that year Brauer also published a solution to Emil Artin's conjecture that the sums of the Artin L-series are entire functions. Brauer had field a Guggenheim fellowship in 1941 and 1942 and spent a part of this time visiting Artin in Bloomington, Indiana. They both knew that the solution of Artin's conjecture rested on the validity of a certain statement about group characters; and at some point in the next few years Brauer realized that this statement could be proved, using methods he had developed for his modular character theory.

By 1953 he embodied the essential idea from these methods in a theorem—perhaps his most widely known—that if θ is a complex valued class function on a finite group G, then θ is a generalized character of G, if its restriction θ is a generalized character of E, for every elementary subgroup E of G. (If X_1, ..., X_s are the irreducible ordinary characters of a group G, then functions of form z 1X_1 +... + $z_s X_s$ with the $z_1$integers—not necessarily positive—are called generalized characters of G. A group E is elementary if. for some prime p, E is the direct product of a p-group and a cyclic group,) From this theorem Brauer also obtained a new proof of a much older conjecture (which he had first solved a few years earlier): that if ∈ is a primitive gth root of unity, where g is the order of G, then Q(∈) is a splitting field for G (that is, for all the irreducible characters of G).

In 1948 Brauer moved from Toron to the University of Michigan at Ann Arbor, and four years later to Harvard. From this period can be dated the first systematic attack on the problem of describing (or classifying) all finite simple groups. In a paper written with his pupil K. A. Fowler. Brauer proved by very elementary means a striking fact: Let G be a simple group of even order, and let x be an involution in G (that is, an element satisfying the conditions $x^2 = 1$, x Let $H = C_G(x)$ be the centralizer of x and let n be the order of H. Then G has a proper subgroup of index less than $1/2n(n + 1)$. This gives hope to a general program announced by Brauer in 1954: Given an abstract group H with an involution x in its center, to find all simple groups G containing H as a subgroup in such a way that $H = C_G(x)$. For the theorem above shows that (with H and x given) the number of isomorphism types of such groups G is finite—although it does not provide practical means of constructing them. It is natural to take for H the centralizer of an involution in some known simple group G_0. It may happen that $G_0$is—up to isomorphism— the only simple group having an involution x such that C_G (x) ≅ H; we have then a characterization of G_0 by an internal property.

By 1954 Brauer (and independently M. Suzuki and G. E. Wall) had found characterizations of this kind for groups of type PSL(2, q). The method in such problems is to build up knowledge of the ordinary character table of *G* from what is given about *H*. This is exactly the kind of work for which Brauer's modular methods were designed, and he used these methods successfully on many simple group characterizations during the next twenty-five years, constantly refining and developing his modular theory. But now new actors began to appear in the drama of the simple groups. Suzuki and Wall used only ordinary character theory in their work on PSL(2.q), and their methods—still following Brauer's program—led eventually to the discovery of new simple groups. Around 1960 J. G. Thompson began a powerful attack on the internal structure of simple groups, using new group-theoretical methods. In 1963 he and W. Feit verified the long-standing conjecture that every simple (noncyclic) group G has even order—which showed that G must possess involutions. so that Brauer's general program will always apply, The Feit-Thompson paper, remarkable for its length and difficulty, again used only ordinary character theory, based on an old theorem of Frobenius.

In the period 1960–1980 dozens of people joined the common effort to find all finite simple groups. Many new simple groups were found. some by application of Brauer's program and others from quite different sources. Brauer was deeply involved in this effort right up to the end of his life but did not see its final success, of which he must be counted one of the chief architects.

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J. A. Green