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(*b*. Bologna, Italy, 15 April 1552; *d*. Bologna, 11 February 1626),

mathematics.

Cataldi was the son of Paolo Cataldi, also of Bologna. Little else is known of his life. His first teaching position (1569–1570) was at the Florentine Academy of Design. He then went to Perugia to teach mathematics at the university; his first lecture was given on 12 May 1572. He also taught at the Academy of Design in Perugia and was a lecturer in mathematics at the Studio di Bologna from 1584 until his death. Cataldi showed his benevolence by giving the superiors of various Franciscan monasteries the task of distributing free copies of his *Pratica aritmetica* to monasteries, seminaries, and poor children.

In the history of mathematics Cataldi is particularly remembered for the *Trattato del modo brevissimo di trovar la radice quadra delli numeri*, finished in 1597 and published in 1613. In this work the square root of a number is found through the use of infinite series and unlimited continued fractions. It represents a notable contribution to the development of infinite algorithms.

In the orientation of mathematical thinking of the late Renaissance and the seventeenth century toward infinitesimal questions, along with geometric methods (such as Cavalieri's principle of indivisibles), in the field of arithmetic, the passage from the finite to the infinite appears in processes of iteration in calculus. In this area Cataldi started with the practical rulesfurnished by ancient treatises on arithmetic for finding the square root of any natural number N that is not a perfect square. These treatises gave first the basic rule for finding the natural number a such that

$a^2 < N$ (a + 1)^2.

Then rules were given for finding approximate rational values of expressed respectively by the formulas

The first of these formulas, which goes back at least as far as Hero of Alexandria (and probably to the Babylonians), coincides with the arithmetic mean 1/2 ($a + N/a$) of the two values a and N/a (which have as their geometric mean , which is being sought). The second formula is obtained by a process of linear interpolation:

$x_1 = a^2 x^2 =(a+1)^2 x = a^2 + r$

$y_1 = a^2 y^2 = a + 1$.

The y corresponding to x is determined by means of the equation

from which one obtains

That is, setting $x = N = a^2 + r$ the result is

Returning to the rounded maximum value (already considered), we write

Starting with a_1 by an analogous procedure we obtain a new value, a_2, a closer approximation than a_1:

Setting , one obtains

By iteration of the indicated procedure we obtain

As Cataldi established, r_n may be rendered as small as one wishes, provided that n is large enough. Theformula

results in

Analogously, in general

from which one obtains

Given , one will have

which guarantees the rapid convergence of the series representing :

Cataldi presents analogous considerations in relation to the other series, which is obtained from the rounded minimum values of :

We shall now see how, starting with the rounded maximum value (already considered) of ,

Cataldi arrives at continued fractions. With the aim of obtaining a better approximation than that reached with a_1 Cataldi adds to the denominator a value x, and then considers the expression.

He observes that this expression has a roundedminimum value of when (and only when) $a + x$ has a rounded maximum value of That is (as can be verified by simple calculation),

when and only when $(a + x)^2 > a^2 + r$. Therefore, given

one will have

That is,

has a rounded minimum value of , while, forreasons analogous to those already given, the roundedmaximum value of the same root will be

and so on, indefinitely. If we write

we will see that the result p_n/q_n for order n can be expressed in terms of the result for order $n - 1$ by means of the formula

From it are derived the recurrent formulas habituallyused by Cataldi:

$p_n = rq_{n-1}$, $q_n = 2aq_{n-1} + p_{n-1}$.

Moreover, Cataldi finds the fundamental relation

$p_nq_{n-1} - p_{n-1}q_n = (-1)^n r^n$,

from which can be obtained the expression of the difference between two consecutive results:

In sum, Cataldi compares the results of the series just studied with the results of the continued fraction, considering the same , and establishes that the results of the series obtained by starting with the rounded minimum values reproduce the results of even orders of the continued fraction (a_2, a_4,...); the results of the series obtained by starting with the maximum values reproduce the results of order

$(2^n - 1)$ (n =1, 2, 3 ⋯).

Having examined Cataldi's contribution to the theory of continued fractions, clearly explained in various writings by E. Bortolotti, it remains for us to consider the place of these contributions in the history of mathematical thought. The question is complex and has given rise to many discussions and polemics. Given the great number of questions that can lead to consideration of continued fractions, hints of the theory of continued fractions are presented many times and in presumably independent ways in the course of mathematical history. It is not our task to reconstruct the history of continued fractions; we shall limit ourselves to indicating some elements of it in order to clarify Cataldi's position. We are led to consideration of this question when we come to Euclid's procedures for determining the greatest common divisor of two natural numbers (bk. VII, prop. 2), which lead to consideration of a limited continued fraction, while the Euclidean criterion for establishing whether two homogeneous magnitudes are commensurable or incommensurable leads in the latter case to an unlimited continued fraction (*Elements*, bk. X, prop. 2); in neither case is the algorithm presented explicitly.

The successive reductions of the continued fraction that is expressed by appear in the work of the Neoplatonic philosopher Theon of Smyrna (second century a.d.), *Expositio rerum mathematicarum ad legendum Platonem utilium*. An examination of this text, however, leads one to conclude that such values were calculated with an aim different from that indicated above. For the sake of brevity we shall omit mention of other appearances of the continued fractions, which can be found in Greek, Indian, and Arabic writings.

In the Renaissance, Rafael Bombelli gave a procedure for the extraction of the square root of a number that is not a square, such that the successive steps in the procedure lead to the calculation of the successive results of a continued fraction. However, Bombelli, who refers to numerical cases, performs the calculations that have been considered, in such a way that the final result retains no traces of the algorithm implicitly defined by the procedure of iteration that is applied (see Bombelli's *L'algebra*, E. Bortolotti, ed. [Bologna, 1929), pp. 26–27).

The use of limited continued fractions in the expression of relationships between large numbers is found in the *Geometria practica* of Daniel Schwenter (1627), published soon after Cataldi's death.

The term "continued fraction" was introduced by John Wallis, who gave a systematic treatment of it in his *Arithmetica universalis* (1655). It contains an example of the development of a transcendental number, under the form that originated with William Brouncker:

A theory of continued fractions was devised by Euler, and Lagrange formulated the theorem concerning the periodic character of continued fractions that represent square roots.

Continued fractions have been of great use to mathematicians in the investigation of the nature of numbers—for instance, Liouville's work on the existence of transcendental numbers (1844) was based on the use of the algorithm considered above.

Cataldi also has a place in the history of the criticism of Euclid's fifth postulate, which led to construction of a non-Euclidean geometry. In his *Operetta delle linee rette equidistanti et non equidistanti*, he attempted to demonstrate the fifth postulate on the basis of remainders. The defect of his argument is found in his definition of equidistant straight lines: "A given straight line is said to be equidistant from another straight line in the same plane when the two shortest lines that are drawn from any two different points on the first line to the second line are equal." Cataldi did not realize that two conditions are imposed on the first, given line, which are not stated as compatible: (1) that it is a locus of points at a constant distance from the second line and (2) that it is straight line. The admission of such a compatibility constitutes a postulate equivalent to Euclid's fifth postulate.

Cataldi's other works, which are mainly didactic, are concerned with theoretical and practical arithmetic, algebra, geometry, and astronomy, and furnish good documentation of the mathematical knowledge of his time.

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