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(*b*. Richmond, Surrey, England 16 August 1821; *d*. Cambridge, England, 26 January 1895),

mathematics, astronomy.

Cayley was the second son of Henry Cayley, a merchant living in St. Petersburg, and Maria Antonia Doughty. He was born during a short visit by his parents to England, and most of his first eight year were spent in Russia. From a small private school in London he moved, at fourteen, to King's College School there. At seventeen he entered Trinity College, Cambridge, as a pensioner, becoming a scholar in 1840. In 1842 Cayley graduated as senior wrangler and took the first Smith's prize. In October 1842 he was elected a fellow of his college at the earliest age of any man of that century. He was tutor there for three years, spending most of his time in research. Rather than wait for his fellowship to expire (1852) unless he entered holy orders or took a vacant teaching post, he entered the law, studying at Lincoln's Inn. He was called to the bar in 1849.

During the fourteen years Cayley was at the bar he wrote something approaching 300 mathematical papers, incorporating some of his best and most original work. It was during this period that he first met the mathematician J. J. Sylvester, who from 1846 read for the bar and, like Cayley, divided his time between law and mathematics. In 1852 Sylvester said of Cayley that he "habitually discourses pearls and rubies," and after 1851 each often expressed gratitude to the other in print for a point made in conversation. That the two men profited greatly by their acquaintance is only too obvious when one considers the algebraic theory of invariants, of which they may not unreasonably be considered joint founders. They drifted apart professionally when Cayley left London to take up the Sadlerian professorship but drew together again when, in 1881–1882, Cayley accepted Sylvester's invitation to lecture at Johns Hopkins University.

In 1863 Cayley was elected to the new Sadlerian chair of pure mathematics at Cambridge, which he held until his death. In September 1863 he married Susan Moline, of Greenwich; he was survived by his wife, son, and daughter. During his life he was given an unusually large number of academic honors, including the Royal Medal (1859) and the Copley Medal (1881) of the Royal Society. As professor at Cambridge his legal knowledge and administrative ability were in great demand in such matters as the drafting of college and university statutes.

For most of his life Cayley worked incessantly at mathematics, theoretical dynamics, and mathematical astronomy. He published only one full-length book, *Treatise on Elliptic Functions* (1876); but his output of papers and memoirs was prodigious, numbering nearly a thousand, the bulk of them since republished in thirteen large quarto volumes. His work was greatly appreciated from the time of its publication, and he did not have to wait for mathematical fame. Hermite compared him with Cauchy because of his immense capacity for work and the clarity and elegance of his analysis. Bertrand, Darboux, and Glaisher all compared him with Euler for his range, his analytical power, and the great extent of his writings.

Cayley was the sort of courteous and unassuming person about whom few personal anecdotes are told; but he was not so narrow in outlook as his prolific mathematical output might suggest. He was a good linguist; was very widely read in the more romantic literature of his century; traveled extensively, especially on walking tours; mountaineered; painted in watercolors throughout his life; and took a great interest in architecture and architectural drawing.

Characteristically, as explained in the bibliography of his writings, Cayley frequently gave abundant assistance to other authors (F. Galton, C. Taylor, R. G. Tait, G. Salmon, and others), even writing whole chapters for them—always without ostentation. Salmon, who corresponded with him for many years, gave *Esse quam videri* as Cayley's motto. Although Cayley disagreed strongly with Tait over quaternions (see below), their relations were always amicable; and the sixth chapter of the third edition of Tait's *Quaternions* was contributed by Cayley, much of it coming verbatim from letters to Tait. Cayley was above all a pure mathematician, taking little if any inspiration from the physical sciences when at his most original. "Whose soul too large for vulgar space, in *n* dimensions flourished," wrote Clerk Maxwell of Cayley. So far as can be seen, this was a more astute characterization than that of Tait, by whom Cayley was seen in a more pragmatic light, "forging the weapons for future generations of physicists." However true Tait's remark, it was not an indication of Cayley's attitude toward his own work.

A photograph of Cayley is prefixed to the eleventh volume of the *Collected Papers*. A portrait by Lowes Dickenson (1874, Volume VI) and a bust by Henry Wiles (1888) are in the possession of Trinity College, Cambridge. A pencil sketch by Lowes Dickenson (1893) is to be found in Volume VII.

Cayley's mathematical style was terse and even severe, in contrast with that of most of his contemporaries. He was rarely obscure, and yet in the absence of peripheral explanation it is often impossible to deduce his original path of discovery. His habit was to write out his findings and publish without delay and consequently without the advantage of second thoughts or minor revision. There were very few occasions on which he had cause to regret his haste. (References below to the *Collected Mathematical Papers*, abbreviated *C. M.P.*, contain the volume number, followed by the number of the paper, the year of original publication, and the page numbers of the reprint.)

Cayley is remembered above all else for his contributions to invariant theory. Following Meyer (1890–1891), the theory may be taken to begin with a paper by Boole, published in 1841, hints of the central idea being found earlier in Lagrange's investigation of binary quadratic forms (1773) and Gauss's similar considerations of binary and ternary forms (1801). Lagrange and Gauss were aware of special cases in which a linear homogeneous transformation turned a (homogeneous) quadratic into a second quadratic whose discriminant is equal to that of the original quadratic multiplied by a factor which was a function only of the coefficients of the transformation. Cauchy, Jacobi, and Eisenstein all have a claim to be mentioned in a general history of the concept of invariance, but in none of their writings is the idea explicit. Boole, on the other hand, found that the property of invariance belonged to all discriminants, and he also provided rules for finding functions of "covariants" of both the coefficients and the variables with the property of invariance under linear transformation.

In 1843 Cayley was moved by Boole's paper to calculate the invariants of *n*th-order forms. Later he published a revised version of two papers he had written. The first, with the title "On the Theory of Linear Transformations" (*C. M. P.*, I, no. 13 [1845], 80–94), dealt only with invariants; the second, "On Linear Transformations" (*C. M. P.*, I, no. 14 [1846], 95–112), introduced the idea of covariance. In this second paper Cayley set out "to find all the derivatives of any number of functions, which have the property of preserving their form unaltered after any linear transformations of the variables." He added that by "derivative" he meant a function "deduced in any manner whatever from the given functions." He also attempted to discover the relations between independent invariants—or "hyperdeterminants," as he called them at first, looking upon algebraic invariance as a generalized form of the multiplication of determinants. When writing the notes to his *Collected Papers*, he remarked that what he had done in this paper was to be distinguished from Gordan's "Ueherschiebung," or derivational theory. Cayley may be regarded as the first mathematician to have stated the problem of algebraic invariance in general terms.

Cayley's work soon drew the attention of many mathematicians, particularly Boole, Salmon, and Sylvester in England and Aronhold, Clebsch, and, later, Gordan in Germany. (Jordan and Hermite followed in France; and Brioschi in Italy was to carry the new ideas into the realm of differential invariants, in the study of which his compatriots later excelled.) Salmon's many excellent textbooks (in particular, see his *Modem Higher Algebra*, 1859, dedicated to Cayley and Sylvester), which were translated into several languages, diffused Cayley's results, to which Cayley himself constantly added. Sylvester was, among other things, largely responsible for the theory's luxuriant vocabulary; and in due course Aronhold related the theory to Hesse's applications of determinants to analytical geometry. The vocabulary of the subject is today one of the greatest obstacles to a discussion of invariant theory, since following Gordan's theorem of 1868 and Hilbert's generalizations of it, the tendency has been away from developing techniques for generating and manipulating a multiplicity of special invariants, each with its own name. Notice, however, that Cayley's "quantic" is synonymous with the "form" of later algebraists. As a typical source of terminological confusion we may take the contravariant (or the curve represented by the contravariant equation), called by Cayley the "Pippian" and known elsewhere (following Cremona) as the "Cayleyan."

Beginning with an introductory memoir in 1854, Cayley composed a series of ten "Memoirs on Quanties," the last published in 1878, which for mathematicians at large constituted a brilliant and influential account of the theory as he and others were developing it. The results Cayley was obtaining impressed mathematicians by their unexpectedness and elegance. To take three simple examples, he found that every invariant vanishes, for a binary p-ic which has a linear factor raised to the rth power, if $2r > p$; that a binary p-ic has a single or no p-ic covariant of the second degree in the coefficients according as p is or is not a multiple of 4; and that all the invariants of a binary p-ic are functions of the discriminant and $p — 3$ anharmonic ratios, each formed from three of the roots together with one of the remaining $p—3$ roots. A more renowned theorem concerned the number of linearly independent seminvariants (or invariants) of degree i and weight w of a binary p-ic. Cayley found an expression giving a number which he proved could not be less than that required; and for a long time he treated this as the required number although admitting his inability to prove as much Sylvester eventually gave the required proof.

An irreducible invariant (covariant) is one that cannot be expressed rationally and integrally in terms of invariants (covariants and invariants) of degree lower in the coefficients than its own, all invariants belonging to the same quantic or quantics. At an early stage Cayley appreciated that there are many cases in which the number of irreducible invariants and covariants is limited. Thus in his "Second Memoir on Quantics" (*C. M. P.*, II, no. 141 [1856], 250–275) he determined the number (with their degrees) of "asyzygetic" invariants for binary forms of orders 2 to 6, and he gave similar results for asyzygclic systems of irreducible covariants. Cayley made the mistake, however, of thinking that with invariants of forms of order higher than 4, the fundamental system is infinite. The error (which arose from his wrongly taking certain syzygies to be independent, thus increasing the number of invariants and covariants allowed for) stood for thirteen years, until Gordan (*Crelle's Journal*, **69** [1869], 323–354) proved that the complete system for a binary quantic of any order has a finite number of members. Hilbert, in 1888 and later, simplified and greatly generalized Gordan's findings.

Perhaps the best known of Cayley's "Memoirs on Quantics" was the sixth (*C. M. P.*, II, no. 158 [1859], 561–592; see also the note on 604–606, where he compares his work with that of Klein, which followed), in which Cayley gave a new meaning to the metrical properties of figures. Hitherto, affine and projective geometry had been regarded as special cases of metric geometry. Cayley showed how it was possible to interpret all as special cases of projective geometry. We recall some of the more important results of earlier geometrical studies. Poncelet (*ca.* 1822) had evolved the idea of the absolute involution determined by the orthogonal lines of a pencil on the line at infinity and having the "circular points" (so called because they are common to all circles in the plane) as double points. Beginning with the idea that perpendicularity could be expressed in terms of the formation of a harmonic range with the circular points, Laguerre (*ca*. 1853) showed that the numerical value of the angle of two lines of the Euclidean plane expressed in radian measure is 1/2i times the natural logarithm of the cross ratio which they form with the lines of their pencil through the circular points. Cayley now found that if *P* and *Q* are two points, and *A* and *B* are two further points in which the line *PQ* cuts a conic, then (if *A* and *B* are a real point pair; otherwise, where they are conjugate imaginaries we multiply by *i*) their separation could be expressed as a rather involved arc cosine function involving the coordinates, which space does not permit to be detailed here (see *C. M. P.* 11, no. 158 [1859], 589). A clear idea of the importance of his paper is obtained if we consider Klein's substitution of a logarithmic function for the arc cosine (which Cayley later admitted to be preferable), in which case

where *c* is a constant for all lines, may be taken as the generalized distance (which we may here call δ[*P*,*Q*]) between *P* and *Q*, in the sense that the following fundamental requirements are met by the function: δ(*P*,*Q*) = 0 if and only if *P* and *Q* are identical; δ(*P*.*Q*) = δ(*Q*,*P*); δ(*p*,*Q*) + δ(*Q*,*R*) ≥ δ(*P*,*R*), the equality holding when *p*,*Q*, and *R* are collinear. Cayley referred to the arbitrarily assumed conic as the "Absolute."

In his definition of distance Cayley has frequently been accused of circularity (recently, for example, by Max Jammer, in *Concepts of Space* [Cambridge, Mass., 1954], p. 156) Cayley anticipated such criticism, however, explaining in his note to the *Collected Papers* that he looked upon the coordinates of points as quantities defining only the ordering of points, without regard to distance. (This note shows that Klein drew his attention to Staudt's work in the same vein, of which he was ignorant when writing the sixth memoir.) Thus if x_a and x_b are coordinates belonging respectively to the points *A* and *B*, the corresponding coordinate of *P* may be written $\lambda_1 x_a+\lambda_2 x_b$, and similarly for the remaining points and coordinates. The function δ(*P, Q*) then reduces to one in which no trace of the ordinary (Euclidean) metric distance remains.

The full significance of Cayley's ideas was not appreciated until 1871, when Klein (*Mathematische Annalen*, **4** [1871], 573–625) showed how it was possible to identify Cayley's generalized theory of metrical geometry with the non-Euclidean geometries of Lobachevski, Bolyai, and Riemann. When Cayley's Absolute is real, his distance function is that of the "hyperbolic" geometry; when imaginary, the formulas reduce to those of Riemann's "elliptic" geometry. (The designations "hyperbolic" and "elliptic" are Klein's.) A degenerate conic gives rise to the familiar Euclidean geometry. Whereas during the first half of the century geometry had seemed to be becoming increasingly fragmented, Cayley and Klein, through the medium of these ideas, apparently succeeded for a lime in providing geometers with a unified view of their subject. Thus, although the so-called Cayley-Klein metric is now seldom taught, to their contemporaries it was of great importance.

Cayley is responsible for another branch of algebra over and above invariant theory, the algebra of matrices. The use of determinants in the theory of equations had by his time become a part of established practice, although the familiar square notation was Cayley's (*C. M. P.* I, no. 1 [1841], 1–4) and although their use in geometry, such as was provided by Cayley from the first, was then uncommon. (They later suggested to him the analytical geometry of *n* dimensions.) Determinants suggested the matrix notation; and yet to those concerned with the history of the "theory of multiple quantity" this notational innovation, even with its derived rules, takes second place to the algebra of rotations and extensions in space (such as was initiated by Gauss, Hamilton, and Grassmann), for which determinant theory provided no more than a convenient language.

Cayley's originality consisted in his creation of a theory of matrices that did not require repeated reference to the equations from which their elements were taken. In his first systematic memoir on the subject (*C. M. P.*, II, no. 152 [1858], 475–496), he established the associative and distributive laws, the special conditions under which a commutative law holds, and the principles for forming general algebraic functions of matrices. He later derived many important theorems of matrix theory. Thus, for example, he derived many theorems of varying generality in the theory of those linear transformations that leave invariant a quadratic or bilinear form. Notice that since it may be proved that there are $n(n + 1)/2$ relations between them, Cayley expressed the n^2 coefficients of the nary orthogonal transformation in terms of $n(n - 1)/2$ parameters. His formulas, however, do not include all orthogonal transformations except as limiting cases (see E. Pascal's *Die Determinanten* [1919], paras, **47** ff.).

The theory of matrices was developed in two quite different ways: the one of abstract algebraic structure, favored by Cayley and Sylvester; the other, in the geometrical tradition of Hamilton and Grassmann. Benjamin Peirce (whose study of linear associative algebras, published in 1881 but evolved by him much earlier, was a strong influence on Cayley) and Cayley himself were notable for their ability to produce original work in both traditions. (It is on the strength of his work on linear associative algebras that Peirce is often regarded as cofounder of the theory of matrices.) In his many informal comments on the relation between matrices and quaternions (see, for example, his long report to the British Association, reprinted in *C. M. P.*, IV, no. 298 [1862], 513–593; and excerpts from his controversial correspondence with his friend P. G. Tail, printed in C. G. Knott's *Life and Scientific Work of P. G. Tait* [Cambridge, 1911], pp. 149–166) Cayley showed a clearer grasp of their respective merits than most of his contemporaries, but like most of them he found it necessary to favor one side rather than the other (coordinates rather than quaternions in his case) in a heated controversy in which practical expediency was the only generally accepted criterion. He had no significant part in the controversy between Tait and J. W. Gibbs, author of the much simpler vector analysis. In passing, we notice Cayley's statement of the origins of his matrices (Knott, *op. cit.*, p. 164, written 1894): "I certainly did not get the notion of a matrix in any way through quaternions: it was either directly from that of a determinant; or as a convenient mode of expression of the equations [of linear transformation)…."

That Cayley found geometrical analogy of great assistance in his algebraic and analytical work—and conversely—is evident throughout his writings; and this, together with his studied avoidance of the highly physical interpretation of geometry more typical of his day, resulted in his developing the idea of a geometry of *n* dimensions. It is not difficult to find instances of the suggested addition of a fourth dimension to the usual trio of spatial dimensions in the work of earlier writers—Lagrange, d'Alembert, and Moebius are perhaps best known. (But only Moebius made his fourth dimension spatial, as opposed to temporal.) Grassmann's theory of extended magnitude, as explained in *Ausdehnungslehre* (1844), may be interpreted in terms of *n*-dimensional geometry; and yet by 1843 Cayley had considered the properties of determinants formed around coordinates in *n*-space. His "Chapter in the Analytical Geometry of (*n*) Dimensions" (*C. M. P.*, I, no. II [1843], 55– 62) might have been considered at the time to have a misleading title, for it contained little that would then have been construed as geometry. It concerns the nonzero solutions of homogeneous linear equations in any number of variables.

By 1846 Cayley had made use of four dimensions in the enunciation of specifically synthetic geometrical theorems, suggesting methods later developed by Veronese (*C. M. P.*, I, no. 50 [1846], 317–328). Long afterward Cayley laid down in general terms, without symbolism, the elements of the subject of "hyperspace" (*cf* his use of the terms "hyperelliptic theta functions," "hyperdetermtnant," and so on) in his "Memoir on Abstract Geometry" (*C. M. P.*, VI, no. 413 [1870], 456–469), showing that he was conscious of the metaphysical issues raised by his ideas in the minds of his followers but that as a mathematician he was no more their slave then than when remarking in his paper of 1846 (published in French): "We may in effect argue as follows, *without having recourse to any metaphysical idea as to the possibility of space of four dimensions* (all this may be translated into purely analytic language)…."

As an example of Cayley's hypergeomelry, we might take the result that a point of ($m - n$)-space given by a set of linear equations is conjugate, with respect to a hyperquadric, to every point whose coordinates satisfy the equations formed by equating to zero a certain simple set of determinants (involving the partial differential coefficients of the hyperquadric function). Cayley and Sylvester subsequently developed these ideas.

In 1860 Cayley devised the system of six homogeneous coordinates of a line, now usually known as Plücker's line coordinates. Plücker, who published his ideas in 1865 (*Philosophical Transactions of the Royal Society*, **155** [1865], 725–791), was working quite independently of Cayley (*C. M. P.*, IV, no. 284 [1860], 446–455, and no. 294 [1862], 490–494), who neglected to elaborate upon his own work. Influenced not by Cayley but by Plücker, Klein (Plücker's assistant at the time of the latter's death in 1868) exploited the subject most fully.

Cayley wrote copiously on analytical geometry, touching on almost every topic then under discussion. Although, as explained elsewhere, he never wrote a textbook on the subject, substantial parts of Salmon's *Higher Plane Curves* are due to him; and without his work many texts of the period, such as those by Clebsch and Frost, would have been considerably reduced in size. One of Cayley's earliest papers contains evidence of his great talent for the analytical geometry of curves and surfaces, in the form of what was often known as Cayley's intersection theorem (*C. M. P.*, I, no. 5 [1843], 25–27). There Cayley gave an almost complete proof (to be supplemented by Bacharach, in *Mathematische Annalen*, **26** [1886], 275–299) that when a plane curve of degree r is drawn through the mn points common to two curves of degrees m and n (both less than r), these do not count for mn conditions in the determination of the curve but for mn reduced by

$$(m + n - r - 1)(m + n - r - 2).$$

(The Cayley-Bacharach theorem was subsequently generalized by Noether. See Severi and Löffler, *Vorlesungen über algehraische Geometric*, ch. 5.) He found a number of important theorems "on the higher singularities of a plane curve" (the title of an influential memoir; *C. M. P.*, V, no. 374 [1866], 520–528), in which they were analyzed in terms of simple singularities (node, cusp, double tangent, inflectional tangent); yet the methods used here did not find permanent favor with mathematicians. A chapter of geometry which he closed, rather than opened, concerns the two classifications of cubic curves: that due to Newton, Stirling, and Cramer and that due to Plücker. Cayley systematically showed the relations between the two schemes (*C. M. P.*, V, no. 350 [1866], 354–400).

It is possible only to hint at that set of interrelated theorems in algebraic geometry which Cayley did so much to clarify, including those on the twenty-eight bitangents of a nonsingular quartic plane curve and the theorem (first announced in 1849) on the twenty-seven lines that He on a cubic surface in three dimensions (*C. M. P.*, I, no. 76 [1849], 445–456). (Strictly speaking, Cayley established the existence of the lines and Salmon, in a correspondence prior to the paper, established their number. See the last page of the memoir and G. Salmon, *The Geometry of Three Dimensions*, 2nd ed. [Dublin, 1865], p. 422.) Although no longer in vogue this branch of geometry, in association with Galois theory, invariant algebra, group theory, and hyperelliptic functions, reached a degree of intrinsic difficulty and beauty rarely equaled in the history of mathematics. The Cayley-Salmon theorem is reminiscent of Pascal's mystic hexagram, and indeed Cremona subsequently found a relation between the two (see B. Segre, *The Nonsingular Cubic Surface* [Oxford, 1942] for a survey of the whole subject). Cayley's twenty-seven lines were the basis of Schläfli's division of cubic surfaces into species; and in his lengthy "Memoir on Cubic Surfaces" Cayley discussed the complete classification with masterly clarity, adding further investigations of his own (*C. M. P.*, VI, no. 412 [1869], 359–455).

As might have been expected from his contributions to the theory of invariants, Cayley made an important contribution to the theory of rational transformation and general rational correspondence. The fundamental theorem of the theory of correspondence is difficult to assign to a particular author, for it was used in special cases by several writers; but Chasles

(*Comptes rendus*, **58** [1864], 175) presented the theorem that a rational correspondence $F(x,y) = 0$ of degree m in x and n in y (x and y being, if necessary, parameters of the coordinates of two points) between spaces or loci in spaces gives in the general case $m + n$ correspondences. (For a history of the subject see C. Segre, "Intorno alia storia del principio di corrispondenza," in *Bibliotheca mathematica*, 2nd ser., **6** [1892], 33–47; Brill and Noether, "Bericht über die Entwicklung der Theorie der algebraischen Funktionen in älterer and neuerer Zeit," in *Jahresbericht der Deutschen Mathematiker-Vereinigung*, **3** [1894], sees. 6, 10.) Soon after this, Cayley generalized Chasles's theorem to curves of any genus (*C. M. P.*, V, no. 377 [1866], 542–545), but his proof was not rigorous and was subsequently amended by A. Brill. The Chasles-Cayley-Brill theorem states that an (m,n) correspondence on a curve of genus p will have $m + n + 2p\gamma$ coincidences, where γ is known as the "value of the correspondence." (The points corresponding to a point P, together with P taken γ times, is to De a group or a so-cauea linear point system.)

Cayley's many additions to the subject of rational correspondences have for the most part passed into anonymity, although the name "Cayley-Plücker equations" is a reminder to geometers of how early appreciated were the connections between the order, the rank, the number of chords through an arbitrary point, the number of points in a plane through which two tangents pass, and the number of cusps of a curve in space and corresponding quantities (class, rank, and so on) of its osculating developable. These equations are all due to Cayley but were deduced from Plücker's equations connecting the ordinary singularities of plane curves.

Cayley devoted a great deal of his time to the projective characteristics of curves and surfaces. Apart from his intricate treatment of the theory of scrolls (where many of his methods and his vocabulary still survive), the Cayley-Zeuthen equations are still a conspicuous reminder of the permanent value of his work. Given an irreducible surface in three-dimensional space, with normal singularities and known elementary projective characters, many other important characteristics may be deduced from these equations, which were first found empirically by Salmonan and later proved by Cayley and Zeuthen. For further details of Cayley's very extensive work in algebraic geometry, an ordered if unintentional history of his thought is to be found almost as a supporting framework for Salmon's *Treatise on the Analytic Geometry of Three Dimensions* (of the several editions the third, of 1882, with its preface, is historically the most illuminating). (For a more general history of algebraic geometry see "Selected Topics in Algebraic Geometry," which constitutes *Bulletin of the National Research Council* [Washington, D.C.], **63** [1929] and supp. **96** (1934), written by committees of six and three, respectively.)

Cayley's wide mathematical range made it almost inevitable that he should write on the theory of groups. Galois's use of substitution groups to decide the algebraic solvability of equations, and the continuation of his work by Abel and Cauchy, had provided a strong incentive to many other mathematicians to develop group theory further. (Thus Cayley wrote "Note on the Theory of Permutations," *CM.P.*, I, no. 72 [1849], 423–424.) Cayley's second paper on the theory (1854), in which he applied it to quaternions, contained a number of invaluable insights and provided mathematicians with what is now the accepted procedure for defining a group. In the abstract theory of groups, where nothing is said of the nature of the elements, the group is completely specified if all possible products are known or determinable. In Cayley's words: "A set ol symbols, 1, α, β,... all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself, belongs to the set, is said to be a *group*." From the first Cayley suggested listing the elements in the form of a multiplication table ("On the Theory of Groups, as Depending on the Symbolic Equation $\theta^n = 1$." *CMP.*, II, no. 125 [1854], 123–130; second and third parts followed, for which see *CM.P.*, II, no. 126 [1854], 131–132, and IV, no. 243 [1859], 88–91). This formulation differed from those of earlier writers to the extent that he spoke only of symbols and multiplication without further defining either. He is sometimes said to have failed to appreciate the step he had taken, but this seems unlikely when we consider his footnote to the effect that "The idea of a group *as applied to* permutations or substitutions is due to Galois..." (italics added). He went on to give what has since been taken as the first set of axioms for a group, somewhat tacitly postulating associativity, a unit element, and closure with respect to multiplication. The axioms are sufficient for finite, but not infinite, groups.

There is some doubt as to whether Cayley ever intended his statements in the 1854 paper to constitute a definition, for he not only failed to use them subsequently as axioms but later used a different and unsatisfactory definition. (See, for instance, an article for the *English Cyclopaedia*, in *C. M. P.*, IV, no. 299 [1860], 594–608: *cf.* the first two of a series of four papers in *CMP.*, X, no. 694 [1878], 401–403.) In a number of historical articles G. A. Miller (see volume I of his *Collected Works* [Urbana, III., 1935]) has drawn attention to the unsatisfactory form of a later definition and indeed has criticized other mathematicians for accepting it: but there are few signs that mathematicians were prepared for the postulational definition until well into the present century. In 1870 Kronecker explicitly gave sets of postulates applied to an abstract finite Abelian group; but even Lie and Klein did most of their work oblivious to the desirability of such sets of axioms, as a result occasionally using the term "group" in what would now be reckoned inadmissible cases.

In addition to his part in founding the theory of abstract groups, Cayley has a number of important theorems to his credit: perhaps the best known is that every finite group whatsoever is isomorphic with a suitable group of permutations (see the first paper of 1854). This is often reckoned to be one of the three most important theorems of the subject, the others being the theorems of Lagrange and Sylow. But perhaps still more significant was his early appreciation of the way in which the theory of groups was capable of drawing together many different domains of mathematics: his own illustrations, for instance, were drawn from the theories of elliptic functions, matrices, quantics, quaternions, homographic transformations, and the theory of equations. If Cayley failed to pursue his abstract approach, this fact is perhaps best explained in terms of the enormous progress he was making in these subjects taken individually.

In 1845 Cayley published his "Mémoire sur les fonctions doublement périodiques," treating Abel's doubly infinite products (*C. M. P.*, I, no. 25 [1845], 156–182; see his note on p. 586 of the same volume). Weierstrass subsequently (1876, 1886) simplified the initial form and in doing so made much of Cayley's work unnecessary (see Cayley's later note, *loc. Cit*.). His work on elliptic functions, pursued at length and recurred to at intervals throughout his life, nevertheless contains ample evidence of Cayley's ability to simplify the work of others, an early instance being his establishment of some results concerning theta functions obtained by Jacobi in his *Fundamenta nova theoriae functionum ellipticarum* of 1829 (*C. M. P.*, I, no. 45 [1847], 290–300). Cayley's only full-length book was on elliptic functions, and he made a notable application of the subject to geometry when he investigated analytically the property of two conics such that polygons may be inscribed by one and circumscribed about the other. The property was appreciated by Poncelet and was discussed analytically by Jacobi (using elliptic functions) when the conics were circles. Using his first paper of 1853 and gradually generalizing his own findings, by 1871 Cayley was discussing the problem of the number of polygons which are such that their vertices lie on a given curve or curves of any order and that their sides touch another given curve or curves of any class. That he was able to give a complete solution even where the polygons were only triangles is an indication of his great analytical skill.

Cayley wrote little on topology, although he wrote on the combinatorial aspect, renewed the discussion of the four-color-map problem, and corresponded with Tait on the topological problems associated with knots. He wrote briefly on a number of topics for which alone a lesser mathematician might have been remembered. He has to his credit an extremely useful system of coordinates in plane geometry which he labeled "circular coordinates" (*C. M. P.*, VI, no. 414 [1868], 498) and which later writers refer to as "minimal coordinates." There is also his generalization of Euler's theorem relating to the numbers of faces, vertices, and edges of the non-Platonic solids. He wrote to great effect on the theory of the numbers of partitions, originated by Euler. (His interest in this arose from his need to apply it to invariant theory and is first evident in his second memoir on quantics, *C. M. P.*, II, no. 141 [1856], 250–281.) His short paper "On the Theory of the Singular Solutions of Differential Equations of the First Order" (*C. M. P.*, VIII, no. 545 [1873], 529–534) advanced the subject considerably and was part of the foundation on which G. Chrystal's first satisfactory treatment of the *p* -discriminant was based (*Transactions of the Royal Society of Edinburgh*, **138** [1896], 803 ff.).

Cayley long exploited the theory of linear differential operators (previously used by Boole to generate invariants and covariants), as when he factored the differential equation $(D^2+pD\ \mathrm{D}+q)\ y=0$ as $(D+\alpha[x])(D+\beta[x])y'=0$, with $\alpha+\beta=p$ and $\alpha\beta+\beta^1=q$ both being theoretically soluble (*C. M. P.*, XII, no. 851 [1886], 403). This technique is linked to that of characterizing invariants and covariants of binary quantics as the polynomial solutions of linear partial differential equations. (The differential operators were in this context known as annihilators, following Sylvester.) He wrote occasionally on dynamics, but his writings suggest that he looked upon it as a source of problems in pure mathematics rather than as a practical subject. Thus in five articles he considered that favorite problem of the time, the attraction of ellipsoids; and in a paper of 1875 he extended a certain problem in potential theory to hyperspace (*C. M. P.*, IX, no. 607 [1875], 318–423). That he kept himself informed of the work of others in dynamics is evident from two long reports on recent progress in the subject which he wrote for the British Association (*C. M. P*, III, no. 195 [1857], 156–204; IV, no. 298 [1862], 513–593).

Cayley wrote extensively on physical astronomy, especially on the disturbing function in lunar and planetary theory; but the impact of what he wrote on the subject was not great, and Simon Newcomb, who spoke of Cayley's mathematical talents with extraordinary deference, did not allude to them in his *Reminiscences of an Astronomer* (London-New York, 1903, p. 280). (It is interesting to note that when he met Cayley at an Astronomical Society Club dinner, Newcomb mistook Cayley's garb for that of an attendant.) Cayley nevertheless performed a great service to his countryman John Couch Adams, who in 1853, taking into account the varying eccentricity of the earth's orbit, had obtained a new value for the secular acceleration of the moon's mean motion. Adams' figure, differing from Laplace's, was contested by several French astronomers, including Ponté -coulant. Cayley looked into the matter independently, found a new and simpler method for introducing the variation of the eccentricity, and confirmed the value Adams had previously found (*C. M. P.*, III, no. 221 [1862], 522–561). Here was yet another instance of the truth of the remark made about Cayley by Sylvester: "… whether the matter he takes in hand be great or small, 'nihil tetigit quod non ornavit'" (*Philosophical Transactions*, **17** [1864], 605), And yet Cayley deserves to be remembered above all not for those parts of mathematics which he embellished, but for those which he created.

BIBLIOGRAPHY

I. Original Works. The great majority of Cayley's mathematical writings (966 papers in all, with some short notes subsequently written about them) are in *The Collected Mathematical Papers of Arthur Cayley*, 13 vols, indexed in a 14th (Cambridge, 1889–1898). The printing of the first seven vols, and part of the eighth was supervised by Cayley himself. The editorial task was assumed by A. R. Forsyth when Cayley died. His excellent biography of Cayley is in vol. VIII, which also contains a complete list of the lectures Cayley gave in Cambridge as Sadlerian professor. The list of writings in vol. XIV includes the titles of several articles which Cayley contributed to the *Encyclopaedia Britannica*. See, e. g., in the 11th ed. "Curve" (in part), "Determinant," "Equation," "Gauss," "Monge," "Numbers, Partition of," and "Surface" (in part). A work in which Cayley's part was not negligible is G. Salmon, *A Treatise on the Higher Plane Curves*, 2nd and 3rd (1879) eds. Upward of twenty sections and the whole of ch. 1 were written by Cayley for the 2nd ed., and further additions were made in the 3rd ed. See Salmon's prefaces for further details. Cayley frequently gave advice and assistance to other authors. Thus he contributed ch. 6 of P. G. Tait's *An Elementary Treatise on Quaternions* (Cambridge, 1890), as well as making improvements. There is no systematic record as such of Cayley's less conspicuous work. He composed a six-penny booklet, *The Principles of Book-Keeping by Double Entry* (Cambridge, 1894). His *An Elementary Treatise on Elliptic Functions* (London, 1876) was issued in a 2nd ed. which, owing to his death, was only partly revised.

II. Secondary Literature. There are few works dealing historically with Cayley's mathematics alone. General histories of mathematics are not listed here, nor are mathematical works in which historical asides are made. The best biographical notice is by A. R. Forsyth, reprinted with minor alterations in *The Collected Mathematical Papers of Arthur Cayley*, VIII (1895), ix-xliv, from the "Obituary Notices" in *Proceedings of the Royal Society*, **58** (1895), 1–43. Forsyth also wrote the article in the *Dictionary of National Biography*, XXII (supp.), 401–402. Another admirable and long obituary notice is by M. Noether, in *Mathematische Annalen*, **46** (1895), 462–480. Written during Cayley's lifetime was G. Salmon's "Science Worthies no. xxii.— Arthur Cayley," in *Nature*, **28** (1883), 481–485. Of general value are Franz Meyer, "Bericht über den gegenwärtigen Stand der invariantentheorie." in *Jahresbericht der Deutschen Mathematiker-Vereinigung*, **1** (1890–1891), 79–288; and A. Brill and M. Noether. "Bericht über die Entwicklung der Theorie der algebraischen Funclionen in älterer and neuerer Zeit." *ibid*., **3** (1894). 107–566. The best specifically historical studies of aspects of Cayley's mathematics are Luboš Nový, ' Arthur Cayley et sa définition des groupes absiraits-finis," in *Acta historiae rerum naturalium necnon technicarum* (Czechoslovak Studies in the History of Science, Prague), spec. issue no. 2 (1966), 105–151; and "Anglická algebraická školá." in *Dějiny věd a techniky*, **1** , no. 2 (1968), 88–105.

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