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(*fl*. ad. 250)

mathematics.

We know virtually nothing about the life of Diophantus. The dating of his activity to the middle of the third century derives exclusively from a letter of Michael Psellus (eleventh century). The letter reports that Anatolius, the bishop of Laodicea since A.D. 270, had dedicated a treatise on Egyptian computation to his friend Diophantus. The subject was one to which, as Psellus states, Diophantus himself had given close attention.[1] This dating is in accord with the supposition that the Dionysius to whom Diophantus dedicated his masterpiece, *Arithmetica*, is St. Dionysius, who, before he became bishop of Alexandria in a.d. 247, had led the Christian school there since 231.[2] An arithmetical epigram of the *Greek Anthology* provides the only further information (if the data correspond to facts): Diophantus married at the age of thirty-three and had a son who died at forty-two, four years before his father died at the age of eighty-four.[3] That is all we can learn of his life, and relatively few of his writings survive. Of these four are known: *Moriastica, Porismata, Arithmetica*, and *On Polygonal Numbers*.

Moriastica. The *Moriastica*, which must have treated computation with fractions, is mentioned only once, in a scholium to Iamblichus' commentary on Nicomachus' *Arithmetica*.[4] Perhaps the *Moriastica* does not constitute an original treatise but only repeats what Diophantus wrote about the symbols of fractions and how to calculate with them in his *Arithmetica*.

Porismata. In several places in the *Arithmetica* Diophantus refers to propositions which he had proved "in the *Porismata*." It is not certain whether it was—as seems more probable—an independent work, as Hultsch and Heath assume, or whether such lemmas were contained in the original text of the *Arithmetica* and became lost with the commentators; the latter position is taken by Tannery, to whom we owe the critical edition of Diophantus.

Arithmetica. The *Arithmetica* is not a work of theoretical arithmetic in the sense understood by the Pythagoreans or Nicomachus. It deals, rather, with logistic, the computational arithmetic used in the solution of practical problems. Although Diophantus knew elementary number theory and contributed new theorems to it, his *Arithmetica* is a collection of problems. In the algebraic treatment of the basic equations, Diophantus, by a sagacious choice of suitable auxiliary unknowns and frequently brilliant artifices, succeeded in reducing the degree of the equation (the unknowns reaching as high as the sixth power) and the number of unknowns (as many as ten) and thus in arriving at a solution. The *Arithmetica* is therefore essentially a logistical work, but with the difference that Diophantus' problems are purely numerical with the single exception of problem V, 30.[5] In his solutions Diophantus showed himself a master in the field of indeterminate analysis, and apart from Pappus he was the only great mathematician during the decline of Hellenism.

Extent of the Work . At the close of the introduction, Diophantus speaks of the thirteen books into which he had divided the work; only six, however, survive. The loss of the remaining seven books must have occurred early, since the oldest manuscript (in Madrid), from the thirteenth century, does not contain them. Evidence for this belief may be found in the fact that Hypatia commented on only the first six books (end of the fourth century). A similarity may be found in the *Conics* of Apollonius, of which Eutocius considered only the first four books. But whereas the latter missing material can be supplied in great part from Arabic sources, there are no such sources for the *Arithmetica*, although it is certain that Arabic versions did exist.

Western Europe learned about a Diophantus manuscript for the first time through a letter to Bianchini from Regiomontanus (15 February 1464), who reported that he had found one in Venice; it contained, however, not the announced thirteen books but only six. In his inaugural address at Padua at about the same time, Regiomontanus spoke of the great importance of this find, since it contained the whole "flower of arithmetic, the *ars rei et census*, called algebra by the Arabs."

Reports concerning the supposed existence of the complete *Arithmetica* are untrustworthy.[6] The question, then, is where one should place the gap: after the sixth book or within the existing books? The quadratic equations with one unknown are missing; Diophantus promised in the introduction to treat them, and many examples show that he was familiar with their solution. A section dealing with them seems to be missing between the first and second books. Here and at other places[7] a great deal has fallen into disorder through the commentators or transcription. For example, the first seven problems of the second book fit much better with the problems of the first, as do problems II, 17, and II, 18. As for what else may have been contained in the missing books, there is no precise information, although one notes the absence, for example, of the quadratic equation system (1)$x^2 \pm y^2 = a$; (2) $xy = b$ which had already appeared in Babylonian mathematics. Diophantus could surely solve this as well as the system he treated in problems I, 27, 30: (1) $x \pm y = a$ (2) $xy = b$, a system likewise known by the Babylonians.

Since in one of the manuscripts the six books are apportioned into seven and the writing on polygonal numbers could be counted as the eighth, it has been supposed that the missing portion was not particularly extensive. This is as difficult to determine as how much—considering the above-mentioned problems, which are not always simple—Diophantus could have increased the difficulty of the problems.[8]

Introduction to the Techniques of Algebra. In the introduction, Diophantus first explains for the beginner the structure of the number series and the names of the powers up to n^6. They are as follows:

n^2 is called square number, τετράγωνος (άρθμός)

n^3 is called cube number, κύβος

n^4 is called square-square number,δυναμοδύναμς

n^5 is called square-cube number,δυναμόκνβος

n^6 is called cube-cube number, κυβόκνβος

The term n^1 however, is expressed as the side of a square number, πλενρὰ τοû τετυαγώνου.[9]

Diophantus introduced symbols for these powers; they were also used—with the exception of the second power—for the powers of the unknowns. The symbols are: for x^2, Δ^{T}(δύεαμs) for x^3, K^{T}; for x^4, $\Delta^{T}\Delta$; for x^5, ΔK^{T}; and for x^6, $K^{T}K$. The unknown x, "an indeterminate multitude of units," is simply called "number" (άριθμός); it is reproduced as an s-shaped symbol, similar to the way it appears in the manuscripts.[10] No doubt the symbol originally appeared as a final sigma with a cross line, approximately like this: s̀; a similar sign is found (before Diophantus) in a papyrus of the early second century.[11] Numbers which are not coefficients of unknowns are termed "units" (μοναδες) and are indicated by M̊. The symbols for the powers of the unknowns are also employed for the reciprocal values $1/x$, $1/x^2$, etc., in which case an additional index, x, marks them as fractions. Their names are patterned on those of the ordinals: for example, $1/x$ is the xth (αραθμοστμν $1/x^2$ the x^2th(δυαμοστόν), and so on. All these symbols—among which is one for the "square number," □^{os}(τετράγωνος)—were read as the full words for which they stand, as is indicated by the added grammatical endings, such as ś^{ot} and śś=άριθμοи. Diophantus then sets forth in tabular form for the various species (εîδος) of powers multiplication rules for the operations $x^m \cdot x^n$ and $x^m \cdot x^{1/n}$; thus—as he states—the divisions of the species are also defined. The sign for subtraction,↑, is also new; it is described in the text as an inverted "psi." The figure is interpreted as the paleographic abbreviation of the verb λείπειν("to want").

Since Diophantus did not wish to write a textbook, he gives only general indications for computation: one should become practiced in all operations with the various species and "should know how to add positive ('forthcoming') and negative ('wanting') terms with different coefficients to other terms, themselves either positive or likewise partly positive and partly negative, and how to subtract from a combination of positive and negative terms other terms either positive or likewise partly positive and partly negative."[12] Only two rules are stated explicitly: a "wanting" multiplied by a "wanting" yields a "forthcoming" and a "forthcoming." multiplied by a "wanting" yields a "wanting." Only in the treatment of the linear equations does Diophantus go into more detail: one should "add the negative terms on both sides, until the terms on both sides are positive, and then again... subtract like from like until one term only is left on each side."[13] It is at this juncture that he promised that he would later explain the technique to be used if two species remain on one side. There is no doubt that he had in mind here the three forms of the quadratic equation in one unknown.

Diophantus employs the usual Greek system of numerals, which is grouped into myriads; he merely—as the manuscripts show—separates the units place of the myriads from that of the thousands by means of a point. One designation of the fractions, however, is new; it is used if the denominator is a long number or a polynomial. In this case the word μορίον (or έν μορίω), in the sense of "divided by" (literally, "of the part"), is inserted between numerator and denominator. Thus, for example, our expression $(2x^3 + 3x^2 + x)/(x^2 + 2x + 1)$ appears (VI, 19)as

One sees that the addends are simply juxtaposed without any plus sign between them. Similarly, since brackets had not yet been invented, the negative members had to be brought together behind the minus symbol: thus, (VI, 22). The symbolism that Diophantus introduced for the first time, and undoubtedly devised himself, provided a short and readily comprehensible means of expressing an equation: for example, $630x^2 + 73x = 6$ appears as to. M̊s̄ (VI, 8). Since an abbreviation is also employed for the word "equals" ('ίσοε),[14] Diophantus took a fundamental step from verbal algebra toward symbolic algebra.

The Problems of the Arithmetica. The six books of the *Arithmetica* present a collection of both determinate and (in particular) indeterminate problems, which are treated by algebraic equations and also by algebraic inequalities. Diophantus generally proceeds from the simple to the more difficult, both in the degree of the equation and in the number of unknowns. However, the books always contain exercises belonging to various groups of problems. Only the sixth book has a unified content. Here all the exercises relate to a right triangle; without regard to dimension, polynomials are formed from the surface, from the sides, and once even from an angle bisector. The first book, with which exercises II, 1–7, ought to be included, contains determinate problems of the first and second degrees. Of the few indeterminate exercises presented there, one (I, 14: x

$+ y = k \cdot xy$) is transformed into a determinate exercise by choosing numerical values for y and k. The indeterminate exercises I, 22–25, belong to another group; these are the puzzle problems of "giving and taking," such as "one man alone cannot buy"—formulated, to be sure, in numbers without units of measure.[15] The second and all the following books contain only indeterminate problems, beginning with those of the second degree but, from the fourth book on, moving to problems of higher degrees also, which by a clever choice of numerical values can be reduced to a lower degree.[16]

The heterogeneity of the 189 problems treated in the *Arithmetica* makes it impossible to repeat the entire contents here. Many who have worked on it have divided the problems into groups according to the degree of the determinate and indeterminate equations. The compilations of all the problems made by Tannery (II, 287–297), by Loria (pp. 862–874), and especially by Heath (*Diophantus*, pp. 260–266) provide an introductory survey. However, the method of solution that Diophantus adopts often yields new problems that are not immediately evident from the statement of the original problem and that should be placed in a different position by any attempted grouping of the entire contents. Nevertheless, certain groups of exercises clearly stand out, although they do not appear together but are dispersed throughout the work. Among the exercises of indeterminate analysis—Diophantus' own achievements lie in this area—certain groups at least should be cited with individual examples:

I. Polynomials (or other algebraic expressions) to be represented as squares. Among these are:

1. One equation for one unknown:

(II, 23; IV, 31) $ax^2 + bx + c = u^2$.

(VI, 18) $ax^3 + bx^2 + cx + d = u^2$.

(V, 29) $ax^4 + b = u^2$.

(VI, 10) $ax^4 + bx^3 + cx^2 + dx + e = u^2$

(IV, 18) $x^6 - ax^3 + x + b^2 = u^2$.

One equation for two unknowns:

(V, 7, lemma 1) $xy + x^2 + y^2 = u^2$.

One equation for three unknowns:

(V, 29) $x^4+y^4 + z^4 = u^2$.

2. Two equations for one unknown ("double equation"):

(II, 11) $a_1x + b_1 = u^2$

$a_2x + b_2 = v^2$

(VI, 12) $a_1x^2 + b_1x = u^2$

$a_2x^2 + b_2x = v^2$.

Two equations for two unknowns:

(II, 24) $(x+y)^2+x = u^2$

$(x+y^2+y = v^2$.

3. Three equations for three unknowns:

(IV, 19) $xy + 1 = u^2$

$yz + 1 = v^2$

$xz + 1 = w^2$.

4. Four equations for four unknowns:

5. Further variations: In V, 5, [17] to construct six squares for six expressions with three unknowns or six squares for six expressions with four unknowns (IV, 20), etc.

II. Polynomials to be represented as cube numbers.

1. One equation for one unknown:

(VI, 17) $x^2 + 2 = u^3$.

(VI, 1) $x^2 - 4x + 4 = u^3$.

2. Two equations for two unknowns:

(IV, 26) $xy + x = u^3$

$xy + y = v^3$.

3. Three equations for three unknowns:

III. To form two polynomials such that one is a square and the other a cube.

1. Two equations for two unknowns:

(IV, 18) $x^3 + y = u^3$, $y^2 + x = v^2$

(VI, 21) $x^3 + 2x^2 + x = \mathrm{u}^3$, $2x^2 + 2x = v^2$.

2. Two equations for three unknowns:

(VI, 17) $xy/2 + z = u^2$, $x + y + z= v^{3.}$

IV. Given numbers to be decomposed into parts.

1. From the parts to form squares according to certain conditions:

(V, 9) $1=x+ y$; it is required that $x + 6= u^2$ and $y + 6 = v^2$.

(II, 14) $20 = x + y$; it is required that $x + u^2 = v^2$ and $y + u^2 = w^2$.

(IV, 31) $1 = x + y$; it is required that $(x + 3)\cdot\ (y + 5) = u^2$.

(V, 11) $1 = x + y + z$; it is required that $x + 3 = u^2 y + 3 = v^2$, and $z + 3 = w^2$

(V, 13) $10 = x + y + z$; it is required that $x + y = u^2 y +z = u^2$, and $z + x = w^2$.

(V,20) it is required that

2. From the parts to form cubic numbers:

(IV, 24) $6 = x + y$; it is required that $xy = \mathrm{u}^3 – \mathrm{u}$.

(IV, 25) $4 = x + y + z$; it is required that $xyz = u^3$, whereby $u = (x – y)+ (y – z) + (x – z)$.

V. A number is to be decomposed into squares.

(II, 8) $16 = x^2 + y^2$

(II, 10) $60 = x^2 - y^2$

(IV, 29) $12 = x^2 + y^2+z^2 + u^2$

(V, 9) $13 = x^2+y^2$, whereby $x^2 > 6$ and $y^2 > 6$.

In the calculation of the last problem Diophantus arrives at the further exercise of finding two squares that lie in the neighborhood of $(51/20)^2$. He terms such a case an "approximation" (παϱισότηs) or an "inducement of approximation" (ἀγωγὴ τῆε παϱιτητοε). Further examples of solution by approximation are:

(V, 10) $9 = x^2 + y^2$, whereby $2 < x^2 < 3$. This is the only instance in which Diophantus represents (as does Euclid) a number by a line segment.

(V, 12) $10 = x + y + z$, where $x > 2$, $y > 3$, and $z > 4$.

(V, 13) $20 = x + y + z$, whereby each part <10.

(V, 14) $30 = x^2 + y^2 + z^2+u^2$, whereby each square <10.

VI. Of the problems formulated in other ways, the following should be mentioned.

(IV, 36) $xy/(x +y) = a$, $yz/(y + z) = b$, and $xz/(x + z) = c$.

(IV, 38) The products

are to be a triangular number $u(u + 1)/2$, a square v^2, and a cube w^3, in that order.

(V, 30) This is the only exercise with units of measure attached to the numbers. It concerns a wine mixture composed of x jugs of one type at five drachmas and y jugs of a better type at eight drachmas. The total price should be $5x + y = u^2$, given that $(x + y)^2 = u^2 + 60$.

Methods of Problem-solving. In only a few cases can one recognize generally applicable methods of solution in the computations that Diophantus presents, for he considers each case separately, often obtaining an individual solution by means of brilliant stratagems. He is, however, well aware that there are many solutions. When, as in III, 5 and 15, he obtains two solutions by different means, he is satisfied and does not arrange them in a general solution—which, in any case, it was not possible for him to do.[18] Of course a solution could not be negative, since negative numbers did not yet exist for Diophantus. Thus, in V, 2, he says of the equation $4 = 4x + 20$ that it is absurd (ἄτοπον) The solution need not be a whole number. Such a solution is therefore not a "Diophantine" solution. The only restriction is that the solution must be rational.[19] In the equation $3x + 18 = 5x^2$(IV, 31), where such is not the case, Diophantus notes: "The equation is not rational" (οὐκ ᾽έστιν ἡ ἲσωσιςρ́ητή); and he ponders how the number 5 could be changed so that the quadratic equation would have a rational solution.

There are two circumstances that from the very beginning hampered or even prevented the achievement of a general solution. First, Diophantus can symbolically represent only one unknown; if the problem contains several, he can carry them through the text as "first, second, etc." or as "large, medium, small," or even express several unknowns by means of one. Mostly, however, definite numbers immediately take the place of the unknowns and particularize the problem. The process of calculation becomes particularly opaque because newly appearing unknowns are again and again designated by the same symbol for x.

Second, Diophantus lacked, above all else, a symbol for the general number n. It is described, for example, as "units, as many as you wish" (V, 7, lemma 1; M̊ ὅσων θέλεις). For instance, nx is termed "x, however great" (II, 9; ὅδσος δήποτε) or "any x" (IV, 39; ἀϱιθμόστις). Nevertheless, Diophantus did succeed, at least in simple cases, in expressing a general number—in a rather cumbersome way, to be sure. Thus in IV, 39, the equation $3x^2 + 12x + 9 = (3 - nx)^2$ yields $x = (12 + 6n)/(n^2 - 3)$; the description reads "x is a sixfold number increased by twelve, which is divided by the difference by which the square of the number exceeds 3."

Among the paths taken by Diophantus to arrive at his solutions, one can clearly discern several methods:

1. For the determinate linear and quadratic equations there are the usual methods of balancing and completion (see, for example, the introduction and II, 8); in determinate systems, Diophantus solves for one unknown in terms of the other by the first equation and then substitutes this value in the second. For the quadratic equation in two unknowns, he employs the Babylonian normal forms; for the equation in one unknown, the three forms $ax^2 + bx = c$, $ax^2 =bx + c$, and $ax^2 + c = bx$. Moreover, his multiplication of the equation by a can be seen from the criterion for rationality $(b/2)^2 + ac = \square$ or, as the case may be, $(b/2)^2-ac = \square$[20].

2. The number of unknowns is reduced. This often happens through the substitution of definite numbers at the beginning, which in linear equations corresponds to the method of "false position." If a sum is to be decomposed into two numbers, for example $x + y = 10$, then Diophantus takes $x = 5 + X$ and $y = 5 - X$ This is also the case with the special cubic equations in IV, I and 2.[21]

3. The degree of the equation is reduced. Either a definite number is substituted for one or more unknowns or else a function of the first unknown is substituted.

(V, 7, lemma 1) $xy + x^2 + y^2 = u^2$; y is taken as 1 and u as x-2; this gives $x^2 + x + 1 = (x - 2)^2$;therefore $x = 3/5$ and $y = 1$, or $x = 3$ and $y = 5$.

(V, 29) $x^4 + y^4 + z^4 = u^2$, with $y^2 = 4$ and $z^2 = 9$; therefore $x^4 + 97 = u^2$. With $u = x^2 - 10$ this yields $20x^2 = 3$. Since 20/3 is not a square, the method of reckoning backward (see below) is employed.

(IV, 37) $60u^3 = v^2$, with $v = 30u$.

(II, 8) $16 - x^2 = (nx - 4)^2$, with $n = 2$. The "cancellation of a species" (see II, 11, solution 2) is possible with $ax^2 + bx + c = u^2$, for example, by substituting $mx + n$ for u and determining the values of m and n for which like powers of x on either side have the same coefficient. Expressions of higher degree are similarly simplified.

(VI, 18) $x^3 + 2 = u^2$, with $x = (X - 1)$, yields $(X - 1)^3 + 2 = u^2$; if $u = (3X/2) + 1$, then $X^3 - 3X^2 + 3X + 1 = (9X^2/4) + 3X + 1$, and hence a first-degree equation.

(VI, 10) $x^4 + 8x^3 + 18x^2 + 12x + 1 = u^2$, where $u = 6x + 1 - x^2$.

4. The double equation. (II, 11) (1) $x + 3 = u^2$,(2) $x + 2 = v^2$; the difference yields $u^2 - v^2 = 1$. Diophantus now employs the formula for right triangles, $m \cdot n = [(m + n)/2]^2 - [(m - n)/2]^2$, and sets the difference $1 = 4 \cdot 1/4$; thus the following results: $u = 17/8$, $v = 15/8$, and $x = 97/64$. Similarly, in II, 13, the difference 1 is given as $2 \cdot 1/2$; in III, 15, $5x + 5 = 5(x + 1)$; and in III, 13, $16x + 4 = 4(4x + 1)$.

5. Reckoning backward is employed if the computation has resulted in an impasse, as above in V, 29;here Diophantus considers how in $20x^2 = 3$ the numbers 20 and 3 have originated. He sets $20 = 2n$ and $3 = n^2 - (y^4 + z^4)$. With $n = y^2 + 4$ and $z^2 = 4$, $3 = 8y^2$ and $20/3 = (y^2 + 4)/4y^2$. Now only $y^2 + 4$ remains to be evaluated as a square. Similar cases include IV, 31, and IV, 18.

6. Method of approximation to limits (V, 9–14). In V, 9, the problem is $13 = u^2 + v^2$, with $u^2 > 6$ and $v^2 > 6$. First, a square is sought which satisfies these conditions. Diophantus takes $u^2 = 6½ + (1/x)^2$. The quadruple $26 + 1/y^2$ (with $y = x/2$) should also become a square. Setting $26 + 1/y^2 = (5 + 1/y)^2$yields $y = 10$, $x^2 = 400$, and $u = 51/20$. Since $13 = 3^2 + 2^2$, Diophantus compares 51/20 with 3 and 2. Thus, 51/20 sr 3 - 9/20 and 51/20 = 2 + 11/20. Since the sum of the squares is not 13 (but 13 1/200), Diophantus sets $(3 - 9x)^2 + (2 + 11x)^2 = 13$ and obtains $x = 5/101$. From this the two squares $(257/101)^2$ and $(258/101)^2$ result.

7. Method of limits. An example is V, 30. The conditions are $(x^2 - 60)/8 < x < (x^2 - 60)/5$. From this follow $x^2 < 8x + 60$ or $x^2 = 8x + n$ $(n < 60)$, and $x^2 > 5x + 60$ or $x^2 = 5x + n$ $(n > 60)$. The values (in part incorrect) assigned according to these limits were no doubt found by trial and error. In IV, 31, the condition is $5/4 < x^2 < 2$. After multiplication by 8^2, the result is $80 < (8x)^2 < 128$; consequently $(8x)^2 = 100$ is immediately apparent as a square;therefore $x^2 = 25/16$. In a similar manner, x^6 is interpolated between 8 and 16 in VI, 21.

8. Other artifices appear in the choice of designated quantities in the exercises. Well-known relations of number theory are employed. For example (in IV, 38), 8 · triangular number + 1 = □, therefore $8[n(n + 1)/2] + 1 = (2n + 1)^2$. In IV, 29, Diophantus applies the identity $(m + n)^2 = m^2 + 2mn + n^2$ to the problem $x^2 + x + y^2 + y + z^2 + z + u^2 + u = 12$. Since $x^2 + x + 1/4$ is a square, $4 \cdot 1/4$ must be added to 12; whence the problem becomes one of decomposing 13 into four squares. Other identities employed include:

(II, 34) $[(m - n)/2]^2 + m \cdot n = [(m + n)/2]^2$

(VI, 19) $m^2 + [((m^2 - 1)/2]^2 = [(m^2 + 1)/2]^2$

(II, 30) $m^2 + n^2 \pm 2mn = \square$

(III, 19) $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$

(V, 15) In this exercise the expressions $(x + y + z)^3 + x$, $(x + y + z)^3 + y$ and $(x + y + z)^3 + z$ are to be transformed into perfect cubes. Hence Diophantus takes $x = 7X^3$ and $(x + y + z) = X$, so the first cube is $(2x)^3$. The other two numbers are $y = 26X^3$ and $z = 63X^3$. From this results $96X^2 = 1$. Here again reckoning backward must be introduced. In 1, 22, in the indeterminate equation $2x/3 + z/5 = 3y/4 + x/3 = 4z/5 + y/4$, Diophantus sets $x = 3X$ and $y = 4$. In VI, 16, a rational bisector of an acute angle of a right triangle is to found; the segments into which the bisector divides one of the sides are set at $3x$ and $3 - 3x$, and the other side is set at $4x$. This gives a hypotenuse of $4-4x$, since $3x{:}4x = (3 - 3x)$: hypotenuse.[22]

In VI, 17, one must find a right triangle for which the area plus the hypotenuse = u^2 and the perimeter = v^3. Diophantus takes $u = 4$ and the perpendiculars equal to x and 2; therefore, the area is x, the hypotenuse = $16 - x$, and the perimeter = $18 = v^3$. By reckoning backward (with $u = m$, rather than $u = 4$) the hypotenuse becomes $m^2 – x$ and the perimeter $m^2 + 2 = v^3$. Diophantus then sets $m = X + 1$ and $v = X - 1$, which yields the cubic equation $X^3 - 3X^2 + 3X - 1 = X^2 + 2X + 3$, the solution of which Diophantus immediately presents (obviously after a factorization): $X = 4$.

It is impossible to give even a partial account of Diophantus' many-sided and often surprising inspirations and artifices. It is impossible, as Hankel has remarked, even after studying the hundredth solution, to predict the form of the hundred-and-first.

On Polygonal Numbers. This work, only fragmentarily preserved and containing little that is original, is immediately differentiated from the *Arithmetica* by its use of geometric proofs. The first section treats several lemmas on polygonal numbers, a subject already long known to the Greeks. The definition of these numbers is new; it is equivalent to that given by Hypsicles, which Diophantus cites. According to this definition, the polygonal number

where a indicates the number of vertices and n the number of "sides" of the polygon.[23] Diophantus then gives the inverse formula, with which one can calculate n from p and a. The work breaks off during the investigation of how many ways a number p can be a polygonal number.

Porisms and Number-theory Lemmas. Diophantus refers explicitly in the *Arithmetica* to three lemmas in a writing entitled "The Porisms," where they were probably proved. They may be reproduced in the following manner:

1. If $x + a = u^2$, $y + a = v^2$, and $xy + a = w^2$, then $v = u + 1$ (V, 3).[24]

2. If $x = u^2$, $y = (u + 1)^2$, and $z = 2\cdot(x + y) + 2$, then the six expressions $xy\,(x + y)$, $xy + z$, $xz + (x + z)$, $xz + y$, $yz + (x + z)$, and $yz + x$ are perfect squares (V, 5).

3. The differences of two cubes are also the sums of two cubes (V, 16). In this case one cannot say whether the proposition was proved.

In solving his problems Diophantus also employs other, likewise generally applicable propositions, such as the identities cited above (see Methods of Problem-solving, §8). Among these are the proposition (III, 15) $a^2 \cdot (a + 1)^2 + a^2 + (a + 1)^2 = \square$ and the formula (III, 19) $(a^2 + b^2)\cdot(c^2 + d^2) = x^2 + y^2$, where $x = (ac \pm bd)$ and $y = (ad \mp bc)$. The formula is used in order to find four triangles with the same hypotenuse. From the numbers chosen in this instance, $a^2 + b^2 = 5$ and $c^2 + d^2 = 13$, it has been concluded that Diophantus knew that a prime number $4n + 1$ is a hypotenuse.[25] In the examples of the decomposition of numbers into sums of squares, Diophantus demonstrates his knowledge of the following propositions, which were no doubt empirically derived: No number of the form $4n + 3$ is the sum of two square numbers (V, 9), and no number of the form $8n + 7$ is the sum of three square numbers (V, 11). Furthermore, every number is the sum of two (V, 9), three (V, 11), or four (IV, 29, and 30; V, 14) square numbers. Many of these propositions were taken up by mathematicians of the seventeenth century, generalized, and proved, thereby creating modern number theory.

In all his multifarious individual problems, in which the idea of a generalization rarely appears, Diophantus shows himself to be an ingenious and tireless calculator who did not shy away from large numbers and in whose work very few mistakes can be found.[26] One wonders what goals Diophantus had in mind in his *Arithmetica*. There was undoubtedly an irresistible drive to investigate the properties of numbers and to explore the mysteries which had grown up around them. Hence Diophantus appears in the period of decline of Greek mathematics on a lonely height as "a brilliant performer in the art of indeterminate analysis invented by him, but the science has nevertheless been indebted, at least directly, to this brilliant genius for few methods, because he was deficient in the speculative thought which sees in the True more than the Correct."[27]

Diophantus' Sources. Procedures for calculating linear and quadratic problems had been developed long before Diophantus. We find them in Babylonian and Chinese texts, as well as among the Greeks since the Pythagoreans. Diophantus' solution of the quadratic equation in two unknowns corresponds completely to the Babylonian, which reappears in the second book of Euclid's *Elements* in a geometric presentation. The treatment of the second-degree equation in one unknown is also Babylonian, as is the multiplication of the equation by the coefficient of x^2. There are a few Greek algebraic texts that we possess which are more ancient than Diophantus: the older arithmetical epigrams (in which there are indeterminate problems of the first degree), the *Epanthema* of Thymaridas of Paros, and the papyrus (Michigan 620) already mentioned. Moreover, knowledge of number theory was available to Diophantus from the Babylonians and Greeks, concerning, for example, series and polygonal numbers,[28] as well as rules for the formation of Pythagorean number triples. A special case of the decomposition of the product of two sums of squares into other sums of squares (see above, Porisms and Lemmas) had already appeared in a text from Susa.[29] One example of indeterminate analysis in an old Babylonian text corresponds to exercise II, 10, in Diophantus.[30] Diophantus studied special cases of the general Pellian equation with the "side and diagonal numbers" $x^2 - 2y^2 = \pm 1$. The indeterminate Archimedean cattle problem would have required a solution of the form $x^2 - ay^2 = 1$. Consequently, Diophantus certainly was not, as he has often been called, the father of algebra. Nevertheless, his remarkable, if unsystematic, collection of indeterminate problems is a singular achievement that was not fully appreciated and further developed until much later.

Influence. In their endeavor to acquire the knowledge of the Greeks, the Arabs—relatively late, it is true—became acquainted with the *Arithmetica*. AlNadīm (987/988) reports in his index of the sciences that Qusṭā ibn Lūqā (*ca*. 900) wrote a *Commentary on Three and One Half Books of Diophantus' Work on Arithmetical Problems* and that Abū'l-Wafāʾ (940–988) likewise wrote *A Commentary on Diophantus' Algebra*, as well as a *Book on the Proofs of the Propositions Used by Diophantus and of Those That He Himself* [Abu'l-Wafā'] *Has Presented in His Commentary*. These writings, as well as a commentary by Ibn al-Haytham on the *Arithmetica* (with marginal notations by Ibn Yūnus), have not been preserved. On the other hand, Arab texts do exist that exhibit a concern for indeterminate problems. An anonymous manuscript (written before 972) treats the problem $x^2 + n = u^2$, $x^2 - n - v^2$ a manuscript of the same period contains a treatise by al-Ḥusain (second half of tenth century) that is concerned with the theory of rational right triangles.[31] But most especially, one recognizes the influence of Diophantus on al-Karajī. In his algebra he took over from Diophantus' treatise a third of the exercises of book I; all those in book II beginning with II, 8; and almost all of book III. What portion of the important knowledge of the Indians in the field of indeterminate analysis is original and what portion they owe to the Greeks is the subject of varying opinions. For example, Hankel's view is that Diophantus was influenced by the Indians, while Cantor and especially Tannery claim just the opposite.

Problems of the type found in the *Arithmetica* first appeared in the West in the *Liber abbaci* of Leonardo of Pisa (1202); he undoubtedly became acquainted with them from Arabic sources during his journeys in the Mediterranean area. A Greek text of Diophantus was available only in Byzantium, where Michael Psellus saw what was perhaps the only copy still in existence.[32] Georgius Pachymeres (1240–1310) wrote a paraphrase with extracts from the first book.[33] and later Maximus Planudes (*ca*. 1255–1310) wrote a commentary to the first two books.[34] Among the manuscripts that Cardinal Bessarion rescued before the fall of Byzantium was that of Diophantus, which Regiomontanus discovered in Venice. His intention to produce a Latin translation was not realized. Then for a century nothing was heard about Diophantus. He was rediscovered by Bombelli, who in his *Algebra* of 1572, which contained 271 problems, took no fewer than 147 from Diophantus, including eighty-one with the same numerical values.[35] Three years later the first Latin translation, by Xylander, appeared in Basel; it was the basis for a free French rendering of the first four books by Simon Stevin (1585). Viète also took thirty-four problems from Diophantus (including thirteen with the same numerical values) for his *Zetetica* (1593); he restricted himself to problems that did not contradict the principle of dimension. Finally, in 1621 the Greek text was prepared for printing by Bachet de Méziriac, who added Xylander's Latin translation, which he was able to improve in many respects. Bachet studied the contents carefully, filled in the lacunae, ascertained and corrected the errors, generalized the solutions, and devised new problems. He, and especially Fermat, who took issue with Bachet's statements,[36] thus became the founders of modern number theory, which then—through Euler, Gauss,[37] and many others—experienced an unexpected development.

NOTES

1. Tannery, *Diophanti opera*, II, 38 f. As an example of "Egyptian analysis" Psellus gives the problem of dividing a number into a determined ratio.

2. Tannery, in his *Mèmoires scientifiques*, II, 536 ff., mentions as a possibility that the *Arithmetica* was written as a textbook for the Christian school at the request of Dionysius and that perhaps Diophantus himself was a Christian.

3. Tannery, *Diophanti opera*, II, 60 ff.

4.*Ibid*., p. 72.

5. V, 30, is exercise 30 of the fifth book, according to Tannery's numbering.

6. Tannery, *Diophanti opera*, II, xxxiv.

7. III, 1–4, belongs to II, 34, 35; and III, 20, 21, is the same as II, 14, 15.

8. Problems such as the "cattle problem" do not appear in Diophantus.

9. Or, as in IV, 1, of a cube number.

10. Tannery, *Diophanti opera*, II, xxxiv.

11. Michigan Papyrus 620, in J. G. Winter, *Papyri in the University of Michigan Collection*, vol. III of *Michigan Papyri* (Ann Arbor, 1936), 26–34.

12. Heath, *Diophantus of Alexandria*, pp. 130–131.

13.*Ibid*.

14. Tannery, *Diophanti opera*, II, xli. There are two parallel strokes joined together.

15. Similar problems exist in Byzantine and in Western arithmetic books since the time of Leonardo of Pisa.

16. Heath (in his *Conspectus*) considers the few determinate problems in bk. II to be spurious. Problems 1, 2, 15, and 33–37 of bk. IV become determinate only through arbitrary assumption of values for one of the unknowns.

17 $x^2y^2 + (x^2 + y^2)$, $y^2z^2 + (y^2 + z^2)$, $z^2x^2 + (z^2 + x^2)$, $x^2y^2 + z^2$, $y^2z^2 + x^2$, $z^2x^2 + y^2$

18. Sometimes Diophantus mentions infinitely many solutions (VI, 12, lemma 2). In VI, 15, lemma, Diophantus presents, besides a well-known solution of the equation $3x^2 - 11 = y^2$ (namely, $x = 5$ and $y = 8$), a second one: $3\cdot (5 + z)^2 - 11 = (8 - 2z.)^2$

19. Sometimes, for example in IV, 14, the integer solution is added to the rational solution.

20. For example, in VI, 6; IV, 31; and V, 10.

21. In IV, 1, the system $x^3 + y^3 = 370$, $x + y = 10$, corresponds to the quadratic system $xy = 21$, $x + y = 10$, which was a paradigm in al-Khwārizmī. Tannery (*Méemoires scientifiques*, II, 89) shows how close Diophantus was to a solution to the cubic equation $x^3 = 3\,px + 2\,q$.

22. Here one sees the application of algebra to the solution of a geometric problem.

23. The nth polygonal number has n “sides.”

24. This is not a general solution; see Tannery, *Diophanti opera*, I, 317.

25. Heath, p. 107.

26. For example, IV, 25, and V, 30; see Heath, pp. 60, 186.

27. Hankel, p. 165; Heath, p. 55.

28. For example, see Hypsicles’ formula used in *On Polygonal Numbers*.

29. See E. M. Bruins and M. Rutten, *Textes mathématiques de Suse*, no. 34 in the series Mémoires de la mission archéologique en Iran (Paris, 1961), p. 117.

30. See S. Gandz, in *Osiris*, **8** (1948), 13 ff.

31. See Dickson, p. 459.

32. Tannery, *Diophanti opera*, II, xviii.

33. *Ibid.*, pp. 78–122. Also in “Quadrivium de Georges Pachymère,” in *Studi e testi*, CXIV (Vatican City. 1940), 44–76.

34. *Ibid*, pp. 125–255.

35. Bombelli and Antonio Maria Pazzi prepared a translation of the first five books, but it was not printed.

36. In his copy of Bachet’s edition Fermat wrote numerous critical remarks and filled in missing material. These remarks appeared as a supplement, along with selections from Fermat’s letters to Jacques de Billy, in Samuel de Fermat’s new edition of Diophantus of 1670.

37. The importance of Diophantus is emphasized by Gauss in the introduction to his *Disquisitiones arithmetice:* “Diophanti opus celebre, quod totum problematis indeterminatis dicatum est, multas quaestiones continet, quae propter difficultatem suam artificiorumque subtilitatem de auctoris ingenio et acumine existimationem haud mediocrem suscitant, praesertim si subsidiorum, quibus illi uti licuit, tenuitatem consideres” (“The famous work of Diophantus, which is totally dedicated to indeterminate problems, contains many questions which arouse a high regard for the genius and penetration of the author, especially when one considers the limited means available to him”).

BIBLIOGRAPHY

The first Western presentation of the Diophantine problems was by Raffaele Bombelli, in his *Algebra* (Bologna, 1572; 2nd ed., 1579). A Latin translation was produced by W. Xylander, *Diophanti Alexandrini Rerum arithmeticarum libri sex, quorum duo* *adjecta habent scholia Maximi Planudis. Item liber de numeris polygonis seu multangulis* (Basel, 1575). The text with a Latin translation was prepared by C.-G. Bachet de Méziriac, *Diophanti Alexandrini Arithmeticorum libri sex, et de numeris multangulis liber unus* (Paris, 1621); a new edition was published by Samuel de Fermat with notes by his father, Pierre de Fermat (Toulouse, 1670). There is also the definitive text with Latin translation by P. Tannery, *Diophanti Alexandrini opera omnia cum Graecis commentariis*, 2 vols. (Leipzig, 1893–1895). An English translation, in a modern rendering, is T. L. Heath, *Diophantus of Alexandria, A Study in the History of Greek Algebra* (Cambridge, 1885); the second edition has a supplement containing an account of Fermat's theorems and problems connected with Diophantine analysis and some solutions of Diophantine problems by Euler (Cambridge, 1910; New York, 1964). German translations are O. Schultz, *Diophantus von Alexandria arithmetische Aufgaben nebst dessen Schrift über die Polygon-Zahlen* (Berlin, 1822); G. Wertheim, *Die Arithmetik und die Schrift über die Polygonalzahlen des Diophantus von Alexandria* (Leipzig, 1890); and A. Czwalina, *Arithmetik des Diophantos von Alexandria* (Göttingen, 1952). A French translation is P. Ver Eecke, *Diophante d'Alexandrie*(Paris, 1959). A Greek text (after Tannery's), with translation into modern Greek, is E. S. Stamatis, ΔιοΦαντου αϱιθμητικα η αλγεϱατων αϱϰαιων ελληνων (Athens, 1963).

On Polygonal Numbers appears in a French trans. by G. Massoutié as *Le traité des nombres polygones* (Mâcon, 1911).

Along with the views of the authors in their text editions and translations, the following general criticisms should be consulted: M. Cantor, *Vorlesungen über Geschichte der Mathematik*, 3rd ed., I (Leipzig, 1907), 463–488; P. Cossali, *Origine, trasporto in Italia, primi progressi in essa dell' algebra* (Parma, 1797), I, 56–95; T. L. Heath, *History of Greek Mathematics* (Oxford, 1921), II, 440–517; B. L. van der Waerden, *Erwachende Wissenschaft*, 2nd ed. (Basel-Stuttgart, 1966), pp. 457–470. See also H. Hankel, *Zur Geschichte der Mathematik in Alterthum und Mittelalter* (Leipzig, 1874), 2nd ed. with foreword and index by J. E. Hofmann (Hildesheim, 1965); F. Hultsch, "Diophantus," in Pauly-Wissowa, V, pt. 1, 1051–1073; G. Loria, *Le scienze esatte nell'antica Grecia* (Milan, 1914), pp. 845–919; E. Lucas, *Recherches sur l'analyse indéterminacutée et l'arithmétique de Diophante*, with preface by J. Itard (Paris, 1967); G. H. F. Nesselmann, *Die Algebra der Griechen* (Berlin, 1842), pp. 244–476; and G. Sarton, *Introduction to the History of Science*, I (Baltimore, 1927), 336 ff.

Special criticism includes P. Tannery, "La perte de sept livres de Diophante," in his *Mémoires scientifiques*, II (Toulouse-Paris, 1912), 73–90; "Étude sur Diophante," *ibid*, 367–399; and "Sur la religion des derniers mathématiciens de l"antiquité," *ibid*., 527–539.

Historical works include I. G. Bašmakova, "Diofant i Ferma," in *Istoriko-matematicheskie issledovaniya*, **17** (1966), 185–204; L. E. Dickson, *History of the Theory of Numbers*, II, *Diophantine Analysis* (Washington, D.C., 1920); T. L. Heath, *Diophantus* (see above), ch. 6 and supplement; K. Reich, "Diophant, Cardano, Bombelli, Viète. Ein Vergleich ihrer Aufgaben," in *Rechenpfennige, Aufsätze zur Wissenschaftsgeschichte* (Munich, 1968), pp. 131–150; P. Tannery, *Diophanti opera* (see above), II, prolegomena; and P. Ver Eecke, *Diophante* (see above), introduction.

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