# Ehresmann, Charles | Encyclopedia.com



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(*b*. Strasbourg, France. 19 April 1905; *d*. Amiens, France, 22 September 1979)

*mathematics*.

Ehresmann's father was a gardener employed by a convent in Strasbourg. Ehresmann's parents spoke only the Alsatian dialect, and until 1918 his schooling was in German. He was educated in the Lycée Kléber at Strasbourg and entered the École Normale Supérieure in 1924. After graduation in 1927 and military service, he taught from 1928 to 1929 at the French Lycée in Rabat, Morocco. In the years 1930 and 1931 he did research at Göttingen, and between 1932 and 1934 at Princeton, earning a doctorate in mathematics at the University of Paris in 1934. From 1934 to 1939 he conducted research at the Centre Nationale de la Recherche Scientifique.

Ehresmann became a lecturer at the University of Strasbourg in 1939, and after the German invasion in 1940, he followed that university when it relocated to Clermont-Ferrand. After 1952 he traveled extensively to many countries, where he often was invited to give courses. He became a professor at the University of Paris in 1955, where a chair of topology was created for him. After his retirement in 1975, he taught at the University of Amiens (where his wife was a professor of mathematics) in a semiofficial position. He was married twice and had one son by his first wife. He died of kidney failure.

Ehresmann was one of the creators of differential topology, which explores the topological properties (in homotopy and homology) of a differential manifold, in relation to its differential structure. In his dissertation and subsequent papers between 1935 and 1939, he explicitly described the homology of classical types of homogeneous manifolds, such as Grassmannians, flag manifolds, Stiefel manifolds, and classical groups. His methods were based on decomposition of these manifolds into cells, even before the general definition of CW complexes had been given. His results later became a useful tool in the theory of characteristic classes.

Between 1939 and 1956 Ehresmann participated in the creation and development of fundamental notions in differential topology: fiber spaces, connections, almost complex structures, jets, and foliations. Fiber spaces, first considered in special cases by Seifert in 1933 and Hassler Whitney in 1935, became a focus of topological researcg around 1940, when their importance was realized. Ehresmann approached the theory of fiber spaces from an original angle. He had become familiar with the theory of what Élie Cartan called connections (generalizing the Levi-Cività parallelism in Riemannian manifolds) and with the "generalized spaces" on which these connections are defined; very few mathematicians understood Cartan's ideas at that time. Ehresmann realized that beneath Cartan's formulas and constructions were two fundamental fiber spaces whose basis was a differential manifold: the tangent bundle and the space of frames, the mutual relations of which were the key to Cartan's theory. This gave Ehresmann a view of the general theory of fiber spaces somewhat different from that of other mathematicians in that field.

Ehresmann's theory emphasized the importance of a group of automorphisms of a fiber and led him to the general concept of principal fiber space, where the fibers themselves are topological groups isomorphic to a fixed group. This notion has acquired a fundamental importance in differential and algebraic geometry. Ehresmann could then precisely describe what may be called a two-way correspondence between general fiber spaces and principal ones over a fixed base; with any fiber space there is associated a well-determined (up to isomorphism) principal fiber space; conversely, with any principal fiber space with fibers isomorphic to a group G, and with any action of G on a space F, there is associated a welldetermined fiber space with fibers isomorphic to F. Thus, the "space of frames" with group GL(n**R** ) is associated to the tangent bundle of a differential manifold of dimension n as principal fiber space.

With the help of these concepts, Ehresmann could, for any fiber bundle E over a differential manifold *M*, give a definition (generalizing Cartan's) of a connection on *E*. Geometrically it amounts to defining, for each $x \in M$ and any point $u_x$ in the fiber $E_x$, a vector subspace $H_u$ of the tangent space to E at the point $u_x$, which is supplementary to $E_x$ in that space and therefore projects isomorphically onto the tangent space to M at x.

When a fiber space E is associated to a principal fiber space p with group G, and G is a subgroup of a group H, it is always possible to consider E as associated to a principal fiber space with group H (extension of G to H). But when K is a subgroup of G, it is not always possible to consider E as associated with a principal fiber space with group K (restriction of G to K). Ehresmann showed that a topological condition must be satisfied: the existence of a section over *M* of the fiber space associated with *p* and with the natural action of *G*| on the homogeneous space G/K. This explains why there are always Riemannian structures on an arbitrary manifold *M*. *E* is then the tangent bundle, G = GL(n, **R** ) and K the orthogonal group

O(n, **R** ); the quotient G/K is then diffeomorphic to an **R** $^{N}$; and for fiber spaces with such fibers, sections over the base always exist, But even for pseudoRiemannian structures, topological conditions on M are necessary.

Ehresmann studied in detail the case in which *M*| has even dimension 2m, E is the tangent bundle so that G = GL(2m, **R** ), and K = GL(m, **c** ). When the restriction of G to K is possible, he said, the structure it defines on *M* is an almost complex| structure. The latter term comes from the fact that when *M* is a complex analytic manifold of complex dimension *m*, the tangent bundle has fibers that are vector spaces over **C** ; an almost complex structure, however, does not always derive from a complex structure, and additional conditions have to be imposed on the differential structure of *M/*. Independently, Heinz Hopf studied almost complex structures, and many other cases of restrictions to classical subgroups of GL(n, **R** ) were considered later.

In 1944, Ehresmann inaugurated the global theory of completely integrable systems of partial differential equations. In his local study of partial differential equations, Cartan had emphasized the advantages that derive from a geometrical conception of such systems, in contrast with their expression in nonintrinsic terms using local coordinates. For ordinary differential equations. this geometrical conception goes back to Henri Poincaré and substitutes for such an equation (with no singularities) on a manifold *M* a field of tangent lines on M—or, in modern terms, a line subbundle of the tangent bundle T(M). The natural generalization is therefore a vector subbundle L of rank p>1 of T(M), and the generalization of the integral curves of a differential equation are the injective immersions f:N→M into M of a manifold N of dimension q ≤ p, such that for y∈N, the image by the tangent map $T_y$ (f) of the tangent space $T_y$ (N) is contained in the fiber $L_f$ (y). Completely integrable systems are those for which there are such maps *f* for manifolds N of maximal dimension *p*, whose images f(N) may contain arbitrary points of M.

The characterization of these systems by local properties was presented in the work of Rudolf Clebsch and Georg Frobenius. Ehresmann initiated the study of their global solutions, in the spirit of the "qualitative" investigations started by Poincaré for *p* = 1, which have become known as the theory of dynamical systems. There are always maximal connected solutions *f* (N); they are called the leaves of the system, forming a partition of M, called a foliation. Ehresmann published only a few papers on that topic; the bulk of the basic notions in the theory was developed, under his guidance, in the dissertation of his pupil Georges Reeb. It was Reeb who obtained the first significant results, in particular the remarkable "Reeb foliation" of the sphere **S** $_{3}$, with a single compact leaf, which later played an important part in the general theory. Until 1960 these papers of Ehresmann and Reeb did not attract much attention, but since then the theory has enjoyed a vigorous and sustained growth that has made it a main branch of differential geometry and differential topology, with recently discovered and surprising links with the theory of C* algebras.

The next theory pioneered by Ehresmann was what he called the theory of jets. Two $C^x$ maps f, g of a manifold M into a manifold N have a contact of order k at a point x∈M where *f(x) = g(x)* if, in local coordinates around x and *f(x)*, their Taylor expansions coincide up to order k. This is independent of the choice of local coordinates and defines an equivalence relation in the set E(M, N) of $C^x$ maps of M into N. Ehresmann called the equivalence class of such a map f the kth jet of f at the point x. He developed the main properties of that notion in a series of notes. It has since been recognized that this notion provides the best frame for an intrinsic conception of general systems of partial differential equations and for Lie pseudogroups (formerly called infinite Lie groups), free from cumbersome computations in local coordinates.

After 1957 Ehresmann became one of the leaders in the new theory of categories, to which he attracted many younger mathematicians and in which his fertile imagination introduced a large number of concepts and problems. Over the next twenty years, he published his papers in that field and those of his school in a periodical of which he was both editor in chief and publisher, *Cahiers de topologie et de géométrie différentielle*.

Ehresmann's personality was distinguished by forthrightness, simplicity, and total absence of conceit or careerism. As a teacher he was outstanding, not so much for the brilliance of his lectures as for the inspiration and tireless guidance he generously gave to his research students, including Reeb and Jacques Feldbau; throughout his career he supervised a large number of doctoral dissertations.

# BIBLIOGRAPHY

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