# Fredholm, (Erik) Ivar | Encyclopedia.com



(*b*. Stockholm, Sweden, 7 April 1866; *d*. Stockholm, 17 August 1927)

*applied mathematics*.

Ivar Fredholm's small but pithy output was concentrated in the area of the equations of mathematical physics. Most significantly, he solved, under quite broad hypotheses, a very general class of integral equations that had been the subject of extensive research for almost a century. His work led indirectly to the development of Hilbert spaces and so to other more general function spaces.

Fredholm was born into an upper-middle-class family; his father was a wealthy merchant and his mother, née Stenberg, was from a cultured family. His early education was the best obtainable, and he soon showed his brilliance. After passing his baccalaureate examination in 1885, he studied for a year at the Polytechnic Institute in Stockholm. During this single year he developed an interest in the technical problems of practical mechanics that was to last all his life and that accounted for his continuing interest in applied mathematics. In 1886 Fredholm enrolled at the University of Uppsala—the only institution in Sweden granting doctorates at that time—from which he received the Bachelor of Science in 1888 and the Doctor of Science in 1898. Because of the superior instruction available at the University of Stockholm, Fredholm also studied there from 1888, becoming a student of the illustrious Mittag-Leffler. He remained at the University of Stockholm the rest of his life, receiving an appointment as lecturer in mathematical physics in 1898 and in 1906 becoming professor of rational mechanics and mathematical physics.

Fredholm's first major work was in partial differential equations. His doctoral thesis, written in 1898 and published in 1900, involved the study of the equation—written in Fredholm's own operator notation—

where $f(\xi,\eta,\zeta)$ is a definite homogeneous form. This equation is significant because it occurs in the study of deformation of anisotropic media (such as crystals) subjected to interior or exterior forces. Initially, Fredholm solved only the particular equations associated with the physical problem; in 1908 he completed this work by finding the fundamental solution to the general elliptical partial differential equations with constant coefficients.

Fredholm's monument is the general solution to the integral equation that bears his name:

In this equation the functions $f$ (called the kernel) and $\psi$ are supposed to be known continuous functions, and $\phi$ is the unknown function to be found. This type of equation has wide application in physics; for example, it can be shown that solving a particular case of (1) is equivalent to solving $\Delta u + \lambda u = 0$ for $u$, an equation that arises in the study of the vibrating membrane.

Equation (1) had long been under investigation, but only partial results had been obtained. In 1823, Niels Abel had solved a different form of (1) that also had a particular kernel. Carl Neuman had obtained, in 1884, a partial solution for (1) by use of an iteration scheme, but he had to impose certain convexity conditions to ensure convergence of his solution. By 1897 Vito Volterra had found a convergent iteration scheme in the case where $f$ has the property that $f(x,y) = 0$ for $y > x$.

Fredholm began work on equation (1) during a trip to Paris in 1899, published a preliminary report in 1900, and presented the complete solution in 1903. His approach to this equation was ingenious and unique. Fredholm recognized the analogy between equation (1) and the linear matrix-vector equation of the form $(I + F)\ U = V$. He defined

where $[f(x_i,y_j)]$ is the determinant of the $n$ X $n$ matrix whose $ij^{th}$ component is $f(x_i,y_j)$. The quantity $D_f$, Fredholm showed, plays the same role in equation (1) that the determinant of $I + F$ plays in the matrix equation; that is, there is a unique solution $\phi$ of equation (1) for every continuous function $\psi$ whenever $D_f$ is not zero. Furthermore, he proved that the homogeneous equation associated with equation (1),

has a nontrivial solution (that is, one not identically zero) if and only if $D_f = 0$.

The analogy between the matrix and integral equations was further pointed up by Fredholm when he defined the $n$th order minor of $f$—denoted by —by an expression similar to that for $D_f$. Then he showed that if $D_f$ is not zero, an explicit representation for the solution of equation (1) similar to Cramer's rule was given by

Fredholm then went on to show that if $D_f$ is equal to zero, the dimension of the null space (the vector space of the set of solutions) of equation (2) is finite dimensional. He did this by setting

where the denominator is a nonvanishing minor of least *n*th (finite) order (which he showed always exists) and where the same minor is denoted in the numerator, but with $\xi_i$ replaced by the variable $x$. Then he proved the set $\{\Phi_i: i = 1,2, \ldots, n\}$ to be a basis for the null space of equation (2). Finally, to solve equation (1) in the case $D_f = 0$, Fredholm first showed that a solution will exist when and only when $\psi(x)$ is orthogonal to the null space of the transposed homogeneous equation—or, equivalently, when

where $\{\Psi_i i = 1,2, ..., n\}$ is a basis for the null space of the equation

In this case, the solutions are not unique but can be represented by

and $\{a_i: i = 1,2,..., n\}$ is a set of arbitrary constants.

Thus, Fredholm proved that the analogy between the matrix equation $(I + F)U = V$ and equation (1) was complete and even included an alternative theorem for the integral equation. Yet he showed more. His result meant that the solution $\phi(x)$ for equation (1) could be developed in a power series in the complex variable $\lambda$

which is a meromorphic function of $\lambda$ for every $\lambda$ satisfying $D_{\lambda f} \neq 0$. (To see this, replace *f(x,y)* with *λf(x,y)* in equation [1].) This result was so important that, unable to prove it, in 1895–1896 in connection with his studies of the partial differential equation $\Delta u + \lambda u = h(x,y)$.

Fredholm's work did not represent a dead end. His colleague Erik Holmgren carried Fredholm's discovery to Gottingen in 1901. There David Hilbert was inspired to take up the study; he extended Fredholm's results to include a complete eigenvalue theory for equation (1). In the process he used techniques that led to the discovery of Hilbert spaces.

# BIBLIOGRAPHY

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