Hadamard, Jacques | Encyclopedia.com

(*b*. Versailles, France, 8 December 1865; *d*. Paris, France, 17 October 1963)

mathematics.

Hadamard was the son of Amédée Hadamard, a Latin teacher in a noted Paris lycée his mother, Claude-Marie Picard, was a distinguished piano teacher. After studying at the École Normale Supérieure from 1884 to 1888, he taught at the Lycée Buffon in Páris from 1890 to 1893 and received his *docteur ès sciences* degree in 1892. Hadamard was a lecturer at the Faculté des Sciences of Bordeaux from 1893 to 1897, lecturer at the Sorbonne from 1897 to 1909, then professor at the Collège de France from 1909 to 1937, at the École Polytechnique from 1912 to 1937, and at the École Centrale des Arts et Manufactures from 1920 to 1937.

Elected a member of the Académie des Sciences in 1912, Hadamard was also an associate member of several foreign academies, including the National Academy of Sciences of the United States, the Royal Society of London, the Accademia dei Lincei, and the Soviet Academy of Sciences. In addition he held honorary doctorates from many foreign universities.

Hadamard's interest in pedagogy led him to write articles about concepts in elementary mathematics that are introduced in the upper classes of the lycée. His *Leçons de géométrie élémentaire* (1898, 1901) still delight secondary school instructors and gifted pupils. The extent of his grasp on all domains of advanced research in France and abroad was evident in his famous seminar at the Collège de France; no branch of mathematics was neglected. The world's most famous mathematicians came there to present their own findings or those related to their specialty. But it was always Hadamard who had the last word and the surest judgment concerning the significance or the potential of the research presented.

Hadamard's first important works were concerned with analytic functions, notably with the analytic continuation of a Taylor series. Although Karl Weierstrass and Charles Méray were the first to define the meaning that must be attributed to the domain of existence of the analytic continuation of a Taylor series, their reflections amounted to a theorem of existence and uniqueness. Before Hadamard, little was known about the nature and distribution of the singularities of the series in terms of the nature of its coefficients, which define the function a priori. His thesis (1892), preceded by several notes in the *Comptes rendus* of the Academy, is one of his most beautiful works. For the first time an ensemble concept was introduced into function theory. In fact the upper limit—made more precise, explained, and applied to the ensemble composed of the coefficients—permits the determination of the radius of convergence (or, rather, its inverse) of the Taylor series. Conditions affecting the coefficients enable one to characterize the singular points on the circle of convergence. Hadamard's famous theorem on lacunary series (*"lacunes à la Hadamard"*) admitting the circle of convergence as a cut, the theorems on polar singularities, the introduction of the concept of *"écart fini"* and of the "order" of a singular point, and the theorem on the composition of singularities (1898), have remained fundamental in function theory. The results have inspired generations of highly talented mathematicians, especially those working at the beginning of the twentieth century and in the years between the two world wars. His *La série de Taylor et son prolongement analytique* (1901) was the "Bible" of all who were fascinated by the subject.

The year 1892 was one of the most fertile in the history of the theory of functions of a complex variable; it also marked the publication of a work by Hadamard that established the connection between the decrease of the modulus of the coefficients of the Taylor series of an integral function and the genus of the function. This work (which received the Grand Prix of the Académie des Sciences) and the results of his thesis (especially those pertaining to polar singularities), applied to Riemann's ζ function, enabled Hadamard in 1896 to solve the ancient and famous problem concerning the distribution of the prime numbers. He demonstrated (in a less explicit form than is shown here but one easily reducible to it) that the function $\pi(x)$ designating the number of prime numbers less than x is asymptotically equal to $x/\log x$. This is certainly the most important result ever obtained in number theory. Charles de La Vallée-Poussin proved the theorem at the same time, but his demonstration is much less simple than Hadamard's. The total result seems to indicate—without, we believe, its ever having been mentioned in writing by Hadamard—that the research in his thesis and in his work on integral functions was implicitly directed toward the ultimate goal of indicating the properties of the function ζ in order to derive from it the theorem on the prime numbers.

Returning to analytic functions, one should mention the 1896 theorem on the maximum modulus of an integral function (or of a holomorphic function in a disk). And, while remaining close to the essential principles of analytic functions but leaving aside those with a complex variable, one must emphasize Hadamard's introduction (1912, 1925) of the idea and of the problem of quasi analyticity, which consists in finding a relationship between the growth of the maxima of the moduli of the derivatives of a function on a segment and the fact of being determined in a unique way by the values that the function and its derivatives take at a point. It should be noted that it was Albert Holmgren's considerations relating to Augustin Cauchy's problem for the equation of heat that had led Hadamard to consider classes of infinitely differentiable functions, not necessarily analytic on a segment but nevertheless possessing the characteristic property of uniqueness on the segment. The idea of quasi analyticity plays a significant role in modern analysis.

It is important to emphasize a subject treated by Hadamard in which, avoiding analysis (under the circumstances, differential geometry) and replacing it with consideration of analysis situs or topology, he was able to display, in one of his most beautiful memoirs (1898), the philosophic character—referring to astronomical ideas—of the fundamental concept of "the problem correctly posed," although no concrete allusion to this term figured in it. This idea of the correctly posed problem played an essential part in Hadamard's later researches on equations with partial derivatives. The importance of analysis situs in the theory of differential equations was shown by Henri Poincaré, whom Hadamard admired greatly and to whose work he devoted several memoirs and monographs (1922, 1923, 1954).

The memoirs in question (1898) treat surfaces of negative curvature having a finite number of nappes extending to infinity. All analytic description is abandoned. On these surfaces the geodesics behave in three different ways: (1) they are closed or asymptotic to other such geodesics; (2) they extend to infinity on one of the nappes; (3) entire segments of these geodesics approach successively a series of closed geodesics, the length of these segments growing toward infinity. The striking thing is that the ensemble E of tangents to the geodesics passing through a point and remaining at a finite distance is perfect and never dense; and in each neighborhood of every geodesic whose tangent belongs to E (neighborhood of directions) there exists a geodesic which extends to infinity in an arbitrarily chosen nappe. In each of these neighborhoods there also exist geodesics of the third category. Hadamard states: "*Any change, however small, carried in the initial direction of a geodesic which remains at a finite distance is sufficient to produce any variation whatsoever in the final aspect of the curve*, the disturbed geodesic being able to take on any one of the forms enumerated above" (*Oeuvres*, p. 772).

But in a physics problem a slight modification in the circumstances at a certain moment ought to have little influence on the solution, since one never possesses conditions which are more than approximate. Hadamard concluded from this that the behavior of a trajectory might well depend on the arithmetic character of the constants of integration. One already sees here the genesis of the idea of the "problem correctly posed" which guided Hadamard in his researches on equations with partial derivatives. The problem of geodesics on the surfaces studied by Hadamard is not a correctly posed mechanics problem.

Hadamard fully set out the idea of the correctly posed problem for equations with partial derivatives in his excellent *Lectures on Cauchy's Problem in Linear Differential Equations* (1922; French ed., 1932). Thus, for Laplace's equation, Dirichlet's problem is a correctly posed problem; on the other hand, for an equation of the hyperbolic type, Cauchy's problem is the one which meets this criterion. These ideas have had a great influence on modern research because they have shown the necessity of introducing different types of neighborhoods and, in consequence, different species of continuity; these conceptions led to general topology and functional analysis. Also in the *Lectures* is the notion of the "elementary solution" which has so much in common with that of "distribution" (or "generalized function"). Also in connection with equations with partial derivatives, one should mention the concept of the "finite portion" of a divergent integral, which plays an essential role in the solution of Cauchy's problem.

Hadamard took a lively interest in Vito Volterra's functional calculus and suggested the term "functional" to replace Volterra's term "line function." Above all, in 1903 Hadamard was able to give a general expression for linear functionals defined for continuous functions on a segment. This was the ancestor of Friedrich Riesz's fundamental formula.

Few branches of mathematics were uninfluenced by the creative genius of Hadamard. He especially influenced hydrodynamics, mechanics, probability theory, and even logic.

BIBLIOGRAPHY

Hadamard's writings were collected as *Oeuvres de Jacques Hadamard*, 4 vols. (Paris, 1968). The years within parentheses in the text will enable the reader to find, in the bibliography of the *Oeuvres, any mémoire* that interests him.

On Hadamard or his work, see Mary L. Cartwright, "Jacques Hadamard," in *Biographical Memoirs of Fellows of the Royal Society*, **2** (Nov. 1965); P. Lévy, S Mandelbrojt, B. Malgrange, and P. Malliavin, *La vie et l'oeuvre de Jacques Hadamard,*, no. 16 in the series L'Enseignement Mathématique (Geneva, 1967); and S. Mandelbrojt and L. Schwartz, "Jacques Hadamard," in *Bulletin of the American Mathematical* Society, **71** (1965).

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