Harish-chandra | Encyclopedia.com

(*b*. Kanpur, Uttar Pradesh, India, 11 October 1923; *d*. Princeton, New Jersey, 16 October 1983),

mathematics, Lie groups.

Harish-Chandra was a major figure in the mathematics of the twentieth century. His work linked algebra, analysis, geometry, and group theory in a fundamental and epoch-making manner that subsequently became the foundation on which modern work in a variety of fields, ranging from differential geometry and mathematical physics to numbertheory, is being carried out.

Life and Career Overview . Harish-Chandra's father, Chandra Kishore, was a civil engineer in what was then known as United Provinces, situated in the Gangetic plains of northern India, and his mother, Satyagati Seth Chandrarani, was the daughter of a lawyer. Thus, Harish-Chandra's early years were spent in a comfortable upper-middle-class family. As is often the case, Harish-Chandra's early years were divided between his parents and grandparents. He was deeply influenced in many aspects of his life later by his father who was deeply religious and of great integrity. He was precocious, starting his seventh grade at the age of nine. Although his health was not robust, he was very successful in the formal aspects of education such as examinations, performing brilliantly. He took an MSc degree from the University of Allahabad at Allahabad in 1943. While he was in Allahabad, he came under the influence of Professor K. S. Krishnan, one of India's most outstanding physicists, and so Harish-Chandra's early interests were in theoretical physics. From Allahabad, Harish-Chandra went to Bangalore in southern India, where he worked with Homi Bhabha, also a theoretical physicist, who would later on become the founder-director of the Tata Institute of Fundamental Research in Mumbai. In 1945 he left Bangalore and went to Cambridge, England, to study at Cambridge University with Paul A. M. Dirac, under whom he wrote a thesis on the representations of the Lorentz group. The years in Cambridge convinced him that his talents were more in mathematics than in physics, and he began his lifelong study of representations of semisimple Lie groups.

Harish-Chandra went to the United States in 1947 where he stayed, except for brief visits to India, until the end of his life. In 1950 he went to Columbia University, where he remained until 1963, when he was offered a permanent position at the Institute for Advanced Study in Princeton, New Jersey. Harish-Chandra was named the I.B.M.-von Neumann Professor of Mathematics at the institute in 1968. He was elected a fellow of the Royal Society in 1973 and a member of the National Academy of Sciences of the United States in 1981. He received honorary doctorates from Delhi University in 1973 and Yale University in 1981. Harish-Chandra married Lalitha Kale of Bangalore, India, while he was on a visit to India in 1952. They had two daughters. His health was never very robust, and starting in 1969 he had several heart attacks that diminished his capacity to work intensely. Unfortunately, medical techniques were still not very advanced even in the United States, and the damage to his heart proved irreversible. He died in 1983 while out on a walk in Princeton.

Mathematics of Lie Groups . Harish-Chandra's work was mostly concerned with representations of semisimple Lie groups and harmonic analysis on them. Starting around 1949 he almost single-handedly erected his monumental theory over the course of the next thirty years or so. The depth and beauty of his results suggest that this is one of the most profound works of twentieth-century mathematics by an individual mathematician, and they make a compelling case for regarding him as one of the greatest mathematicians of this era.

The theory of group representations (homomorphisms of the group into the group of invertible linear transformations of a complex vector space) originated in the late nineteenth century with Georg Frobenius. If G is the group and $L(G \to GL(V))$ is the representation with dim $(V)<\infty$, Frobenius introduced the numerical function $\Theta_L(g) = \mathrm{Tr}(L(g))$ on G, called the character of the representation L, which determined the representation up to equivalence. Then, in the 1920s, Hermann Weyl, building on earlier work of Issai Schur for the orthogonal groups and his own work with F. Peter, developed a complete theory of representations of arbitrary compact groups. Then, in the 1930s, Fourier analysis, which hitherto had been confined to the analysis of functions on a torus (Fourier series) or analysis of functions on $\mathbf{R}^n$(Fourier integrals), was extended to all locally compact abelian groups by Andrei. Weil and independently by Mark G. Krein, and Israel Gel'fand. All of these developments could be seen in a unified manner as harmonic analysis on the groups in question, and the central question emerged as the expansion of the delta function at the identity element of the group as a linear combination of the characters of irreducible representations of the group. For U^1, the circle group, and $\mathbf{R}$, this expansion takes the familiar form

and for a locally compact abelian group G,

where $\hat{G}$ is the dual group of continuous homomorphisms of G into U^1. This formula, known as the Plancherel formula, takes, for compact G, the form where Θ_ω is the character of the representations in the class ω.

From this perspective, toup G is to determine all the functions on the group that are the characters of the irreducible representations, in terms of the structural data of the group. For G=SU(2) of 2×2 unitary matrices of determinant one, let ; then the irreducible characters are given by

Since any element of the group is conjugate to some u_θ, this formula determines the character on the full group. Since $\Theta_n(1) = n$, the Plancherel formula becomes

The formula (1) is a special case of the Weyl character formula valid for any compact connected Lie group G. The elements of G are conjugate to elements of a maximal torus T, the irreducible characters are parametrized by the characters of T that are positive in a suitable ordering, and they are given on T by

where W is the Weyl group acting on T, ∊, T is generic, and ϱ is a special character of T. Weyl also obtained a formula for the dimension of the irreducible representation that has the character Θ_ξ.

The growth of quantum mechanics, where symmetries of quantum systems are typically implemented by unitary operators in the Hilbert space of quantum states, gave a great impetus to the theory of infinite dimensional unitary representations of groups. For the Poincaré group, Eugene P. Wigner classified in 1939 all the physically important irreducible unitary representations, leading to the classification of free elementary particles by mass and spin. Then Gel'fand and Dmitri A. Raikov proved in 1943 that any locally compact group has enough irreducible unitary representations to separate points. The theory of representations and harmonic analysis on general locally compact groups began in earnest after this. Valentine Bargmann, following a suggestion of Wolfgang Pauli, developed the theory for the simplest such group, the group SL (2,**R**) of 2 × 2 real matrices of determinant 1. Independently, Gel'fand and Mark Naimark worked out the theory for the complex classical simple Lie groups of Élie Cartan, especially SL (n,C). These works gave a glimpse of a completely new landscape of infinite dimensional unitary representations containing analogs of the Frobenius-Weyl character theory, as well as the Plancherel formula.

Relation between Lie Group and Lie Algebra . This was the situation when Harish-Chandra began his odyssey. In his characteristic manner, he started on a theory of representations and Fourier analysis for all real semisimple Lie groups. His initial papers were dominated by the infinitesimal point of view, where the Lie algebra and its universal enveloping algebra were at the center of the stage. His 1951 paper on the enveloping algebra, cations of the Universal Enveloping Algebra of a Semisim-ple Lie Algebra," for which he received the Cole Prize of the American Mathematical Society in 1954, was perhaps the first one in which representations of infinite dimensional associative algebras were considered. In it he proved the fundamental theorems of semisimple Lie algebras, earlier obtained by Cartan using classification, by general algebraic methods. The techniques and concepts of this paper would play a critical role later in the 1960s in the theory of infinite dimensional (Kac-Moody) Lie algebras, and in the 1980s in the theory of quantum groups.

Harish-Chandra then turned his attention to the study of infinite dimensional representations of real semi-simple Lie groups. The method of passing to the Lie algebra, so effective in the finite dimensional case, is a much more subtle one in the infinite dimensional situation. Nevertheless, by a brilliant use of his idea of analytic vectors, Harish-Chandra showed that the correspondence between Lie algebra representations and Lie group representations remained particularly close even in the infinite case. In particular, by such methods he was led to one of his greatest discoveries, namely, that one can associate a character to infinite dimensional irreducible representations also. More precisely, he showed that for any unitary and irreducible representation L, and a smooth function f with compact support on the group G, the operator $L(f) := \int_G f(x)L(x)dx$ is of trace class and its trace $\Theta_L(f)$ is a distribution on G, the distribution character of the representation L. L may even be a Banach space representation satisfying some mild conditions. The distribution Θ_L is invariant (under all inner automorphisms of G) and determines L up to a very sharp equivalence (unitary equivalence when L is unitary, for instance) and is thus the correct generalization of the Frobenius-Weyl character.

In a long series of remarkable papers totaling several hundreds of pages in length, Harish-Chandra answered fundamental questions about the characters and discovered the formulae for the most crucial ones for reaching an explicit Plancherel formula for all real semisimple groups. Because it is not easy to use the condition that the distribution in question is the character of an irreducible unitary representation, Harish-Chandra had the insight to see that almost all of the properties of the character should flow from the fact that it is an eigendistribution of the bi-invariant (i.e., invariant under left and right translations) differential operators on G. More precisely, let **3** be the algebra of bi-invariant differential operators. By virtue of the identification of **3** with the center of the universal enveloping algebra of the Lie algebra of G, **3** acts on the smooth vectors of the irreducible representation L through a homomorphism ξ (**3** →**C**), and the distribution character Θ_L satisfies the differential equations

everywhere on the group G. He now proved the remarkable theorem (the regularity theorem) that any invariant distribution Θ, which has the property that the space spanned by the derivatives partial; (z) Θ (z∊**3**) is of finite dimension, is a function, that is, there is a function θ, which is locally integrable on G and analytic on a dense open set of it, such that

(f smooth and of compact support on G).

Other proofs have become available in the early 2000s, but they all have to rely on deep theories of differential operators such as D-modules.

Once the regularity theorem is proved, the next step in the Harish-Chandra program became that of writing the formula for the irreducible characters on the group. Very early on he had realized that the irreducible unitary representations of G (at least those that would play a role for harmonic analysis on $L^2(G)$) come in several "series" associated to the various conjugacy class of Cartan subgroups of G. The Cartan subgroups, the analogs in the noncompact case of the maximal tori of compact groups, are abelian subgroups with the property that a generic point of the group can be conjugated to be in one of them. Up to conjugacy there are only finitely many of these, and at most one can be compact. The work of Bargmann for SL (2,**R**), and his own extensions of it to the case when G/K is Hermitian symmetric (K is the maximal compact subgroup of G), led him to the fundamental insight that the series of representations corresponding to a compact Cartan subgroup B (when there is one) are parametrized by characters of B and have the special property of occurring as discrete direct summands of the regular representation of G, hence the name *discrete series* for these, and further that these characters are given on B by a very close variant of Weyl's formula in the compact case. If A is a non compact Cartan subgroup, one can associate a suitable subgroup M of G with discrete series and use a very direct procedure to build the series corresponding to this Cartan subgroup. This perspective thus placed the discrete series at the very foundation of the theory and highlighted the fact that they should be constructed before anything can be done.

Harish-Chandra began by constructing the characters of the discrete series, in the first place, as invariant eigendistributions. Because the invariant eigendistributions are functions by his regularity theorem, it is enough to specify them on the Cartan subgroups of G. He then proved that if B is a compact Cartan subgroup and ξ is a generic character of B, there is exactly one invariant eigendistribution Θ_ξ on the group that is given by Weyl's formula (3) on the compact Cartan subgroup and verifies a suitable boundedness condition on the other Cartan subgroups. The Harish-Chandra formula for Θ_ξ on B is given by

where W_G is the subgroup of W that arises from elements of G. Now G = SL (2,**R**) has 2 conjugacy classes of Cartan subgroups whose representatives can be taken to be the compact one B of the rotations and the non compact one A of diagonal matrices . In this case the distributions are the Θ_n ($n=\pm 1,\pm 2, \ldots$) with ($\theta\, 0, \pi, t\, 0$)

In particular $|\Theta_n(\pm h_t)|\,|\,e^t - e^{-t}|^{-1}$ is the boundedness condition. In the general case there is an invariant analytic function D (*discriminant*) such that $D=|\Delta|^2$ on any Cartan subgroup, and the boundedness condition is

Note that W_G={1} and so there is no alternating sum as in the case of SU(2). Harish-Chandra's method for continuing the character to the other Cartan subgroups was to use the differential equations satisfied by the distribution at the interfaces of the Cartan subgroups and show that the boundedness condition (5) forced the continuation to be unique. The author will not comment here on the very beautiful but difficult analytic methods Harish-Chandra discovered to prove that the invariant eigendistributions Θ_ξ are precisely the characters of the discrete series. In particular, this part of his work implied that the discrete series occurs if and only if one of the Cartan subgroups is compact. The characters of the other series could now be expressed explicitly. For instance, for G = SL (2,**R**), the Cartan subgroup A gives rise to the characters that vanish on B and are given on A by

The third step in the program was then to obtain the Plancherel formula for the group. This involved new ideas, especially in dealing with the continuous part of the decomposition of θ. Harish-Chandra discovered the general principle that the measure that should be used in the Plancherel formula to combine the matrix coefficients can be obtained from the asymptotic expansions of these eigen-functions at infinity on the group. This principle, linking the Plancherel measure with the asymptotics of the matrix coefficients, is a far-reaching generalization of a result of H. Weyl, who had discovered it in his work on the eigenfunction expansions of singular differential operators on a half line. For G = SL (2,**R**), the Plancherel formula becomes

The matrix coefficients defined by a suitable vector in the representation corresponding to say have the asymptotics

where the $c^{\pm}(\lambda)$ are rational fractions involving classical Gamma functions, and

Harish-Chandra then turned his attention to the semisimple groups defined over a p-adic field. This was not merely an idle generalization but essential for number theory. In fact, he himself had pioneered some of the most fundamental work on the arithmetic of semisimple groups in his paper with Armand Borel, where they proved that if **G** is a semisimple algebraic matrix group defined over the field **Q** of rational numbers, and $G_{\mathbf{Z}}$ is the sub group of integral matrices, then the space $G_{\mathbf{R}}/G_{\mathbf{Z}}$ has finite volume. For the harmonic analysis of the natural representation of G in $L_2(G_{\mathbf{R}}/G_{\mathbf{Z}})$, which is important in number theory, it turned out to be essential to understand the representation theory and harmonic analysis of the groups $G_{\mathbf{Q_p}}$, the groups of p-adic points of the algebraic group **G** .

In his work on the representation theory of the p-adic groups, Harish-Chandra was guided by the same approach that served him so well in the case of real groups. He called this the philosophy of cusp forms. But the discrete series for p-adic groups is much more arithmetic and less accessible than in the real case, although he showed that the main results for the continuous spectrum go through in the p-adic case. Harish-Chandra was very fond of the idea that the representation theory of all the p-adic completions of an algebraic semisimple group defined over **Q** ought to be based on the same set of principles, and he called it the Lefschetz principle. Its full effectiveness can be seen only by constructing the discrete series for the *p*-adic groups and by going over to the adelic groups. Unfortunately, this was not given to him to accomplish, illness overcoming his ability to work at an intense level in the last years of his life.

Personality . In his creative life, Harish-Chandra opted for intense penetration of a few areas as opposed to extensive knowledge, while in his personal life, his temperament preferred the plain over the ornate. His lifestyle was very simple, even ascetic, involving, especially in his younger years, periods of absolute stillness and concentration stretching for hours at a time; in later years, with the increasing uncertainty of his health, he had to moderate this aspect of his life, but he still had in him the passion for great bursts of work even in later years, one of which was responsible for his fatal heart attack. In his early years he was a good painter, and over the years came to admire intensely Van Gogh and Cezanne. He was conscious of his powers but was modest in a truly deep sense. His personality and achievements compelled others to devote themselves to problems that he considered important. In an age where collaboration and multiple-authorship are the norm, he was a singular figure, working solo to overcome Himalayan obstacles. His work is a faithful reflection of his personality—lofty, intense, uncompromising. It will be a long time before anyone remotely resembling him will arise in the history of mathematics.

BIBLIOGRAPHY

There is no official archive containing Harish-Chadra's unpublished works. The complete bibliography of his works can be found in the Collected Papers *cited here (Vol I).*

WORKS BY HARISH-CHANDRA

"On Some Applications of the Universal Enveloping Algebra of a Semisimple Lie Algebra." *Transactions of the American Mathematical Society* 70 (1951): 28–96.

"Representations of a Semisimple Lie group in a Banach Space." *Transactions of the American Mathematical Society* 75 (1953): 185–243.

With Armand Borel. "Arithmetic Subgroups of Algebraic Groups." *Annals of Mathematics* 75 (1962): 485–535.

"Invariant Eigendistributions on a Semisimple Lie Group." *Transactions of the American Mathematical Society* 119 (1965): 457–508.

"Discrete Series for Semisimple Lie Groups, I: Construction of Invariant Eigendistributions" *Acta Mathematica* 113 (1965): 241–318.

"Discrete Series for Semisimple Lie Groups, II: Explicit Determination of the Characters." *Acta Mathematica* 116 (1966): 1–111.

"Harmonic Analysis on Semisimple Lie Groups." *Bulletin of the American Mathematical Society* 76 (1970): 529–551.

"Harmonicanalysis on Reductive P-adic Groups." In *Harmonic Analysis on Homogeneous Spaces*, edited by Calvin C. Moore. Proceedings of Symposia in Pure Mathematics, XXVI. Providence, RI: American Mathematical Society, 1973.

"Harmonic Analysis on Real Reductive Groups, I: The Theory of the Constant Term." *Journal of Functional Analysis* 19 (1975): 104–204.

"Harmonic Analysis on Real Reductive Groups, II: Wave Packets in the Schwartz Space." *Inventiones Mathematicae* 36 (1976): 1–55.

"Harmonic Analysis on Real Reductive Groups, III: The Maass-Selberg Relations and the Plancherel Formula." *Annals of Mathematics* 104 (1976) 117–201.

Collected Papers/Harish-Chandra. 4 vols. Edited by V. S. Varadarajan. New York: Springer-Verlag, 1984.

OTHER SOURCES

Herb, Rebecca A. "An Elementary Introduction to Harish-Chandra's Work." In *The Mathematical Legacy of Harish-Chandra*, edited by Robert S. Doran and V. S. Varadarajan. Proceedings of Symposia in Pure Mathematics, vol. 68. Providence, RI: American Mathematical Society, 2000.

Howe, Roger. “The Work of Harish-Chandra on Reductive P-adic Groups.” In *Collected Papers/Harish-Chandra*, edited by V. S. Varadarajan, vol. 1. New York: Springer-Verlag, 1984.

Langlands, Robert P. “Harish-Chandra (1923–1983).” *Biographical Memoirs of Fellows of the Royal Society* 31 (November 1985): 199–225.

Varadarajan, V. S. “The Theory of Characters and the Discrete Series for Semisimple Lie Groups.” In *Harmonic Analysis on Homogeneous Spaces*, edited by Calvin C. Moore. Proceedings of Symposia in Pure Mathematics, vol. 26. Providence, RI: American Mathematical Society, 1973.

——. *Harmonic Analysis on Real Reductive Groups*. Lecture Notes in Mathematics, vol. 576. Berlin: Springer-Verlag, 1977.

—— *An Introduction to Harmonic Analysis on Semisimple Lie Groups*. Cambridge Studies in Advanced Mathematics, no. 16. Cambridge, UK: Cambridge University Press, 1989.

——. “Harish-Chandra, His Work, and Its Legacy.” In *The Mathematical Legacy of Harish-Chandra*, edited by Robert S. Doran and V. S. Varadarajan. Proceedings of Symposia in Pure Mathematics, vol. 68. Providence, RI: American Mathematical Society, 2000.

Wallach, Nolan R. “Some Additional Aspects of Harish-Chandra’s Work on Real Reductive Groups.” In *Collected Papers/Harish-Chandra*, edited by V. S. Varadarajan, vol. 1. New York: Springer-Verlag, 1984.

——. *Real Reductive Groups*. 2 vols. Boston: Academic Press, 1988–1992.

V. S. Varadarajan