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mathematics.

Hodge came from a solid middle-class background. His father, Archibald James Hodge, was a searcher of records (an office concerned with land titles), and his mother, Janet Vallance, the daughter of a prosperous proprietor of a confectionery business. He was educated at George Watson's Boys College, Edinburgh University (1920+1923, M.A.), and St. John's College, Cambridge (1923+1926, B.A.). On 27 July 1929 he married Kathleen Anne Cameron; they had one son and one daughter. A fellow of the Royal Society since 1938. Hodge was physical secretary between 1957 and 1965 and was a foreign associate of many academies, including the U.S. National Academy of Sciences. He was a vice president of the International Mathematical Union (1954+ 1958) and master of Pembroke College, Cambridge (1958+1970). He also was responsible for organizing the International Congress of Mathematicians at Edinburgh in 1958. A cheerful and energetic person, he was both popular and effective in his numerous administrative roles.

On leaving Cambridge, where he had already started to specialize in algebraic geometry. Hodge took up his first teaching appointment at Bristol in 1926, and, influenced by a senior colleague. Peter Fraser, he made strenuous efforts to master the work of the Italian algebraic geometers. In the course of his reading, he soon came across a problem that determined the course of his work for years to come.

In one of his papers, Francesco Severi mentioned the importance of knowing whether a nonzero double integral of the first kind could have all its periods zero. By a stroke of luck, Hodge saw a paper by Solomon Lefschetz in the 1929 Annals of Mathematics in which purely topological methods were used to obtain the period relations and inequalities for integrals on a curve. It was clear to Hodge that Lefschetz's methods could be extended to surfaces to solve Severi's problem, and he found it incredible that this should have escaped Lefschetz's notice. It did not take Hodge long to work out the details and write' On Multiple Integrals Attached to an Algebraic Variety.' This was the turning point inn his career, and therefore it is perhaps appropriate to describe the essence of his proof and its relation to Lefschetz's paper.

If $\omega_1, \ldots \omega_g$ are a basis of the holomorphic differentials on a curve X of genus g, we have $\omega_1 \Lambda \omega_i = 0$ for simple reasons of complex dimension. If $[\omega_i]$ denotes the 1–dimensional cohomology class defined by ω_i (that is, given by its periods), it follows that the cup product $[\omega_i]$ U $[\omega_i]$= 0. If we spell this out in terms of the periods, we obtain the Riemann bilinear relations for the ω_i. This is the modern approach and the essential content of Lefschetz's 1929 paper, though it must be borne in mind that co homology had not yet appeared and that the argument had to be expressed in terms of cycles and intersection theory. It was Lefschetz himself who led the developments in algebraic topology that enable us to express his original proof so succinctly.

Similarly, the Riemann inequalities arise from the fact that, for any holomorphic differential $\omega \neq 0$, $i\omega \lambda \varpi$ is a positive volume and thus $i \int_x \omega \lambda \varpi > 0$. In cohomological terms this implies in particular that $[\omega[\cup [\omega[\neq 0$, and hence that $[\omega[\neq 0$,; in other words ω cannot have all its periods zero. If we now replace X by an algebraic surface and ω by a nonzero holomorphic 2-form (a "double integral," in the classical terminology), exactly the same argument applies and disposes of Severi's question.

In addition to the primitive state of topology at this time, complex manifolds (other than Riemann surfaces) were not conceived of in the modern sense. The simplicity of the proof indicated above owes much to Hodge's work in later years, which made complex manifolds familiar to the present generation of geometers. All this must have had something to do with Lefschetz's surprising failure to see what Hodge saw. In fact, however, this was no simple omission on Lefchetz's part, and he took a great deal of convincing on this point. At first he insisted publicly that Hodge was wrong, and he wrote to him demanding that the paper be withdrawn. In May 1931 Lefschetz and Hodge had a meeting in Max Newman's rooms at Cambridge. There was a lengthy discussion leading to a state of armed neutrality and an invitation to Hodge to spend the next academic year at Princeton. After Hodge had been at Princeton for a month. Lefschetz conceded defeat and, with typical generosity, publicly retracted his criticisms of Hodge's paper. Thereafter Lefschetz became one of Hodge's strongest supporters and fully made up for his initial skepticism. His support proved crucial when in 1936 Hodge was elected to the Lowndean chair of astronomy and geometry at Cambridge.

The publication of' On Multiple Integrals …' opened many doors for Hodge. In November 1930 he was elected to a research fellowship at St. John's College, Cambridge, and shortly afterward was awarded an 1851 Exhibition Scholarship. He was thus in a position to take up Lefschetz's invitation to spend a year in Princeton.

In 1931 Princeton was a relatively small university with a very distinguished academic staff. In mathematics, pioneering work was being done in the new field of topology by Oswald Veblen, James W. Alexander, and lefschetz. Although Hodge never became a real expert in topology, he regularly attended the Princeton seminars and picked up enough general background for his subsequent work.

Lefschetz was, without doubt, the mathematician who exerted the strongest influence on Hodge's work.In Bristol, after his early encounter with Lefschetz's *Annals* paper. Hodge had proceeded to read his Borel tract, *L'analyse situs et la géométrie algébrique*, and was completely won over to the use of topological methods in the study of algebraic integrals. At Princeton, Lefschetz propelled Hodge further along his chosen path. They had frequent mathematical discussions in which Lefschetz's fertile imagination would produce innumerable ideas, most of which would turn out to be false but the rest of which would be invaluable.

Hodge in due course became Lefschetz's successor in algebraic geometry. Whereas the Princeton school inherited and developed Lefschetz's contributions in topology, his earlier fundamental work on the homology of algebraic varieties was somewhat neglected, probably because it was ahead of its time. Hodge's work was complementary or dual to that of Lefschetz, providing an algebraic description of the homology instead of a geometric one. The fact that Lefschetz's theory has now been restored to a central place in modern algebraic geometry is entirely due to the interest aroused by its interactions with Hodge's theory.

Recognizing that he was no longer an expert on algebraic geometry, Lefschetz persuaded Hodge to spend a few months at Johns Hopkins, where Oscar Zariski was the leading light. The visit had a great impact on Hodge's future. In the first place he and his wife formed a close lifelong friendship with the Zariskis. He also was impressed with the new algebraic techniques that Zariski was developing, and in later years he devoted much time and effort to mastering them. Although their technical involvement with algebraic geometry was different, Zariski and Hodge felt a common love for the subject and had serious mathematical discussions whenever they met They also maintained an intermittent correspondence for more than forty years.

Princeton. By the time Hodge came to Princeton, his mathematical ideas, arising from' On Multiple Integrals….' had already progressed very significantly. In studying integrals in higher dimensions, he soon put his finger on the crucial point. Whereas for the Riemann surface of a curve the number of holomorphic 1-forms is half the first Betti number, there is no corresponding relation in higher dimensions for holomorphic p-forms with $p > 1$. Hodge discussed this point on a number of occasions with Peter Fraser until one day Fraser pointed out Georges de Rham's dissertation, which had just arrived in the Bristol University library. In later years Hodge described this as a stroke of good fortune; although it did not solve his problem, it helped him to see what was involved. In de Rham's theory, valid on any real differentiable manifold, the main result is that there always exists a closed p-form ω with prescribed periods, and that ω is unique modulo derived forms. On a Riemann surface there are natural choices given by the real and imaginary parts of the holomorphic differentials, and Hodge was looking for an appropriate generalization to higherdimensional algebraic varieties. He saw that the real and imaginary parts of a holomorphic 1-form on a Riemann surface are in some sense duals of one another, and he had a hunch that there should be an analogous duality in general. More precisely, for each p-form ω there should be an $(n-p)$- form $*\omega$ (n being the dimension of the manifold); the preferred forms, later called harmonic. Would be those satisfying $d\omega = 0$ and $d(*\omega) = 0$. The main theorem to be proved would be the existence of a unique harmonic form with prescribed periods.

Once established at Princeton, Hodge tried to clarify and develop these vague ideas. He soon realized that the relationship of ω to $*\omega$ was a kind of orthogonality and was able to make this precise in euclidean space and, more generally, on a conformally flat manifold. He then attempted to prove the existence theorem by generalizing the classical Dirichlet methods. Thenext stage was to try to remove the restriction of conformal flatness, but for this Hodge needed to become familiar with classical Riemannian geometry. This was his major preoccupation and achievement during his stay in Baltimore. At the same time he also came across a paper by the Dutch mathematician Gerrit Mannoury in which an explicit and convenient metric was introduced on complex projective space, and hence on any projective algebraic manifold. This metric proved of fundamental importance for all of Hodge's subsequent work.

Cambridge, 1932–1939. On returining to Cambridge, Hodge continued his efforts to prove the existence theorem for harmonic forms on a general Riemannian manifold. His first version was, in his words, ' crude in the extreme', Hermann Weyl found it hard to judge whether the proof was complete or, rather, how much effort would be needed to make it complete. Nevertheless, Hodge was convinced that he was on the right track, and his next step was to apply his theory in detail to algebraic surfaces. Using the Mannoury metric, he proceeded to study the harmonic forms and found the calculations much simpler than he had expected. Finally, to his great surprise, he discovered that his results gave a purely topological interpretation of the geometric genus (the number of independent holomorphic 2-forms)

This was a totally unexpected result; and when it was published ('The Geometric Genus of a Surface as a Topological Invariant, ' 1933), it created quite a stir among algebraic geometers. In particular it convinced even the most skeptical of the importance of Hodge's theory and became justly famous as of Hodge's signature theorem.' Twenty years later it played a key role in Friedrich Hirzebruch's work on the Riemann-Roch theorem, and it remains one of the highlights of the theory of harmonic forms. It is intimately involved in the spectacular results (1983–1988) of Simon Donaldson on the structure of four-dimensional manifolds. Essentially Donaldson's work rests on a nonlinear generalization of the Hodge theory.

After his success with the signature theorem, Hodge worked steadily, polishing his theory and developing its applications to algebraic geometry. He also began to organize a connected account of all his work as an essay for the Adams Prize. He was awarded that prize in 1937, but the magnum opus took another three years to complete and finally appeared in book form *(The Theory and Applications of Harmonic Integrals*) in 1941. In the meantime he had published another approach to the existence theorem that had been suggested to him by Hans Kneser. It involved the use of the parametrix method of F.W. Levi and Devid Hilbert and was, as Hodge said, superior in all respects to his first attempt. Unfortunately this version, reproduced in his book, contained a serious error that was pointed out by Bohnenblust. Thenecessary modifications to provide a correct proof were made by Hermann Weyl at Princeton and independently by Kunihiko Kodaira in Japan.

Hodge freely admitted that he did not have the technical analytical background necessary to deal adequately with his existence theorem. He was only too pleased when others, better-qualified analysts than himself, completed the task, This left him free to devote himself to the applications in algebraic geometry which was what really interested him.

In retrospect it is clear that the technical difficulties in the existence theorem required not significant new ideas but a careful extension of classical methods. Thereal novelty, which was Hodge's major contribution, was in the conception of harmonic integrals and their relevance to algebraic geometry. This triumph of concept over technique is reminiscent of a similar episode in the work of Hodge's great predecessor Bernhard Riemann.

Wartime Cambridge . By the spring of 1940, Hodge had completed the manuscript of his book on harmonic integrals and felt he had exhausted his ideas in that area for the time being, He was therefore looking for a new field. On the other hand, his increasing administrative commitment to Pembroke College and Cambridge University left him less time and energy to devote to mathematics. These two factors help to explain the shift in his interests over the next decade. For some time he had been aware of the powerful algebraic techniques that had been introduced into algebraic geometry by Bartel L. van der Waerden and Zariski. These ideas had had little impact on British geo meters, so Hodge felt a duty to interpret and explain the new material to his colleagues. In this he was motivated by a desire to make amends to the Baker school of geometry at Cambridge for the sharp change of direction that his work on harmonic integrals had produced. He thus conceived the idea of writing a book that would replace Henry Baker's *Principles of Geometry*. Although not as demanding as original research. This task soon proved too much for his unaided effort. He therefore enlisted the assistance of Daniel Pedoe, thus beginning a collaborative enterprise that lasted for ten years and led to their three-volume *Methods of Algebraic Geometry*.

Although this book discharged Hodge's obligations to classical algebraic geometry and contained much useful material, it did not achieve its main objective of converting British geometers to modern methods, principally because it was overtaken by events. By the time it appeared, algebraic geometry was exploding with new ideas, and entirely new foundations were being laid. In addition, Hodge did not have the elegance and fluency of style that make algebra palatable. He recognized his limitations as an algebraist, however and despite his admiration for, and interest in, Zariski's work, he eventually returned to his' first love, ' the transceendtal theory.

Postwar Cambridge. Stimulated by a visit to Harvard in 1950, Hodge directed his research interests back to harmonic integrals. He wrote a number of papers that, though not sensational, were a steady development of his original ideas. In particular his paper' A Special Type of Kähler Manifold' led, a few years later, to Kodaira's final characterization of projective algebraic manifolds. The manifolds singled out by Hodge were, for a few years, known as Hodge manifolds; ironic ally, Kodaira's proof that Hodge manifolds are algebraic led to their disappearance.

Hodge also took an interest in the theory of characteristic classes and wrote a paper ("The Characteristic Classes on Algebraic Varieties") to bridge the gap between the algebraic-geometric classes of John A. Todd and the topological classes introduced by Shiing Shen Chem. He clearly saw the significance of this work at an early stage. and subsequent developments have fully justified him.

The early 1950's saw a remarkable influx of new topological ideas into algebraic geometry. In the hands of Henri Cartan, Jean-Pierre Serre, Kodaira, Donald Spencer, and Hirzebruch. these led in a few years to spectacular successes and the solution of many classical problems, such as the Riemann-Roch theorem in higher dimensions. The great revival of transcendental methods provided by sheaf theory and its intimate connection with harmonic forms naturally aroused Hodge's interest. He made strenuous efforts to understand the new methods and eventually saw that they could be applied to the study of integrals of the second kind. At this time (early 1954) he was busy preparing a talk to be delivered at Princeton in honor of Lefschetz's seventieth birthday. so he suggested that the present author, one of his research students, might try to develop the ideas further and see if they led to a complete treatment of integral of the second king. It did not take long to see that one obtained a very elegant and satisfactory theory in this way, and Hodge was given a complete manuscript a few days before his departure for Princeton. He was thus able to describe the results at the Princeton conference ("Integrals of the Second Kind on an Algebraic Variety").

Mathematical Assessment. Hodge's mathematical work centered so much on the one basic topic of harmonic integrals that it is easy to assess the importance of his contributions and to measure their impact. The theory of harmonic integrals can be roughly divided into two parts, the first dealing with real Riemannian manifolds and the second dealing with with complex, and particularly algebraic, manifolds, Thesewukk be considered separately.

For a compact Riemannian manifold (Without boundary), Hodge defined a harmonic form as one satisfying the two equations $d\phi = 0$ and $d^*\phi = 0$, where d is the exterior derivative and d^* its adjoint with respect to the Riemannian metric. An equivalent definition proposed later by Andèbe Weil, is $\Delta\phi= 0$, where $\Delta = dd^* + d^*d$. Hodge's basic theorem asserts that the space) $\mathcal{H}^q$ of harmonic q-forms is naturally isomorphic to the q-dimensional cohomology of X (or dual to the q-dimensional homology). The deep impression. As mentioned earlier, Hermann Weyl, the foremost mathematician of the time, was so impressed that he assisted Hodge with the technicalities, of the proof. At the International congress of Mathematicians in 1954, Weyl said that in his opinion, Hodge's *Harmonic Integrals* was' one of the great landmarks in the history of science in the present century.'

As an analytical result in differential geometry, one might have expected Hodge's theorem to have been discovered by an analyst, a differential geometer, or even a mathematical physicist (since in Miknowski space the equations $d\omega = d^*\omega = 0$ are simply Maxwell's equations). In fact Hodge knew little of the relevant analysis, no Riemannian geometry, and only a modicum of physics. His insight came entirely from algebraic geometry, where many other factors enter to complicate the picture.

The long-term impact of Hodge's theory on differential geometry and analysis was substantial. In both cases it helped to shift the focus from purely local problems to global problems of geometry and analysis' in the large.' Together with Marston Morse's work on the calculus of variations, it set the stage for the new and more ambitious global approach that has dominated much of mathematics ever since.

One of the most attractive applications of Hodge's theory is to compact Lie groups, in which, as Hodge showed in his book, the harmonic forms can be identified with the bi-invariant forms. Another significant in his book, due to Salomon Bochner, showed that suitable curvature hypotheses implied the vanishing of appropriate homology groups, the point being that the corresponding Hodge-Laplacian Δ was positive definite.

If we turn now to a complex manifold X with a Hermitian metric, we can decompose any differential r-form $\phi 4;$ in the form

where ϕ^{p+q} involues. in local coordinates $(z1 \ldots, z_n)$, p of the differentials dz and of the conjugate differentials $dz1$ (and is said to be of type $(p.q)$). In general, if $\Delta d\phi = 0$, so that $d\phi$ is harmonic, the components $d\phi^{p.q}$ need not be harmonic, the of Hodge's remarkable discoveries that for the Mannoury metric projective algebraic manifold (induced by the standard emetic on projective space,) the $d\phi^{p.q}$ are in fact harmonic. This property the Mannoury metric is a consequence of what is now known as the Käher condition: that the 2-form ω (of type (1.1)) associated to the materic ds^2, by the formula,

is closed (that is, $d\omega = 0$).

As a consequence we obtain a direct sum decomposition

of the space of harmonic forms and consequently, by Hodge's main theorem, a corresponding decomposition of the cohomology groups. In particular the Betti numbers $h^r = dim\mathcal{H}^r\ (X, C)$ are given by

Where $h^{p.q} = dim\mathcal{H}^{p.q}$. Moreover, as Hodge showed, the number h_{p+q} depend only on the the complex strumembedding (which defines the metric).

In this way Hodge obtained new numerical invariants of algebraic manifolds. As the $h^{p.q}$ satisfy certain symmetries, namely

$h^{p.q} = h^{q.p}$ and $h^{p.q} = h^{n-p.n-q}$.

various simple consequences are immediately deduced for the Betti numbers. Thus h^{24+1} is always even, generalizing the well-known property of a Riemann surface but also showing that many evendimensional real manifolds cannot carry a complex algebraic structure.

The fact that one obtains in this way the new intrinsic invariants $h^{p.q}$ is a first vindication of the Hodge theory as applied to algebraic manifolds. It shows that the apparently strange idea of introducing an auxiliary metric into algebraic geometry does in fact produce significant new information. For Riemann surfaces the complex structure defines a conformal structure, and hence the Riemannian metric is not far away; but in higher dimensions this relation with conformal structures breaks down and makes Hodge's success all the more suprising. Only in the 1950's, with the introduction of sheaf theory, was an alternative and more intrinsic definition given for the Hodge numbers, namely

$h^{p.q} = dim\ h^q\ (X, \Omega^p)$.

where Ω^p is the sheaf of holomorphic p-forms.

A further refinement of Hodge's theory involved the use of the basic 2-form ω, which is itself harmonic. Hodge showed that every harmonic r-form ϕ has a further decomposition of the form

where ϕ_s is an' effective' harmonic (r –2s) form. This decomposition is the analogue of the results of Lefschetz that relate the homology of X to the homology of its hyperplane sections, ϕ playing the role of a hyperplane.

The theory of harmonic forms thus provides a remarkably rich and detailed structure for the cohomology of algebraic manifolds. This' Hodge structure' has been at the basis of a vast amount of work since the mid-1930's, and it has become abundantly clear that it will in particular play a key role in future work on the theory of moduli.

One problem that Hodge recognized as of fundamental importance is the characterization of the homology classes carried by algebraic subvarieties of X. For divisors (varieties of dimension n –1) this problem had been settled by Émile Picard and Lefschetz, and Hodge saw what the appropriate generalization should be. For many years he attempted to establish his conjecture, but it eluded all his efforts. The conjecture is that a rational cohomology class in H^{2q} (X, Q) is represented by an algebraic subvariety if and only if its harmonic form is of type $(q. q)$. Thenecessity of this condition is easy; the difficulty lies in the converse, which asserts the existence of a suitable algebraic subvariety. This' Hodge conjecture' has achieved a considerable status, almost on a par with the Riemann hypothesis or the Poincaré conjecture. Its central importance is fully recognized, but no solution is in sight.

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