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(*b*. Düsseldorf, Germany, 25 April 1849; *d*. Göttingen, Germany, 22 June 1925)

mathematics.

Klein graduated from the Gymnasium in Düsseldorf. Beginning in the winter semester of 1865-1866 he studied mathematics and physics at the University of Bonn, where he received his doctorate in December 1868. In order to further his education he went at the start of 1869 to Göttingen, Berlin, and Paris, spending several months in each city. The Franco-Prussian War forced him to leave Paris in 1870. After a short period of military service as a medical orderly, Klein qualified as a lecturer at Göttingen at the beginning of 1871. In the following year he was appointed a full professor of mathematics at Erlangen, where he taught until 1875. From 1875 to 1880 he was professor at the Technische Hochschule in Munich, and from 1880 to 1886 at the University of Leipzig. From 1886 until his death he was a professor at the University of Göttingen. He retired in 1913 because of poor health. During World War I and for a time thereafter he gave lectures in his home. In August 1875 Klein married Anne Hegel, a granddaughter of the philosopher; they had one son and three daughters.

One of the leading mathematicians of his age, Klein made many stimulating and fruitful contributions to almost all branches of mathematics, including applied mathematics and mathematical physics. Moreover, his extensive activity contributed greatly to making Göttingen the chief center of the exact sciences in Germany. An opponent of one sided approaches, he possessed an extraordinary ability to discover quickly relationships between different areas of research and to exploit them fruitfully.

On the other hand, he was less interested in work requiring subtle and detailed calculations, which he gladly left to his students. In his later years Klein’s great organizational skill came to the fore, enabling him to initiate and supervise large-scale encyclopedic works devoted to many areas of mathematics, to their applications, and to their teaching. In addition Klein became widely known through his many books based on his lectures dealing with almost all areas of mathematics and with their historical development in the nineteenth century.

Klein’s extraordinarily rapid development as a mathematician was characteristic. At first he wanted to be a physicist, and while still a student he assisted J. Plücker, in his physics lectures at Bonn. At that time Plücker, who had returned to mathematics after a long period devoted to physics, was working on a book entitled *Neue Geometrie des Raumes, gegründet auf der geraden Linie als Raumelement*. His sudden death in 1868 prevented him from completing it, and the young Klein took over this task. Klein’s dissertation and his first subsequent works also dealt with topics in line geometry. The new aspects of his efforts were that he worked with homogeneous coordinates, which Plücker did only occasionally; that he understood how to apply the theory of elementary divisors, developed by Weierstrass a short time before, to the classification of quadratic straight line complexes (in his dissertation); and that he early viewed the line geometry of P_3 as point geometry on a quadric of P_5, which was a completely new conception.

In 1870 Klein and S. Lie (*see Werke*, I, 90-98) discovered the fundamental properties of the asymptotic lines of the famous Kummer surface, which, as the surface of singularity of a general quadratic straight-line complex, occupied a place in algebraic line geometry. Here and in his simultaneous investigations of cubic surfaces (*Werke*, II, 11-63) there is evidence of Klein’s special concern for geometric intuition, whether regarding the forms of plane curves or the models of spatial constructions. A further result of his collaboration with Lie was the investigation, in a joint work, of the so-called W-curves (*Werke*, I, 424-460). These are curves that admit a group of projective transformations into themselves.

Klein’s most important achievements in geometry, however, were the projective foundation of the non-Euclidean geometries and the creation of the “Erlanger Programm.” Both of these were accomplished during his enormously productive youth.

Hyperbolic geometry, it is true, had already been discovered by Lobachevsky (1829) and J. Bolyai (1832); and in 1868, shortly before Klein, E. Beltrami had recognized that it was valid on surfaces of constant negative curvature. Nevertheless, the non-Euclidean geometries had not yet become common knowledge among mathematicians when, in 1871 and 1873, Klein published two works entitled *Über die sogenannte nicht-euklidische Geometrie* (*Werke*, I, 254-351). His essential contribution here was to furnish so-called projective models for three types of geometry: hyperbolic, stemming from Bolyai and Lobachevsky; elliptic, valid on a sphere on which antipodal points have been taken as identical; and Euclidean. Klein based his work on the projective geometry that C. Staudt had earlier established without the use of the metric concepts of distance and angle, merely adding a continuity postulate to Staudt’s construction. Then he explained, for example, plane hyperbolic geometry as a geometry valid in the interior of a real conic section and reduced the lines and angles to cross ratios. This had already been done for the Euclidean angle by Laguerre in 1853 and, more generally, by A. Cayley in 1860; but Klein was the first to recognize clearly that in this way the geometries in question can be constructed purely projectively. thus one speaks of Klein models with Cayley-Klein metric.

The conceptions grouped together under the name "Erlanger Programm" were presented in 1872 in "Vergleichende Betrachtungen über neuere geometrische Forschungen" (*Werke*, I, 460-498). This work reveals the early familiarity with the concept of group that Klein acquired chiefly through his contact with Lie and from C. Jordan. The essence of the "Erlanger Programm" is that every geometry known so far is based on a certain group, and the task of the geometry in question consists in setting up the invariants of this group. The geometry with the most general group, which was already known, was topology; it is the geometry of the invariants of the group of all continuous transformations—for example, of the plane. Klein then successively distingusihed the projective, the affine, and the equiaffine or principal group of the particular dimension; in certain cases the succeeding group is a subgroup of the previous one. To these groups belong the projective, affine, and equiaffine geometries with their invariants, whereby the equiaffine geometry is the same as the Euclidean elementary geometry.

The non-Euclidean geometries accounted for with the aid of the Cayley-Klein models, as well as the various types of circular and spherical geometries devised by Moebius, Laguerre, and Lie, could likewise be viewed as the invariant theories of certain subgroups of the projective groups. In his later years Klein returned to the "Erlanger Programm" and, in a series of works (*Werke*, I, 503-612), showed how theoretical physics, and especially the theory of relativity, which had emerged in the meantime, can be understood on the basis of the ideas presented there. The "Programm" was translated into six languages and guided much work undertaken in the following years: for example, the analytic geometry of Lothar Heffter, school instruction, and the lifelong efforts of W. Blaschke in differential geometry. Only later in the twentieth century was it superseded.

Klein considered his work in function theory to be the summit of his work in mathematics. He owed some of his greatest successes to his development of Riemann's ideas and to the intimate alliance he forged between the latter and the conceptions of invariant theory, of number theory and algebra, of group theory, and of multidimensional geometry and the theory of differential equations, especially in his own fields, elliptic modular functions and automorphic functions.

For Klein the Riemann surface is no longer necessarily a multisheeted covering surface with isolated branch points on a plane, which is how Riemann presented it in his own publications. Rather, according to Klein, it loses its relationships to the complex plane and then, generally, to three-dimensional space. It is through Klein that the Riemann surface is regarded as an indispensable component of function theory and not only as a valuable means of representing multivalued functions.

Klein provided a comprehensive account of his conception of the Riemann surface in 1882 in *Riemanns Theorie der algebraischen Funktionen und ihre Integrade*. In this book he treated function theory as geometric function theory in connection with potential theory and conformal mapping—as Rriemann had done. Moreover, in his efforts to grasp the actual relationships and to generate new results, Klein deliberately worked with spatial intiution and with concepts that were borrowed from physics, especially from fluid dynamics. He repeatedly stressed that he was much concerned about the deficiencies of this method of demonstration and that he expected them to be eliminated in the future. A portion of the existence theorems employed by Klein had already been proved, before the appearance of the book by Klein, by H. A. Schwarz and C. Neumann. Klein did not incorporate their results in his own work: He opposed the spirit of the reigning school of Berlin mathematicians led by Weierstrass, with its abstract-critical, arithmetizing tendency; Riemann's approach, which inclined more toward geometry and spatial representation, he considered more fruitful. The rigorous foundation of his own theorems and the fusion of Riemann's and Weierstrass' concepts that Klein hoped for and expected found its expression—still valid today—in 1913 in H. Weyl's *Die idee der Riemannschen Fläche*.

A problem that greatly interested Klein was the solution of fifth-degree equations, for its treatment involved the simultaneous consideration of algebraic group theory, geometry, differential equations, and function theory. Hermite, Kronecker, and Brioschi had already employed transcendental methods in the solution of the general algebraic equation of the fifth degree. Klein succeeded in deriving the complete theory of this equation from a consideration of the icosahedron, one of the regular polyhedra known since antiquity. These bodies sometimes can be transformed into themselves through a finite group of rotations. The icosahedron in particular allows sixty such rotations into itself. If one circumscribes a sphere about a regular polyhedron and maps it onto a plane by stereographic projection, then to the group of rotations of the polyhedron into itself there corresponds a group of linear transformations of the plane into itself. Klein demonstrated that in this way all finite groups of linear transformations are obtained, if the so-called dihedral group is added. By a dihedron Klein meant a regular polygon with *n* sides, considered as rigid body of null volume.

Through the relationships of the fifth-degree equations to linear transformations and through the joining of his investigations with H. A. Schwarz's theory of triangular functions, Klein was led to the elliptic modular functions, which owe their name to their occurrence in elliptic functions. He dedicated a long series of basic works to them and, with R. Fricke, presented the complete theory of these functions in two extensive volumes that are still indispensable for research. Individual aspects of the theory were known earlier. It was a question here of holomorphic function in the upper half-plane $\mathcal{H}$ with a pole at infinity, which remain invariant under the transformations of the modular group Γ:

If one sets z = x + iy, then the set F of points z, with and additionally, $x \leqslant 0$ if $x^2 + y^2 = 1$, has at every point in $\mathcal{H}$ exactly one point equivalent to that point under Γ. F is a fundamental domain for Γ relative to $\mathcal{H}$. It had already been recognized as uch by Gauss. In 1877, somewhat later than Dedekind and and independently of him, Klein discovered the fundamental invariant $J(\tau)$, which assume each value in F exactly once and by means of which all modular functions are representable as rational functions.

Klein next investigated the subgroups Γ_1 of Γ with finite index, their fundamental domains, and the related functions. He thus arrived at algebraic function fields, which he investigated with the concept and method of Riemann's function theory. The Abelan integrals and differentials, and thereby the modular forms, as a generalization of the modular functions, lead to the modular functions on Γ_1. We also owe to Klein the congruence group. These are subgroups Γ_1 of Γ that contain the group of all transformations

;$a \equiv d \equiv \pm 1, b \equiv c \equiv 0 \bmod m$

for fixed natural number m. The least possible m for a group Γ_1 Klein designated as the level of the group. The congruence groups are intimately related to basic theorems of number theory. The theory of modular functions was further developed by direct students of Klein, such as A. Hurwitz and R. Fricke, and most notably by Erich Hecke; its application to several variables was due especially to David Hilbert and Carl Ludwig Siegel.

From the modular functions Klein arrived at the automorphic functions, which, along with the former, include the singly and doubly periodic functions. Automorphic functions are based upon arbitrary groups Γ of linear transformations that operate on the Riemann sphere or on a subset thereof; they have interior points in their domain of definition that have neighborhoods in which no two points are equivalent under Γ. They also possess a fundamental domain F. Klein studied the various types of networks produced from F by the action of Γ. A primary role is played by the *Grenzkreisgruppen*, by means of whcih the net fills the interior of a circle that goes into itself under Γ; and under them there are again finitely many generators. The groups lead to algebraic function fields, and thus Klein could apply the ideas of Riemann that he had further developed. At the same time as Klein and in competition with him, Poincaré developed a theory of automorphic functions. In opposition to Klein, however, he established his theory in terms of analytic expressions—called, accordingly, Poincaré series. The correspondence between the two mathematicians during 1881 and 1882, which was beneficial to both of them, can be found in volume III of Klein's *Gesammelte mathematische Abhandlungen*.

The path from automorphic functions to algebraic functions may be traveled in both directions—that is the essence of the statements that Klein termed the "fundamental theorems," which were set forth by both himself and Poincaré in reciprocally influential works. Among the fundamental theorems, for example, is the following portion of the *Grenzkreistheorem:* Let $f(w, z)$ be an irreducible polynomial in w and z in the field of the complex numbers. Then one obtains all solution pairs of the equation $f(w, z) = 0$ in the form $w = g_1(t)$, $z = g_2(t)$, where $g_1(t)$ and $g_2(t)$ are rational functions in t, or doubly periodic functions, or automorphic functions under a *Grenzkreis* group, according to whether the Riemann surface corresponding to $f(w, z) = 0$ is of genus 0, 1, or higher than 1.The variable t is said to be *Grenzkreis* uniformizing; it is well defined up to a linear transformation. Klein, like Poincaré, worked with the fundamental theorems without being able to prove them fully. This was first accomplished at the beginning of the twentieth century by Paul Koebe. The progress made in the theory of automorphic functions since the 1930's is due primarily to W. H. H. Petersson (*b*. 1902).

In the 1890's Klein was especially interested in mathematical physics and engineering. One of the first results of this shift in interest was the textbook he composed with A. Sommerfeld on the theory of the gyroscope. It is still the standard work in this field of mechanics.

Klein was not pleased with the increasingly abstract nature of contemporary mathematics. His longstanding concern with applications was further strengthened by the impressions he received during two visits to the United States. He sought, on the one hand, to awaken a greater feeling for applications among pure mathematicians and, on the other, to lead engineers to a greater appreciation of mathematics as a fundamental science. The first goal was advanced by the founding, largely through Klein's initiative, of the Göttingen Institute for Aeronautical and Hydrodynamical Research; at that time such institutions were still uncommon in university towns. Moreover, at the turn of the century he took an active part in the major publishing project *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*. He himself was editor, along with Konrad Müller, of the four-volume section on mechanics.

What a fruitful and stimulating teacher Klein was can be seen from the number—forty-eight—of dissertations prepared under his supervision. Starting in 1900 he began to take a lively interest in mathematical instruction below the university level while continuing to pursue his academic functions. An advocate of modernizing mathematics instruction in Germany, in 1905 he played a decisive role in formulating the "Meraner Lehrplanentwürfe." The essential change recommended was the introduction in secondary schools of the rudiments of differential and integral calculus and of the function concept. In 1908 at the International Congress of Mathematicians in Rome, Klein was elected chairman of the International Commission on Mathematical Instruction. Before World War I, the German branch of the commission published a multivolume work containing a detailed report on the teaching of mathematics in all types of educational institutions in the German empire.

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