# Laguerre, Edmond Nicolas | Encyclopedia.com

Complete Dictionary of Scientific Biography COPYRIGHT 2008 Charles Scribner's Sons
9-11 minutes

---

(*b*. Bar-le-Duc, France, 9 April 1834; *d*. Bar-le-Duc, 14 August 1886)

*mathematics.*

Laguerre was in his own lifetime considered to be a geometer of brilliance, but his major influence has been in analysis. Of his more than 140 published papers, over half are in geometry; in length his geometrical work represents more than two-thirds of his total output. He was also a member of the geometry section of the Academy of Sciences in Paris.

There was no facet of geometry which did not engage Laguerre's interest. Among his works are papers on foci of algebraic curves, on geometric interpretation of homogeneous forms and their invariants, on anallagmatic curves and surfaces (that is, curves and surfaces which are transformed into themselves by inversions), on fourth-order curves, and on differential geometry, particularly studies of curvature and geodesics. He was one of the first to investigate the complex projective plane.

Laguerre also published in other areas. Geometry led him naturally to linear algebra. In addition, he discovered a generalization of the Descartes rule of signs, worked in algebraic continued fraction, and toward the end of his life produced memoirs on differential equation and elliptical function theory.

The young Laguerre attended several public schools as he moved from place to place for his health. His education was completed at the ècold Polytechnique in Paris, where he excelled in modern languages and mathematics. His overall showing, however, was relatively poor: he ranked forty-sixth in his class. Nevertheless, he published his celebrated "On the Theory of Foci" when he was only nineteen.

In 1854 Laguerre left school and accepted a commission as an artillery officer. For ten years, while in the army, he published nothing. Evidently he kept on with his studies, however, for in 1864 he resigned his commission and returned to Paris to take up duties as a tutor at the école Polytechnique. He remained there for the rest of his life and in 1874 was appointed *examinateur*. In 1883 Laguere accepted, concurrently, the chair of mathematical physics at the Collège de France. At the end of February 1886 his continually poor health broke down completely; he returned to Bar-le-Due, where he died in August. Laguerre was pictured by his comtemporaries as a quiet, gentle man who was passionately devoted to his research, his teaching, and the education of his two daughters.

Although his efforts in geometry were striking, all Laguerre's geometrical production—with but one exception—is now unknown except to a few specilists. Unfortunately for Laguerre's place in history, this part of his outpur has been largely absorbed by later theories or hasd passed into the general body of geometry without acknowledgement. For example, his work on differential inveriants in included in the more comprehensive Lie group theory. Laguerre's one theorem of geometry which is still cited with frequency is the discovery—made in 1853 in "On the Rheory of Foci" —that in the complex projective plane the angel between the lines *a* and *b* which intersect at the point *O* is given by the formula

where the numerator is the cross ratio of *a, b* and lines joining *O* to the circular points as infinity: $I = (i, 1, 0)$ and $J = (-i, 1, 0)$.

Actually, Laguerre proved more. He showed that if a system of angles *A*, *B*, *C*,… in a plane is related by a function *F* (*A*, *B*, *C*,…) = 0, and if the system is transformed into another, *A*', *B*', *C*',…, by a homographic (cross ratio-preserving) mapping, then *A*', *B*', *C*',…, satisfies the relation

where α, □, γ,…are cross ratios, as in expression (1).

This theorem is commonly cited as being an inspiration for Arthur Cayley when he introduced a metric into the projective plane in 1859 and for Felix Klein when he improved and extended Cayley's work in 1871.[1] These assertions appear to be false. These is no mention of Laguerre in Cayley, and Cayley was meticulous to the point of fussiness in the assigning of proper credit. Klein is specific; he states that Laguerre's work was not known to him when he wrote his 1871 paper on non-Eucliden geometry.[2] Presumably the Laguerre piece was brought to Klein's attention after his own publication.

Nevertheless, Laguerre's current reputation rests on a very solid foundation: his discovery of the set of differential equations (Laguerre's equations)

$xy'' + (1 + x)y' - ny = 0, (n = 0, 1, 2, \ldots)$ (2)

and their polynomial solutions (Laguerre's polynomials)

These ideas have been enlarged so that today generalized Laguerre equations are usually considered. They have the form

$xy'' + (s + 1 - x)y' + ny = 0, (n = 0, 1, 2, \ldots)$ (4)

and have as their solutions the generalized Laguerre polynomials,

which also are frequently written as

The alternating sign of (5) not present in (3) is due to the change of signs of the coefficients of the *y* and *y*' terms in (4). The Laguerre functions are defined from the polynomials by setting

If $s = 0$, the notations $L_n(x)$ and are often used. These functions and polynomials have wide uses in mathematical physics and applied mathematics—for example, in the solution of the Schrödinger equations for hydrogen-like atoms and in the study of electrical networks and dynamical systems.[3]

Laguerre studied the Laguerre equation in connection with his investigations of the integral

and published the results in 1879.

He started by setting

from which the relation

was obtained by integration by parts. Observe that as *n* increase beyond bound in (7), the infinite series obtained diverges for every *x*, since the *n* th term fails to go to zero. Nerertheless, for large-value *x* the first few terms can be utilized in (8) to give a good approximation to the integral (6).

Next, Laguerre set

where i is a polynomial, to be determined, of degree I, which is at most $n/2$; □ is another unknown polynomial; and $\{1/x^{2m+1}\}$ is a power series in $1/x$ whose first term is $1/x^{2m+1}$. He then showed that *f* and □$(x)$ satisfy

$x[\square'(x)f(x) - f'(x)\square(x) - \square(x)f(x)] + f^2(x) = A,$

where *A* is a constant. This was used to show that *f* is a solution of the second-order differential equation

$xy'' + (x + 1)y' - my = 0.$ (10)

Another solution, linearly independent of *f*, is

Substitution of *f* back into (10) shows, by comparison of coefficients, that it must satisfy (Laguerre's polynomial)

These results were combined by Laguerre to obtain the continued fraction representation for (6)

Then Laguerre proved that the *m* th approximate of the fraction could be written as , where $f_m(x)$ is the Laguerre polynomial of degree *m* and $\square_m$ is the associated numerator in expression (9). From this the convergence of the fraction in (12) was established.

Finally, Laguerre displayed several properties of the set of polynomials. He proved that the roots of $f_m(x)$ are all real and unequal, and that a quasiorthogonality conditions is satisfied, that is,

where $\delta^{mn} \neq 0$ if $m \neq n$, 1 if $m = n$. Furthermore, from (13) he proved that if □$(x)$ is "any" function, then □ has an expansion as a series in Laguerre polynomials,

The coefficients, $A_n$, are given by the formula

In particular,

which led Laguerre to the following inversion: if (14) is symbolically written as $\square(x) = \theta(f)$, then $\theta(-x) = \square(-f)$.

This memoir of Laguerre’s is significant not only because of the discovery of the Laguerre equations and polynomials and their properties, but also because it contains one of the earliest infinite continued fractions which was known to be convergent. That it was developed from a divergent series is especially remarkable.

What, then, can be said to evaluate Laguerre’s work? That he was brilliant and innovative is beyond question. In his short working life, actually less than twenty-two years, the produced a quantity of first-class papers. Why, then, is his name so little known and his work so seldon cited? Because as brilliant as Laguerre was, he worked only on details—significant details, yet nevertheless details. Not once did not he step back to draw together various pieces and put them into a single theory. The result is that his work has mostly come down as various interesting speical cases of more general theories discovered by others.

## NOTES

1. Arthur Cayley, “Sixth Memoir on Quantics” (1859), in *Collected Works*, II (Cambridge, 1898), 561–592; Felix Klein, “Uber die sogenannte nicht-Euklidische Geometrie,” in *Mathematische Annalen*, *4* (1871), 573–625, also in his *Gesammelte mathematische Abhandlungen*, I (Berlin, 1921), 244–305.

2. Klein *Gesammelte mathematische Abhandlungen*, I, 242.

3. V. S. Aizenshtadt, *et al., Tables of Laguerre Polynomials and Functions*, translated by Prasenjit Basu (Oxford, 1966); J. W. Head and W. P. Wilson, *Laguerre Functions. Tables and Properties*, ITS Monograph 183 R (London, 1961).

## BIBLIOGRAPHY

Laguerre’s works were brought together in his *Oeuvres*, 2 vols. (Paris, 1898), with an obituary by Henri Poincaré. This ed. teems with errors and misprints.

See also Arthur Erdelyi, *et al., Higher Transcendental Functions* (New York, 1935).

Michael Berndopf