# Lobachevsky, Nikolaiivanovich | Encyclopedia.com

Complete Dictionary of Scientific Biography COPYRIGHT 2008 Charles Scribner's Sons
29-37 minutes

---

(*b*. Nizhni Novgorod [now Gorki], Russia, 2 December 1792; *d*. Kazan, Russia, 24 February 1856)

*mathematics*.

Lobachevsky was the son of Ivan Maksimovich Lobachevsky, a clerk in a land-surveying office, and Praskovia Aleksandrovna Lobachevskaya. In about 1800 the mother moved with her three sons to Kazan, where Lobachevsky and his brothers were soon enrolled in the Gymnasium on public scholarships. In 1807 Lobachevsky entered Kazan University, where he studied under the supervision of Martin Bartels, a friend of Gauss, and, in 1812, received the master's degree in physics and mathematics. In 1814 he became an adjunct in physical and mathematical sciences and began to lecture on various aspects of mathematics and mechanics. He was appointed extraordinary professor in 1814 and professor ordinarius in 1822, the same year in which he began an administrative career as a member of the committee formed to supervise the construction of the new university buildings. He was chairman of that committee in 1825, twice dean of the department of physics and mathematics (in 1820–1821 and 1823–1825), librarian of the university (1825–1835), rector (1827–1846), and assistant trustee for the whole of the Kazan educational district (1846–1855).

In recognition of his work Lobachevsky was in 1837 raised to the hereditary nobility; he designed his own familial device (which is reproduced on his tombstone), depicting Solomon's seal, a bee, an arrow, and a horseshoe, to symbolize wisdom, diligence, alacrity, and happiness, respectively. He had in 1832 made a wealthy marriage, to Lady Varvara Aleksivna Moisieva, but his family of seven children and the cost of technological improvements for his estate left him with little money upon his retirement from the university, although he received a modest pension. A worsening sclerotic condition progressively affected his eyesight, and he was blind in his last years.

Although Lobachevsky wrote his first major work, *Geometriya,* in 1823, it was not published in its original form until 1909. The basic geometrical studies that it embodies, however, led Lobachevsky to his chief discovery—non-Euclidean geometry (now called Lobachevskian geometry)—which he first set out in "Exposition succincte des principes de la géométrie avec une démonstration rigoureuse du théorème des parallèles," and on which he reported to the Kazan department of physics and mathematics at a meeting held on 23 February 1826. His first published work on the subject, "O nachalakh geometrii" ("On the Principles of Geometry"), appeared in the *Kazanski vestnik,* a journal published by the university, in 1829–1830; it comprised the earlier "Exposition."

Some of Lobachevsky's early papers, too, were on such nongeometrical subjects as algebra and the theoretical aspects of infinite series. Thus, in 1834 he published his paper "Algebra ili ischislenie konechnykh" ("Algebra, or Calculus of Finites"), of which most had been composed as early as 1825. The first issue of the *Uchenye zapiski*("Scientific Memoirs") of Kazan University, founded by Lobachevsky, likewise carried his article "Ob ischezanii trigonometricheskikh strok" ("On the Convergence of Trigonometrical Series"). The chief thrust of his scientific endeavor was, however, geometrical, and his later work was devoted exclusively to his new non-Euclidean geometry.

In 1835 Lobachevsky published a long article, "Voobrazhaemaya geometriya" ("Imaginary Geometry"), in the *Uchenye zapiski*. He also translated it into French for Crelle's *Journal für die reine und angewandte Mathematik*. The following year he published, also in the *Uchenye zapiski,* a continuation of this work, "Primenenie voobrazhaemoi geometrii k nekotorym integralam" ("Application of Imaginary Geometry to Certain Integrals"). The same period, from 1835 to 1838, also saw him concerned with writing *Novye nachala geometrii s polnoi teoriei parallelnykh* ("New Principles of Geometry With a Complete Theory of Parallels"), which incorporated a version of his first work, the still unpublished *Geometriya*. The last two chapters of the book were abbreviated and translated for publication in Crelle's *Journal* in 1842. *Geometrische Untersuchungen zur Theorie der Parallellinien,* which he published in Berlin in 1840, is the best exposition of his new geometry; following its publication, in 1842, Lobachevsky was, on the recommendation of Gauss, elected to the Güttingen Gesellschaft der Wissenschaften. His last work, *Pangéométrie,* was published in Kazan in 1855–1856.

**Lobachevskian Geometry.** Lobachevsky's non-Euclidean geometry was the product of some two millennia of criticism of the *Elements*. Geometers had historically been concerned primarily with Euclid's fifth postulate: If a straight line meets two other straight lines so as to make the two interior angles on one side of the former together less than two right angles, then the latter straight lines will meet if produced on that side on which the angles are less than two right angles (it must be noted that Euclid understood lines as finite segments). This postulate is equivalent to the statement that given a line and a point not on it, one can draw through the point one and only one coplanar line not intersecting the given line. Throughout the centuries, mathematicians tried to prove the fifth postulate as a theorem either by assuming implicitly an equivalent statement (as did Posidonius, Ptolemy, Proclus, Thabit ibn Qurra, Ibn al-Haytham, Saccheri, and Legend re) or by directly substituting a more obvious postulate for it (as did al-Khayyāmī, al-Tüsī, and Wallis).

In his early lectures on geometry, Lobachevsky himself attempted to prove the fifth postulate; his own geometry is derived from his later insight that a geometry in which all of Euclid's axioms except the fifth postulate hold true is not in itself contradictory. He called such a system "imaginary geometry," proceeding from an analogy with imaginary numbers. If imaginary numbers are the most general numbers for which the laws of arithmetic of real numbers prove justifiable, then imaginary geometry is the most general geometrical system. It was Lobachevsky's merit to refute the uniqueness of Euclid's geometry, and to consider it as a special case of a more general system.

In Lobachevskian geometry, given a line *a* and a point *A* not on it (see Fig. 1), one can draw through *A* more than one coplanar line not intersecting *a*. It

follows that one can draw infinitely many such lines which, taken together, constitute an angle of which the vertex is *A*. The two lines, *b* and *c*, bordering that angle are called parallels to *a* and the lines contained between them are called ultraparallels, or diverging lines; all other lines through *A* intersect *a*. If one measures the distance between two parallel lines on a secant equally inclined to each, then, as Lobachevsky proved, that distance decreases indefinitely, tending to zero, as one moves farther out from *A*, Consequently, when representing Lobachevskian parallels in the Euclidean plane, one often draws them conventionally as asymptotic curves (see Fig. 2). The

angle $\Pi(x)$ between the perpendicular $x$ from point *A* to line *a* and a parallel drawn through *A* is a function of $x$. Lobachevsky showed that this function, named after him, can be expressed in elementary terms as

clearly,$\Pi(0) = \pi/2$ and, for $x > 0$, $\Pi(x) < \pi/2$ Lobachevsky later proved that the distance between two diverging lines, where distance is again measured on a secant equally inclined to each, tends to infinity as one moves farther out from *A*; the distance has a minimum value when the secant is perpendicular to each line, and this perpendicular secant is unique.

A comparison of Euclidean and Lobachevskian geometry yields several immediate and interesting contrasts. In the latter, the sum of the angles of the right triangle of which the vertices are the point *A*, the foot of the perpendicular $x$ on *a* and a point on *a* situated at such a distance from $x$ that the hypotenuse of the triangle lies close to a parallel through *A* is evidently less than two right angles. Lobachevsky proved that indeed for all triangles in the Lobachevskian plane the sum of the angles is less than two right angles.

In addition to pencils of intersecting lines and pencils of parallel lines, which are common to both the Euclidean and Lobachevskian planes, the latter also contains pencils of diverging lines, which consist of all perpendiculars to the same line. In both planes, the circle is the orthogonal trajectory of a pencil of intersecting lines, but the orthogonal trajectories of the other pencils differ in the Lobachevskian plane. That of a pencil of parallel lines (see Fig. 3) is called

a horocycle, or limit circle; as the name implies, it is the curve toward which the circular trajectory of a pencil of intersecting lines tends as the intersection point tends to infinity. The orthogonal trajectory of a pencil of diverging lines (see Fig. 4) is called an equidistant, or hypercycle; it is the locus of points of the Lobachevskian plane that are equally distant from the common perpendicular of the diverging lines in the pencil, and this perpendicular is called the base of the equidistant.

On the basis of these orthogonal trajectories, Lobachevsky also constructed a space geometry. By rotating the circle, the horocycle, and the equidistant about one of the orthogonal lines, he obtained, respectively, a sphere, a horosphere (or limit sphere), and an equidistant surface, the last being the locus of the points of space equally distant from a plane. Although geometry on the sphere does not differ from that of Euclidean space, the geometry on each of the two sheets of the equidistant surface is that of the Lobachevskian plane, the geometry on the horosphere that of the Euclidean plane.

Working from the geometry (and, hence, trigonometry) of the Euclidean plane on horospheres, Lobachevsky derived trigonometric formulas for triangles in the Lobachevskian plane. In modern terms these formulas are:

where $q$ is a certain constant of the Lobachevskian plane.

Comparing these formulas with those of spherical trigonometry on a sphere of radius $r$, that is,

Lobachevsky discovered that the formulas of trigonometry in the space he defined can be derived from formulas of spherical trigonometry if the sides $a$, $b$, $c$ of triangles are regarded as purely imaginary numbers or, put another way, if the radius $r$ of the sphere is considered as purely imaginary. Indeed, (2) is transformed into (1) if it is supposed that $r = qi$ and the correlations cos $ix$ , cosh $x$ and sin $ix$ = sinh $x$ are employed. In this Lobachevsky saw evidence of the noncontradictory nature of the geometry he had discovered. This idea can be made quite rigorous by introducing imaginary points of Euclidean space, producing the so-called complex Euclidean space; defining in that space the sphere of purely imaginary radius; and considering a set of points on that sphere that have real rectangular coordinates $x$ and $y$ and purely imaginary coordinate $z$, (Such a set of points in the complex Euclidean space was first considered in 1908 by H. Minkowski, in connection with Einstein's special principle of relativity; it is now called pseudo-Euclidean space.) The sphere of the imaginary radius in this space thus appears to be a hyperboloid of two sheets, each of which is a model of the Lobachevskian plane, while construction of this model preves it to be noncontradictory in nature.

Lobachevskian geometry is further analogous to spherical geometry. The area of triangle *S* in Lobachevsky's plane is expressed through its angles (in radian measure) by the formula

whereas the area of spherical triangle *S* is expressed its angles (by the same measure) as

Formula (3) is transformed into formula (4) if $r = qi$

Lines of the Lobachevskian plane are represented on the sphere of imaginary radius of the pseudo-Euclidean space by sections with planes through the center of the sphere. Intersecting lines can then be represented by sections with planes intersecting on a line of purely imaginary length; parallel lines by sections with planes intersecting on lines of zero length (isotropic lines); and diverging lines by sections with planes intersecting on lines of real length. Circles, horocycles, and equidistants of the Lobachevskian plane are represented by sections of the sphere of imaginary radius with planes that do not pass through its center.

If the sides $a$, $b$, $c$ of the triangle are very small or the numbers $q$ and $r$ very large, it is necessary to consider only the first members in the series

If it is assumed for formulas (1) and (2), above, that $\sinh x = x$ and $\cosh x = \cos x = 1$, $\sin x = x$ or and these formulas become

in the Euclidean plane. Euclidean geometry may then be considered as a limiting case of both spherical and Lobachevskian geometry as $r$ and $q$ tend to infinity, or as a particular case for $r = q = \infty$. In analogy to the number , which represents the curvature of the sphere, the number — is called the curvature of the Lobachevskian plane; the constant $q$ is called the radius of curvature of this plane.

Lobachevsky recognized the universal character of his new geometry in naming it "pangeometry." He nevertheless thought it necessary to establish experimentally which geometry—his or Euclid's— actually occurs in the real world. To this end he made a series of calculations of the sums of the angles of triangles of which the vertices are two diametrically opposed points on the orbit of the earth and one of the fixed stars Sirius, Rigel, or 28 Eridani. Having established that the deviation of these sums from π is no greater than might be due to errors in observation, he concluded that the geometry of the real world might be considered as Euclidean, whence he also found "a rigorous proof of the theorem of parallels" as set out in his work of 1826. In explaining his calculations (in "O nachalakh geometrii" ["On the Principles of Geometry"]), Lobachevsky noted that it is possible to find experimentally the deviation from π of the sum of the angles of cosmic triangles of great size; in a later work (*Novye nachala geometrii s polnoi teoriei parallelnykh* ["New Principles of Geometry With a Complete Theory of Parallels"]) he moved to the opposite scale and suggested that his geometry might find application in the "intimate sphere of molecular attractions."

In his earlier papers Lobachevsky had defined imaginary geometry on an a priori basis, beginning with the supposition that Euclid's fifth postulate does not hold true and explaining the principle tenets of his new geometry without defining it (although he did describe the results of his experiment to prove his theorem of parallels). In "Voobrazhaemaya geometriya" ("Imaginary Geometry"), however, he built up the new geometry analytically, proceeding from its inherent trigonometrical formulas and considering the derivation of these formulas from spherical trigonometry to guarantee its internal consistency. In the sequel to that paper, "Primenenie voobrazhaemoi geometrii k nekotorym integralam" ("Application of Imaginary Geometry to Certain Integrals"), he applied geometrical considerations in Lobachevskian space to the calculation of known integrals (in order to make sure that their application led to valid results), then to new, previously uncalculated integrals.

In *Novye nachala geometrii s polnoi teoriei parallelnykh* ("New Principles of Geometry With a Complete Theory of Parallels"), Lobachevsky, after criticizing various demonstrations of the fifth postulate, went on to develop the idea of a geometry independent of the fifth postulate, an idea presented in his earliest *Geometriya* (of which a considerable portion was encompassed in the later work). The last two chapters of the book—on the solution of triangles, on given measurements, and on probable errors in calculation—were connected with his attempts to establish experimentally what sort of geometry obtains in the real world. Lobachevsky's last two books, *Geometrische Untersuchungen* and *Pangéométrie*, represent summaries of his previous geometrical work. The former dealt with the elements of the new geometry, while the latter applied differential and integral calculus to it.

At the same time as Lobachevsky, other geometers were making similar discoveries. Gauss had arrived at an idea of non-Euclidean geometry in the last years of the eighteenth century and had for several decades continued to study the problems that such an idea presented. He never published his results, however, and these became known only after his death and the publication of his correspondence. Janos Bolyai, the son of Gauss's university comrade Farkas Bolyai, hit upon Lobachevskian geometry at a slightly later date than Lobachevsky; he explained his discovery in an appendix to his father's work that was published in 1832. (Since Gauss did not publish his work on the subject, and since Bolyai published only at a later date, Lobachevsky clearly holds priority.)

It may be observed that Lobachevsky's works in other areas of mathematics were either directly relevant to his geometry (as his calculations on definite integrals and probable errors of observation) or results of his studies of foundations of mathematics (as his works on the theory of finites and the theory of trigonometric series). His work on these problems again for the most part paralleled that of other European mathematicians. It is, for example, worth noting that in his algebra Lobachevsky suggested a method of separating roots of equations by their repeated squaring, a method coincident with that suggested by Dandelin in 1826 and by Gräffe in 1837. His paper on the convergence of trigonometric series, too, suggested a general definition of function like that proposed by Dirichlet in 1837. (Lobachevsky also gave a rigorous definition of continuity and differentiability, and pointed out the difference between these notions.)

**Recognition of Lobachevskian Geometry** . Lobachevsky's work was little heralded during his lifetime. M. V. Ostrogradsky, the most famous mathematician of the St. Petersburg Academy, for one, did not understand Lobachevsky's achievement, and published an uncomplimentary review of "O nachalakh geometrii" ("On the Principles of Geometry"); the magazine *Syn otechestva* soon followed his lead, and in 1834 issued a pamphlet ridiculing Lobachevsky's paper. Although Gauss, who had received a copy of the *Geometrische Untersuchungen* from Lobachevsky, spoke to him flatteringly of the book, studied Russian especially to read his works in their original language, and supported his election to the Göttingen Gesellschaft der Wissenschaften, he never publicly commented on Lobachevsky's discovery. His views on the new geometry became clear only after the publication, in 1860–1865, of his correspondence with H. C. Schumacher. Following this, in 1865, the English algebraist Cayley (who had himself paved the way for the theory of projective metrics in 1859 with his "Sixth Memoir Upon Quantics") brought out his "A Note Upon Lobachevsky's Imaginary Geometry" from which it is evident that he also failed to understand Lobachevsky's work.

The cause of Lobachevskian geometry was, however, furthered by Hoüel, one of its earliest proponents, who in 1866 brought out a French translation of *Geometrische Untersuchungen*, with appended extracts from the Gauss-Schumacher correspondence. The following year he also published Bolyai's appendix on non-Euclidean geometry, which was translated into Italian by Battaglini in 1867. Hoüel's own *Notices sur la vie et les travaux de N. I Lobachevsky* appeared in 1870. In the meantime, Lobachevsky's *Geometrische Untersuchungen* had been translated into Russian by A. V. Letnikov; it was published in 1868 in the newly founded Moscow magazine *Matematichesky sbornik*, together with Letnikov's article "O teorii parallelnykh linii N. I. Lobachevskogo" ("On the Theory of Parallel Lines by N. I. Lobachevsky").

These translations and reviews were soon augmented by extensions of Lobachevskian geometry itself. In 1858 Beltrami published in the *Giornale di matematiche* his "Saggio di interpretazione della geometria noneuclidea," in which he established that the intrinsic geometry of the pseudosphere and other surfaces of constant negative curvature coincides with the geometry of part of the Lobachevskian plane and that an interpretation of the whole Lobachevskian plane can be constructed in the interior of a circle in the Euclidean plane. This interpretation can be derived by projecting a hemisphere of imaginary radius in pseudo-Euclidean space from its center onto a Euclidean plane tangent to this hemisphere. The article was translated into French by Hoüel in 1869 for publication in the *Annales scientifiques de l'École Normale Supérieure*.

In 1870 Weierstrass led a seminar on Lobachevsky's geometry at the University of Berlin; the young Felix Klein was one of the participants. Weierstrass' own contribution to the subject, the so-called Weierstrass coordinates, are essentially rectangular coordinates of a point on a hemisphere of imaginary radius in pseudo-Euclidean space. Klein compared Lobachevskian geometry with Cayley's projective metrics to establish that Lobachevsky's geometry is in fact one of Cayley's geometries, which Cayley himself (although he was acquainted with Lobachevskian geometry) had failed to notice. The Lobachevskian plane can be regarded as the interior domain of a conic section on a projective plane; when this conic section is a circle, representation of the Lobachevskian plane on the projective plane coincides with Beltrami's interpretation. It thus follows that motions of the Lobachevskian plane are represented in Beltrami's interpretation by projective transformations of the plane mapping a circle into itself. Lines of the Lobachevskian plane are represented in Beltrami's interpretation by chords of the circle, while parallel lines are represented by chords intersecting on the circumference, and diverging lines by non-intersecting chords. There are analogous interpretations for three-dimensional and multidimensional Lobachevskian spaces, in which Beltrami's circle or Klein's conic section is replaced by a sphere or oval quadric, e.g., ellipsoid.

Poincaré made two important contributions to Lobachevskian geometry at somewhat later dates. In 1882 he suggested an interpretation of the Lobachevskian plane in terms of a Euclidean semiplane; in this intrepretation, motions of the Lobachevskian plane are represented by inversive transformations, mapping the border of the semiplane into itself, while Lobachevskian lines are mapped by perpendiculars to this border or by semicircles with centers on it. An important property of this interpretation is that angles in the Lobachevskian plane are mapped conformally. Taking as given that inversive transformations on the plane are represented as fractional linear functions of a complex variable, Poincaré used such an interpretation for the theory of automorphic functions.

Poincaré's interpretation is often described in other terms, as when a semiplane is mapped by inversive transformation into the interior of a circle. An interpretation in terms of the circle can also be presented by projecting the hemisphere of imaginary radius in pseudo-Euclidean space from one of its points onto the Euclidean plane perpendicular to the radius through this point. This projection is analogous to stereographic projection, through which the preservation of angles in Poincaré's interpretation is explained. In 1887 Poincaré suggested a second interpretation of the Lobachevskian plane, this one in terms of a two-sheeted hyperboloid coinciding with that on the hemisphere of imaginary radius as already defined.

A further important development of Lobachevsky's geometry came from Riemann, whose address of 1854, *Uber die Hrpothesen, welche der Geometrie zu Grundliegen,* was published in 1866, Developing Gauss's idea of intrinsic geometry of a surface, Riemann presented a notion of multidimensional curved space (now called Riemannian space). Riemannian space of constant positive curvature (often called Riemannian elliptic space) is represented by the geometry on a sphere in fourth-dimensional or multidimensional Euclidean space or in the space with a projective metric in which an imaginary quadric plays the part of a conic section (or Klein's quadric). It subsequently, then, became clear that Lobachevskian space (often called hyperbolic space) is the Riemannian space of constant negative curvature. It was with this in mind that Klein, in his memoir of 1871, "Über die sogenannte nicht-euklidische Geometrie," chose to consider all three geometries—Euclidean, elliptic, and hyperbolic—from a single standpoint, whereby groups of motions of all three are subgroups of the group of projective transformations of projective space.

Here the extension of the concept of space, which was introduced by Lobachevsky and Riemann, merges with the concept of group, which was introduced by Galois, a synthesis that Klein developed further in 1872 in *Vergleichenden Betrachtungen über neuere geo-metrische Forschungen* (also known as the Erlanger Programm) in which he arrived at a general view of various geometries as theories of invariants of various continuous groups of transformations. A new stage in the development of mathematics thus began, one in which the mathematics of antiquity and the Middle Ages (that is, the mathematics of constant magnitudes) and the mathematics of early modern times (that is, the mathematics of variable magnitudes) were replaced by modern mathematics—the mathematics of many geometries, of many algebras, and of many mathematical systems having no classical analogues. A number of these systems found applications in modern physics-pseudo-Euclidean geometry, one of the projective metrics, figures in the special theory of relativity; Riemannian geometry appears in the general theory of relativity; and group theory is significant to quantum physics.

The presence of specifically Lobachevskian geometry is felt in modern physics in the isomorphism of the group of motions of Lobachevskian space and the Lorentz group. This isomorphism opens the possibility of applying Lobachevskian geometry to the solution of a number of problems of relativist quantum physics. Within the framework of the general theory of relativity, the problem of the geometry of the real world, to which Lobachevsky had devoted so much attention, was solved; the geometry of the real world is that of variable curvature, which is on the average much closer to Lobachevsky's than to Euclid's.

# BIBLIOGRAPHY

I. Original Works. Lobachevsky's writings have been brought together in *Polnoe sobranie sochinenii* ("Complete Works"), **5** vols. (Moscow-Leningrad, 1946–1951): vols. III, geometrical works; vol **IV** , algebraical works; vol, V, works on analysis, the theory of probability, mechanics, and astronomy. His geometrical works were also collected as *Polnoe sobranie sochinenii po geometrii* ("Complete Geometrical Works"), 2 vols. (Kazan, 1883–1886): vol. I, works in Russian; vol. II, works in French and German.

Translations include J. Hoüel, *Études géométriques sur la théorie des parallèls* (Paris, 1866; 2nd ed., 1900); G. B. Halsted, *Geometrical Researches on the Theory of Parallels*, Neomonic ser, no, **4** (Austin, Tex., 1892; repr. Chicago London, 1942); and *New Principles of Geometry With Complete Theory of Parallels,* Neomonic ser. no. **5** (Austin, Tex., 1897); F. Engel, *Zweigeometnsche Abhand-lungen* (Leipzig, 1898); F. Mallieux, *Nouveaux principes de la géométrie avec une théorie complète des parallèles* (Brussels, 1901); and H. Liebmann, *Imaginären Geometrie Anwendungen der imaginäre Geometrie auf einige Integrale* (Leipzig, 1904).

Secondary Literature. Among works on Lobachevsky are the following: A. A. Andronov, "Gde i kogda rodilsya N. I. Lobachevsky?"("Where and When Was Lobachevsky born?"), in *Istoriko-matematicheskie issledovaniya, **9** (1956); R. Bonola, La geometria non-euclidea; Esposizione storico-critico del suo sviluppo*(Bologna, 1906);translated into German as Die *nicht-euklidische*(Bologna, 1906); translated into German as Die *nicht-euktidische Geometric. Historisch-kritische Darstellung ihrer Entwicklung*, 3rd ed. (Leipzig, 1921); translated into Russian as *Neevklidoua geometriya. Kritiko-istorkheskoe issledovanie e razuitiya* (St. Petersburg, 1910); translated into English as *Non-Euclidean Geometry. A Critical and Historical Study of Its Developments* (New York, 1955); J. L. Coolidge, *The Elements of Non-Euclidean Geometry*(Oxford, 1909); H. S. M, Coxeter, *Non-Euclidean Geometry* 3rd ed. (Toronto, 1957); V. M. Gerasimova, *Ukazatel literatury pa geometrii Lobachevskogo I razvitiyu e idei* ("A Guide to Studies on Lobachevskian Geometry and the Development of Its Ideas"; Moscow, 1952); B. V. Gnedenko, "O rabotakh N. I. Lobachevskogo po teorii veroyatnostci" ("On Lobachevsky's Works on the Calculus of Probabilities"), in *Istoriko-matematkheskk issledovaniya,***2** (1949); N. I. Idelson, "Lobachevskii-astronom" ("Lobachevsky as an Astronomer"), *ibid*.

Also of value are V. F. Kagan, *Lobachevsky* 2nd ed. (Moscow-Leningrad, 1948); and *Osnovaniya geometrii* ("Foundations of Geometry"), **2** vols. (I, Moscow-Leningrad, 1949; U, Moscow, 1956); E. K. Khilkevich, "Iz istoriirasprostraneniya i razvitiya idei N. I. Lobachevskogo v 60-70-kh godakh XIX stoletiya" ("The History of the Spread and Development of N. I. Lobachevsky's Ideas in the 60th to 70th Years of the Nineteenth Century"), in *Istoriko-nratematicheskie issledoraniya,***2** (1949); F. Klein, *Vorlesungen über nichteuktidisehe Geometrie* 3rd ed. (Berlin, 1928), also translated into Russian as Neevklidova geometriya (Moscow-Leningrad, 1936); B, L. Laptev, "Teoriya paralletnykh pryamykh v rannikh rabotakh

Lobachevskogo” (Theory of Parallel Lines in Lobachev-sky’s Early Works“), in *lstoriko-matematichiskie issledo-vanuta***4** (1951); H. Liebmann, Nicht-euclidische Geometries 3rd ed. (Berlin, 1923); G. L. Luntz, “O rabotakh N. I. Lobachevskogo po matematicheskomu analizu” (“On N. I. Lobacbevsky’s Work in Mathematical Analysis”), in *Istoriko-matematicheskie issledovamya,***2** (1949); M, B. Modzalevsky, *Materialy dlya biografii N. I. Lobachemkogo* (“Materials for N. I. Lobachevsky”s Biography“; Moscow-Leningrad, 1948); V. V. Morozov, “Ob algebraicheskikh rukopisyakh N. I. Lobachevskogo” (“On Algebraic Manuscripts of N. I. Lobachevsky”), in *lstoriko-matematicheskie issledovamya,***4** (1951); and A, P. Norden, ed., *Ob osnovam’yakh geometrii*. *Sbornik klassicheskikh robot po geometrii Lobachevskogo i razvitiyu e idei* (“On Foundations of Geometry. Collection of Classic Works on Lobachevskian Geometry and the Development of Its Ideas”; Moscow, 1956).

See also B. A. Rosenfeld, *Neevklidovy geometrii* (Moscow, 1955); “Interpretacii geometrii Lobachevskogo” (“Interpretations of Lobachevskian Geometry”), in *istoriko-maiematicheskie issledovaniya,***9** (1956); and *Neevklidovy prostranstca* (“Non-Euclidean Spaces”; Moscow, 1968); D. M. Y. Sommerville, *Bibliography of Non-Euclidean Geometry* (London, 1911); and *The Elements of Non-Euclidean Geometry* (London, 1914); S. A. Yanovskaya, “O mirovozzrenii N. I. Lobachevskogo“(“On N. I. Lobachevsky’s Outlook”), in *Istoriko-matematicheskie isslcdovaniya,***4** (1951); and A. P. Youschkevitch and 1. G. Bash-makova, “Algcbra ill vychislenie konechnykh’ N. I. Lobachevskogo“(“‘Algebra or Calculus of Finites’ by N. I. Lobachevsky”), in *Istoriko-inatenraticheskie issledovaniya,***2** (1949).

B. A. Rosenfeld