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(*b*. Yaroslavl, Russia, 6 June 1857; *d*. Odessa, U.S.S.R., 3 November 1918)

mathematics, mechanics.

Lyapunov was a son of the astronomer Mikhail Vasilievich Lyapunov, who worked at Kazan University from 1840 until 1855 and was director of the Demidovskt Lyceum in Yaroslavl from 1856 until 1863; his mother was the former Sofia Aleksandrovna Shilipova. Lyapunov's brother Sergei was a composer; another brother, Boris, was a specialist in Slavic philology and a member of the Soviet Academy of Sciences.

Lyapunov received his elementary education at home and later with an uncle, R. M. Sechenov, brother of the physiologist I. M. Sechenov, With R. M. Sechenov's daughter Natalia Rafailovna (whom he married in 1886), he prepared for the Gymnasium. In 1870 Lyapunov's mother moved to.Nizhny Novgorod (now Gorky) with her children. After graduating from the Gymnasium in Nizhny Novgorod 876, Lyapunov enrolled in the Physics and ematics Faculty of St. Petersburg University, where P. L. Chebyshev greatly influenced him.

Upon graduating from the university in 1880, Lyapunov remained, upon the recommendation of D. K. Bobylev, in the department of mechanics of Petersburg University in order to prepare for a essorial career. In 1881 he published his first scientific papers, which dealt with hydrostatics: "O ravnovesii tyazhelykh tel v tyazhelykh zhid-kostyakh, soderzhashchikhsya v sosude opredelennoy formy" ("On the Equilibrium of Heavy Bodies in Heavy Liquids Contained in a Vessel of a Certain Shape") and "O potentsiale gidrostaticheskikh davleny" ("On the Potential of Hydrostatic Pressures").

In 1882 Chebyshev posed the following question: "It is known that at a certain angular velocity ellipsoidal forms cease to be the forms of equilibrium of a rotating liquid. In this case, do they not shift into some new forms of equilibrium which differ little from ellipsoids for small increases in the angular velocity?" Lyapunov did not solve the question at the time, but the problem led him to another that became the subject of his master's dissertation, *Ob ustoychivosti ellipsoidalnykh* form *ravnovesia vrashchayushcheysya zhidkosti* ("On the Stability of Ellipsoidal Forms of Equilibrium of a Rotating Liquid"; 1884), which he defended at St. Petersburg University in 1885.

In the autumn of that year Lyapunov began to teach mechanics at Kharkov University as a *Privatdozent*. For some time he was completely occupied by the preparation of lectures and by teaching, but in 1888 his papers on the stability of the motion of mechanical systems having a finite number of degrees of freedom began to be published. In 1892 he published the classic *Obshchaya zadacha ob ustoychivosti dvizhenia* ("The General Problem of the Stability of Motion"), and in the same year he defended it as his doctoral dissertation at Moscow University; N. E. Zhukovsky was one of the examiners. Lyapunov studied this field until 1902.

In 1893 Lyapunov became a professor at Kharkov. In addition to mechanics he taught mathematics courses. He was also active in the Kharkov Mathematical Society. From 1891 until 1898 he was its vice-president, and from 1899 until 1902 its president and the editor of its *Soobshchenia* ("Reports").

At Kharkov, Lyapunov conducted investigations in mathematical physics (1886-1902) and the theory of probability (1900-1901) and obtained outstanding results in both. At the beginning of 1901 he was elected an associate member of the St. Petersburg Academy of Sciences, and at the end of that year he became an academician in applied mathematics, a chair which had remained vacant for seven years, since the death of Chebyshev.

In St Petersburg, Lyapunov devoted himself completely to scientific work. He returned to the problem that Chebyshev had placed before him and, in an extensive series of papers which continued until his death, developed the theory of figures of equilibrium of rotating heavy liquids and of the stability of these figures.

In 1908 Lyapunov attended the Fourth International Congress of Mathematicians in Rome. He was involved in the publication of the complete collected works of Euler (*L. Euleri Opera omnia*) and was an editor of volumes XVIII and XIX of the first (mathematical) series, which appeared in 1920 and 1932. Lyapunov's scientific work received wide recognition. He was elected an honorary member of the universities of St. Petersburg, Kharkov, and Kazan, a foreign member of the Accademia del Lincei (1909) and of the Paris Academy of Sciences (1916), and a member of many other scientific societies.

In the summer of 1917 Lyapunov went to Odessa with his wife, who suffered from a serious form of tuberculosis. He began to lecture at the university: but in the spring of the following year his wife's condition rapidly deteriorated and she died on 31 October 1918. On that day Lyapunov shot himself and died three days later, without regaining consciousness. In accordance with a wish stated in a note that he had left, he was buried with his wife.

Lyapunov and A. A. Markov, who had been schoolmates at St. Petersburg University and, later, colleagues at the Academy of Sciences, were Chebyshev's most prominent students and representatives of the St. Petersburg mathematics school. Both were outstanding mathematicians and both exerted a powerful influence on the subsequent development of science. Lyapunov concentrated on three fields: the stability of equilibrium and motion of a mechanical system having a finite number of degrees of freedom; the stability of figures of equilibrium of a uniformly rotating liquid; the stability of figures of equilibrium of a rotating liquid.

Lyapunov's papers on the stability of systems having a finite number of degrees of freedom, among which his doctoral dissertation occupies a central position, belong equally to mathematics and to mechanics. They contain thorough analyses of a great many problems in the theory of ordinary differential equations.

The mathematical formulation of a problem closely related historically to investigations in celestial mechanics of the eighteenth and nineteenth centuries (the problem of the stability of the solar system) is the following: Given a system of n ordinary, first-order differential equations for n functions $x_k(t)$ of the independent variable t (time). It is assumed that the equations are solved with respect to the derivative dx_k/dt on the left side and that the right sides are power series with respect to $x_1, x_2, \ldots x_n$ without free members, such that the equations have the obvious null solution $x_1 = x_2 = \ldots = x_n = 0$. The cofficents of the series can depend on t. The solution the system is completely defined by assignment of values of the unknown functions x_k for some $t = t_0$. The stability of the null solution, according to Lyapunov, is stability in the infinite time interval $t \geq t_0$ with respect to the initial data. In other words, stability consists in the fact that, for $t \geq t_0$, the solution of the system $x_k(t)$ will be sufficiently small in absolute value for sufficiently small absolute values of the initial data $x_k(t_0)$. The mechanical systems described above, often called dynamic systems, play a fundamental role in dynamics.

When the corresponding system of differential equations is integrated and its solution is found in simple form, the investigation of stability presents no difficulty. As a rule, however, this integration is

impracticable. Therefore, mathematicians generally use the approximation method, which consists in replacing the right sides of the equations by the system of linear members of their expansions into power series. In this manner the task devolves into a study of the stability of a linear system of differential equations; this substantially simplifies the problem, especially when the coefficients of the linear system are constants. It remained unclear, however, whether the replacement of the given system by a linear one was valid and, if valid, under what conditions. Use of a second or somewhat higher-order approximation (that is, the retention of second- or somewhat higherorder members on the right side) enables one to improve the accuracy of knowledge about functions $x_k(t)$ in a finite time interval but gives no new basis for any conclusions about stability in the infinite interval $t \geq t_0$. As Lyapunov noted, the only attempt at a rigorous solution of the question had been made by Poincaré a short time earlier in the special cases of second-order and, in part, third-order systems.

With the aid of new methods that he had created, Lyapunov himself solved, for extremely general assumptions, the question of when the first approximation solves the problem of stability. He thoroughly examined the cases, especially important in practice, when the coefficients of the series on the right side of the equations are constants (the "established motion") or are periodic functions of time t, having one and the same period. For example, if, given constant coefficients, the real parts of all the roots of the system's characteristic equation (an nth degree, "secular" algebraic equation) are negative, the solution of the initial system is stable; if, however there is among the roots one having a positive real part, then the solution is unstable. The first approximation, however, does not permit one to solve the problem of stability if the characteristic equation, while not having roots with positive real numbers, has roots the real parts of which are equal to zero. Here, the cases when the characteristic equation has one root equal to zero, or when it has two purely imaginary conjugate roots, are of special interest; these cases were exhaustively investigated by Lyapunov. In the case of periodic coefficients Lyapunov examined the possibilities arising in two especially interesting instances: when one of the roots of the characteristic equation is equal to unity and when two imaginary conjugate roots have a modulus equal to unity.

Among many other results is the proof of the theorem of the instability of motion if the force function of the forces acting on the system is not a maximum. Several of Lyapunov's articles dealing with a detailed analysis of the solution of homogeneous linear second-order equations having periodic coefficients (1896-1902) have the same orientation.

The ideas of this cycle of papers, especially Lyapunov's doctoral dissertation, are related to Poincaré's investigation. Specific results obtained by both these scholars coincide, but they do not deal with the basic content or the basic methods of their works. In particular, Poincaré made wide use of geometrical and topological concepts, while Lyapunov used purely analytical methods. Both Poincaré's papers and Lyapunov's works are fundamental to the qualitative theory of ordinary differential equations. At first Lyapunov's theory of the stability of mechanical systems did not receive the wide response given to Poincaré's more general ideas. But, from the early 1930's the number of papers directly related to Lyapunov's investigations increased very rapidly, especially with the growth of the significance of problems concerning the stability of motion in modern physics and engineering, primarily because of the study of fluctuations in various mechanical and physical systems. Problems of stability arise in the determination of the work regimen of various machines, in the construction of airplanes, electrical engineering, and ballistics.

Lyapunov studied figures of equilibrium of a uniformly rotating liquid over a period of thirty-six years. In the lecture "O forme nebesnykh tel" ("On the Shape of Celestial Bodies"; 1918) he said:

According to a well-known hypothesis, each such body was initially in a liquid state; it took its present form before solidification, having previously received an unchanging form as the result of internal friction. Assuming this, the shape of a celestial body must be one of those which can be assumed by a rotating liquid mass, the particles of which mutually attract one another according to Newton's law, or, at least, must differ little from such a figure of equilibrium of a rotating liquid [*Izbrannye trudy,* p. 303],

Lyapunov mentioned the mathematical difficulty of studying equilibrium figures, a study which entails the solution of nonlinear integral equations.

It was Newton, his eighteenth-century successors Maclaurin and d'Alembert, and others who established that ellipsoids of rotation could be figures of equilibrium of homogeneous rotating liquids. Later, Jacobi demonstrated that certain triaxial ellipsoids could also be such figures. Other scholars also studied this problem. When, in 1882, Chebyshev placed before Lyapunov the question concerning the possibility of the existence of other equilibrium figures that are close to ellipsoidal, Lyapunov could solve the problem only in the first approximation. Believing it impossible to judge the existence of new figures according to the first approximation, he put off definitive solution to the question. In his master's thesis of 1884 he guardedly mentioned that certain algebraic surfaces which were close to earlier-known ellipsoids of equilibrium, satisfy the conditions of equilibrium in the first approximation. On the other hand, in this thesis he examined the problem of the stability of the Mac-laurin and Jacobi ellipsoids, injecting clarity and rigor into the statement of the problem, defining for the first time the concept of stability for a continuous medium.

In a series of papers written between 1903 and 1918, Lyapunov moved deeply into the investigation of Chebyshev's problem and related questions. In *Recherches dam la théorie de fa figure des corps célestes* and *Sur l' équation de Clairaut…* he proved the existence of nearly spherical figures of equilibrium for a sufficiently slowly rotating nonhornogeneous liquid, and investigated the solutions to the integral-difierential equations arising from this, which contain the unknown function both under the integral sign and under the sign of the derivatives; the first of these equations is "Clairaufs equation." In *Sur an probléeme de Tchéhycheff* and *Sur les figures d'éiquiiihre pen différentes des ellipsoids…* it was established that among ellipsoids of equilibrium there are "ellipsoids of bifurcation'1 and that, in addition to ellipsoidal figures close to these, there also exist close, nonellipsoidal figures of equilibrium. A number of methods are entailed in the consistent determination the equations of these figures. Finally, in the thumously published *Sur certaines séries de figures d'équilibre d'un liquide hétérogéne en rotation,* it is proved that each Maclaurin and Jacobi ellipsoid differing from the ellipsoids of bifurcation engenders new figures of equilibrium that are close to the Maclaurin and Jacobi ellipsoids in form; their density is not considered constant but, rather, as weakly varying.

In *Problém de minimum dans une question stabilité des figures d'équilibre d'une masse fluide en rotation,* Lyapunov further developed and made more precise the theory of stability stated in his master's thesis and investigated the stability of the new, nearly ellipsoidal, figures of equilibrium that he had previously found.

In solving all these problems and the corresponding nonlinear integral and integral-differential equations, Lyapunov had to overcome great mathematical difficulties. To this end he devised delicate methods of approximation, the convergence of which he proved with the rigor of contemporary mathematics; generalized the concept of the integral (in the direction of the Stieltjes-Riemann integral); and proved a number of new theorems on spherical functions.

Lyapunov's works again approach those of Poincaré. At a certain angular velocity, figures of equilibrium that Poincaré called "pear-shaped" branch oif from Jacobi ellipsoids. The astronomer G. H. Darwin encountered the problem of the stability of pear-shaped forms in his hypothesis concerning the origin of double stars arising from the division of a rotating liquid mass into two bodies. Poincaré, who examined the question within tlie limits of the second approximation, stated the hypothesis that these forms were stable; and Darwin, using Poineare's general theory seemingly confirmed this opinion, which was indispensable to his cosmogonical hypothesis, by calculations. Lyapunov's calculations, which were based on precise formulas and evaluations, led him to the opposite conclusion-that the pear—shaped figures were unstable. In 1911 Poincaré stated that he was not sure of the correctness of his prior opinion, but that to solve the question would require one to begin very complex computations again. In 1912 Lyapunov published the necessary calculations in the third part of *Sur certaines séries de figures d'équilihre …*, but no one sought to verify them. In 1917 Sir James Jeans confirmed Lyapunov's results, having discovered the defect in Darwin's computations.

During the period 1886-1902, Lyapunov devoted veral works to mathematical physics; *Sur certaines estions qui se rattachent au probléme de Dirichlet* (1898) is fundamental among these. Here, for the first time, a number of the basic properties of the potentials of simple and double layers were studied with utter rigor and the necessary and sufficient conditions, under which the function that solves Dirichlet\s problem within a given range has normal derivatives over the limiting range of the surface, were indicated. These investigations created the foundation of a number of classic methods for solving boundary-value problems. In addition, Lyapunov's works on mathematical physics brought that area of analysis to the attention of a number of Kharkov mathematicians, especially V. A. Steklov, Lyapunov's student.

Finally, in two works that arose from a course in the theory of probability taught by Lyapunov, he substantially generalized Laplace's limit theorem in application to sums of random independent values. Chebyshev gave the first such generalization of this theorem in 1887, indicating the possibility of its proof, in the form given by him, by the method of moments; and Markov carried out the full proof on this basis in 1898. In "Sur une proposition de la théorie des prohabilités" (1900) and "Nouvelle forme du théoréme sur la limite de probabiltié" (1901), Lyapunov proved the central limit theorem by the method of characteristic functions, which has subsequently assumed a fundamental place in the theory of probability; moreover, he did so under much broader conditions than Markov had used. Some lime later, however, Markov proved the central limit theorem under Lyapunov's conditions by using the method of moments. These works by Lyapunov also served as the starting point for many later investigations.

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