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(*b*. Ryazan, Russia, 14 June 1856; *d*. Petrograd [now Leningard], U.S.S.R., 20 May 1922)

mathematics.

Markov's father, Andrei Grigorievich Markov, a member of the gentry, served in St. Petersburg in the Forestry Department and managed a private estate. His mother, Nadezhda Petrovna, was the daughter of a state employee. Markov was in poor health and used crutches until he was ten years old. He early manifested a talent for mathematics in high school but was not diligent in other courses. In 1874 Markov entered the mathematics department of St. Petersburg University and enrolled in a seminar for superior students, led by A. N. Korkin and E. I. Zolotarev. He had met them in his high school days after presenting a paper on integration of linear differential equations (which contained results already known). He also attended lectures by the head of the St. Petersburg mathematical school, P. L. Chebyshev, and afterward became a consistent follower of his ideas.

In 1878 Markov graduated from the university with a gold medal for his thesis, "Ob integrirovanii differentsialnykh uravenii pri pomoshchi nepreryvnykh drobei" ("On the Integration of Differential Equations by Means of Continued Fractions") and remained at the university to prepare for a professorship. In 1880 he defended his master's thesis, "O binarnykh kvadratichnykh formakh polozhitelnogo opredelitelia" ("On the Binary Quadratic Forms With positive Determinant" *Izbrannye trudy*, pp. 9–83), and began teaching in the university as a docent. In 1884 he defended his doctoral dissertation, devoted to continued fractions and the problem of moments. In 1883 he married Maria Ivoanovna Valvatyeva, the daughter of the proprietress of the estate managed by his father. They had been childhood friends, and Markov had helped her to learn mathematics. Later he proposed to her, but her mother agreed to the marriage only after Markov strengthened his social position.

For twenty-five years Markov combined research with intensive teaching at St. Petersburg University. In 1886 he was named extraordinary professor and in 1893, full professor. In this period he studied many questions: number theory, continued fractions, functions least deviating from zero, approximate quadrature formulas, integration in elementary functions, the problem of moments, probability theory, and differential equations. His lectures were distinguished by an irreproachable strictness of argument, and he developed in his students that mathematical cast of mind that takes nothing for granted. He included in his courses many recent results of investigations, while often omitting traditional questions. The lectures were difficult, and only serious students could understand them. He stated his opinions in a peremptory manner and was extremely exacting with his associates. During his lectures he did not brother about the order of equations on the blackboard, nor about his personal appearance. He was also a faculty adviser for a student mathematical circle. Nominated by Chebyshev, Markov was elected in 1886 an adjunct of the St. Petersburg Academy of Sciences; in 1890 he became an extraordinary academician and in 1896 an ordinary academician. In 1905, after twenty-five years of teaching, Markov retired to make room for younger mathematicians. He was named professor emeritus, but still taught the probability course at the university, by his right as an academician. At this time his scientific interests concentrated on probability theory and in particular on the chains later named for him.

A man of firm opinions, Markov participated in the liberal movement in Russia at the beginning of the twentieth century. In a series of caustic letters to academic and state authorities, he protested against the overruling, at the czar's order, of the election in 1902 of Maxim Gorky to the St. Petersburg Academy, he refused to receive decorations (1903), and he repudiated his membership in the electorate after the illegal dissolution of the Second State Duma by the government (1907). The authorities preferred not to respond to these declarations, considering them the extravagances of an academician. In 1913, when officials pompously celebrated the three-hundredth anniversary if the House of Romanov, Markov organized a celebration of the two-hundredth anniversary of the law of large numbers (in 1713 Jakob I Bernoulli's *Ars conjectandi* was posthumoulsy published).

In September 1917 Markov asked the Academy to send him to the interior of Russia, and he spent the famine winter in Zaraisk, a little country town. There he voluntarily taught mathematics in a secondary school without pay. Soon after his return to Petrograd, his health declined sharply and he had an eye operation. In 1921 he continued lecturing, scarcely able to stand. He died after several months of intense suffering.

Markov belonged to Cheybshev's scientific school and, more than others, was faithful to the creed and the principles of his master. He inherited from Chebyshev an interest in concrete problems; a simplicity of mathematical procedures; a need to solve problems effectively, whether simple or algorithmic; and a desire to obtain exact limits for asymptotic results. These views coexisted with an underestimation of the role of some new general concepts in contemporary mathematics, namely of the axiomatic method and of the theory of functions of complex variable. Characteristic of Markov was the adherence to a chosen method of investigation and maintenance of his own view of what is valuable in science. He once said, “Mathematics is that which Gauss, Chebyshev, Lyapunov, Steklov, and I study” (N. M. Guenter, “O pedagogicheskoi deyatelnosti A. A. Markov” p. 37).

The principal aim of most of Markov’s works in number theory and function theory was to evaluate the exact upper or lower bounds for various quantities (quadratic forms, integrals, derivatives). In probability theory it was at first to apply the bounds for integrals to the proof of the central limit theorem outlined by Chebyshev; later it was to discover new phenomena satisfying this theorem. Markov’s work in various branches of mathematics is also united by systematic use of Chebyshev’s favorite method of continued fractions, which became the principal instrument in Markov’s investigations.

Markov’s work in number theory was devoted mostly to the problem of arithmetical minima of indefinite quadratic forms studied previously in Russia by Korkin and Zolotarev (the topic goes back to Gauss and Hermite). These two authors had shown that if one excludes the form $f(x, y) = x^2 - xy - y^2$ (and the forms equivalent to it) for which min , then for the remaining binary forms $f(x, y) = ax^2 + 2bxy + cy^2$ with $d = b^2 - ac > 0$, one has min $|f| \leq$. By means of continued fractions Markov showed in his master’s thesis (*Izbrannye trudy*, pp. 9–83) that 4/5 and 1/2 are the first two terms of an infinite decreasing sequence $\{N_k\}$ converging to 4/9,such that (1) for every N_k there exists a finite number of nonequivalent binary forms whose minimum is equal to and (2) if the minimum of any indefinite binary form is more than , then it is equal to one of the values of . To the limiting value there correspond infinitely many nonequivalent forms. Following the traditions of the Petersburg mathematical school, Markov also computed the first twenty numbers of $\{N_k\}$ and the forms corresponding to them. In 1901–1909 he returned to the problem of extrema of indefinite quadratic forms. He found the first four extremal forms of three variables (one of them was known to Korkin) and two extremal forms of four variables, and published a long list of ternary forms with $d \leq 50$. Markov’s works on indefinite forms were continued both in the Soviet Union and in the West. Another problem of number theory was considered by Markov in his paper “Sur les nombres entiers dépendents d’une racine cubique d’un nombre entier ordinaire” (*Izbrannye trudy*, pp. 85–133). Following Zolotarev’s ideas, Markov here obtained the final result for decomposition into ideal prime factors in the field generated by and calculated the units of these fields for all $\boldsymbol{A} \leq 70$.

The next area of Markov’s work concerned the evaluation of limits of functions, integrals, and derivatives. The problem of moments was the most notable among these topics. From a work of J. Bienaymé presented to the Paris Academy of Sciences in 1833 (republished in Liouville’s *Journal de mathématiques pures et appliquées* in 1867), Chebyshev borrowed the problem of finding the upper and lower bounds of an integral

(1)

of a nonnegative function f with given values of its moments

(2)

and the idea of applying the solution of this problem of moments to prove limit laws in probability theory. In 1874 Chebyshev published, without proofs, inequalities providing upper and lower bounds for integral (1) for some special values of a and $x (A < a < x < B)$. These bounds were expressed through the convergents of the continued fraction into which the series $\sum m_k / Z^{k+1}$ formally decomposes. The proofs of Chebyshev’s inequalities appeared in 1884 in Markov’s memoir “Démonstration de certaines inégalités de M. Tchebycheff” (*Izbrannye trudy po teorii nepreryvnykh drovei* …, pp. 15–24). The same inequalities with the same proofs were published at almost the same time by the Dutch mathematician Stieltjes. Markov claimed priority, to which Stieltjes replied that he could not have known of Markov’s paper and that Chebyshev’s work had indeed escaped his attention. Later Markov and Stieltjes studied the problem of moments largely side by side and sometimes one would find new proofs of the other’s already published results. Both used continued fractions in their investigations and developed their theory further; but a difference in their methodological approaches manifested itself: Markov was mostly interested in the case of finite numbers of given moments and he studied the problem entirely within the limits of classical calculus; Stieltjes paid more attention to the problem of given infinite sequences of moments, and, seeking the most adequate formulation of the problem, introduced a generalization of the classical integral—the so-called Stieltjes integral.

In his doctoral dissertation Markov solved the question of the upper and lower bounds of integral (1) in the case when the first N moments are known. In subsequent papers he generalized the problem by allowing the appearance of an additional factor $\Omega(x)$ under integral (1); allowing, instead of power moments (2), moments relative to arbitrary functions $\lambda_k(x)$; and substituting the condition $c \leq f(x \leq C)$ for $f(x) \geq 0$. In other papers he investigated the distribution of the roots of the denominators of the convergents of the continued fraction mentioned above and the convergence of this fraction. The last question is closely related to the uniqueness of the solution of the Stieltjes problem of moments (finding a function, given its infinite sequence of power moments). In 1895, in his memoir “Deux démonstrations de la convergence de certains fraction continues” (*Izbrannye trudy po teorii nepreryvnykh drovei* …, pp. (106–119), Markov obtained the following sufficient condition for the convergence, and therefore for the uniqueness of the Stieltjes problem, of functions defined on $[0, \infty)$: Further results were obtained by O. Perron, H. Hamburger, F. Riesz, and T. G. Carleman.

Markov solved in 1889 another problem on extremal values which arose from the needs of chemistry in “Ob odnom voprose D. I. Mendeleeva” (“On a Question of D. I. Mendeleev” *Izbramye trudy po teorii neprryunykh drouei* …, pp. 51–75). Here Markov found the maximum possible value of the derivative $f'(z)$ of a polynomial $f(z)$ of degree $\leq n$ on an interval $[a, b]$, provided that $|f(z)| \leq L$ on $[a, b]$. (This maximum value is equal to $2n^2L|(b - a)$.) Markov’s result was generalized in 1892 by his younger brother Vladimir (who died five years afterward), and it was later extended for other cases by S. N. Bernstein and N. I. Akhiezer. Markov also worked on some other, practical extremal problems, namely the mapping of a part of a surface of revolution onto a plane with minimal deformations and the joining of two straight lines with a smooth curve having minimal curvature. The question of Mendeleev can be reformulated as a question about the maximum deviation of the polynomial $f(z)$ from zero, and it is therefore closely related to Chebyshev's theory of polynomials deviating least from zero and to some other topics connected with this theory, such as orthogonal polynomials (particularly Hermite and Legendre polynomials and the distribution of their roots), interpolation, and approximate quadrature formulas.

Markov obtained new results in all these areas; but unlike Chebyshev, who also studied quadrature formulas, Markov found in his formulas the expression of the remainder term. For example, in his doctoral dissertation he derived the remainder term of a quadrature formula originating with Gauss. Among other topics related to approximation calculus, Markov considered summation and improving the convergence of series. Evidence of Markov's liking for computation are his tables of the integral of probabilities calculated to eleven decimal places. Markov paid much attention to interpolation, summation, transformations of series, approximate calculation of integrals, and calculation of tables in his *Ischislenie konechnykh raznostei* ("Calculus of Finite Differences"). The difference equations themselves occupy a modest place in this book, which contains characteristic connections with the work of Briggs, Gauss, and Euler and many carefully calculated examples. Markov also obtained some results in the theory of differential equations—on Lamé's equation and the equation of the hyper-geometric series—partly overlapping results of Felix Klein, and results concerning the possibility of expressing integrals in terms of elementary functions.

Markov's work in probability theory produced the greatest effect on the development of science. The basic achievements in probability theory by the middle of the nineteenth century were the law of large numbers, presented in its simplest version by Jakob I Bernoulli, and the central limit theorem (as it is now called) of de Moivre and Laplace. Satisfactory proofs under sufficiently wide assumptions had not been found, however, nor had the limits of their applicability. Through their closely interrelated works on these two laws Chebyshev, Markov, and Lyapunov created the foundation for the modernization of probability theory. In 1867 Chebyshev had found an elementary proof of the law of large numbers and turned to demonstrating the central limit theorem, using the solution of the problem of moments. The BienayméChebyshev problem mentioned above, translated into probability language, becomes a problem about the exact limits for the distribution function $F_\xi(x)$ of a random variable ξ with N given first moments $m_k = E\xi^k$. Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be a sequence of independent random variables with zero means (the case of nonzero $E\xi_n$ can be easily reduced to the considered one). According to Chebyshev's approach, one must show (a) that for every k the kth moment m_k of the normalized sum.

tends to the corresponding moment m_k of the standard Gaussian distribution

if $n \to \infty$, and (b) that if $m_k \to \mu_k$ for all K, then $F\zeta\eta(x) \to \Phi(x)$. When Markov published (1884) the proofs of Chebyshev's inequalities concerning the moments, Chebyshev began to work faster. In 1886 he showed that if $m_k = \mu_k$, then $F(x) = \Phi(x)$ (for him, but not for Markov, it was equivalent to assertion [b]); and in 1887 he published a demonstration of point (a) based on incorrect manipulations with divergent series.

Markov decided to turn Chebyshev's argument into a correct one and fulfilled this aim in 1898 in the paper "Sur les racines de l'équation *Izbrannye trudy po teorii nepreryvnykh drovei* …, pp. 231–243; $e^{x2}(d^n e^{-x2}/dx^n) = 0$" *Izbrannye trudy po teorii nepreryunykh deovei* …, pp. 231–243; *Izbrannye trudy*, pp. 253–269) and in his letters to Professor A. V. Vassilyev at Kazan University, entitled "Zakon bolshikh chisel i sposob naimenshikh kvadratov" ("The Law of Large Numbers and the Method of Least Squares" *Izbrannye trudy*, p. 231–251). In the first of his letter Markov defined his aim thus:

The theorem which Chebyshev is proving … has been regarded as true for a long time, but is established by an extremely inaccurate procedure. I do not say proved because I do not recognize inaccurate proofs … The known derivation of the theorem is inaccurate but simple. The derivation by Chebyshev on the contrary is very complicated, for it is based on preliminary investigations.… Therefore the question arises as to whether Chebyshev's derivation differs from the previous one only by its intricacy but is analogous to it in essentials, or whether one can make this derivation accurate. Your essay on Chebyshev's works strengthened my long-standing desire to simplify and at the same time to make quite accurate Chebyshev's analysis" [*Izbrannye trudy* p. 231].

In his letters to Vassilyev, Markov established an arithmetical proof of convergence $m_k \to \mu^k$ (assertion [a]) under the following conditions: (1) for every k the sequence $E\xi_1{}^k, E\xi;_2{}^k, \ldots$ is bounded and (2) var $(\xi;_1 + \ldots + \xi;_n) \geq cn$ for all n and some fixed $c > 0$. The corresponding calculation based on the expansion of the polynomial $(x_1 + \ldots + x_n)^k$ is maintained in all subsequent works by Markov on the limit theorem. In the article "Sur les racines …" Markov proved that $F(x) \to \Phi(x)$ if $m_k \to \mu_k$ (assertion [b]) by means of further analysis of Chebyshev's inequalities and continued fractions. He showed by examples that assumption (2), the need for which was unnoticed by Chebyshev, cannot be omitted.

In 1900 Markov published *Ischislenie veroyatnostei* ("Probability Calculus"). This book played an important role in modernizing probability theory. Characteristic features of the book are the inclusion of recent results obtained by Markov, rigorous proofs, elaborate references to classical works of the eighteenth century which for Markov had contemporary as well as historical importance, many numerical examples, and a polemical tone (Markov never missed an opportunity to mention an incorrectly solved example from another author and to correct the error).

But the triumph of the method of moments lasted only a short time. In 1901 Lyapunov, who was less influenced by their master, Chebyshev, and prized more highly the "transcendental" means (in Chebyshev's words) of the complex variable, played on Markov what he termed "a great dirty trick" (V. A. Steklov, "A. A. Markov," p. 178). Lyapunov discovered a new way to obtain and prove the limit theorems—the method of characteristic functions. The principal idea of this much more flexible method consists in assigning to the distribution of a random variable ξ; not the the sequence of moments $\{m_k\}$ but the characteristic function $\phi(t) = Ee^{il\xi}$ and deducing the convergence of distributions from convergence of characteristic functions. Lyapunov proved the central limit theorem (for independent summands with zero means) by his method under the conditions that (1) all moments $d_n = E \mid \xi;_n\xi;^{2+5}$ are finite for some $\delta > 0$, and

(2)

which are near to necessary and sufficient ones. Although the second conditions of Markov and Lyapunov are of a similar character (both require rapid growth of the variance of the sum), the first condition of Lyapunov is incomparably wider than Chebyshev-Markov's, because it does not require even the existence of moments of the third and subsequent orders.

Markov struggled for eight years to rehabilitate the method of moments and was at last successful. In the memoir "Teorema o predele veroyatnosti dlya sluchaev akademika A. M. Lyapunova" ("Theorem About the Limit of Probability for the Cases of Academician A. M. Lyapunovrdquo;; *Izbrannye trudy*, pp. 319–337), included in the third edition of his *Ischislenie veroyatnostei*, Markov proved Lyapunov's result by using the new procedure of truncating the distributions, thus permitting one to reduce the general case to the case of bounded moments of every order. This procedure is still a useful device, but the method of moments could not stand the competition of the simpler and more universal method of characteristic functions. Also in the third edition of *Ischislenie veroyatnostei*, Markov showed, by means of truncating, that the law of large numbers is true for a sequence $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ of independent random variables if for any $p > 1$ the moments $E \mid \xi_n \mid^p$ are bounded. (Chebyshev had proved the case $p = 2$.) Markov also deduced here the convergence of distributions from the convergence of moments for the cases when the limiteing distribution is not Gaussian but has the density $Ae^{-x2} \mid x \mid \upsilon$ or $Ae^{-x}x^5$ ($x \geq 0$) (The theorem was demonstrated for other continuous limiting distributions in 1920 by M. Pólya).

In his efforts to establish the limiting laws of probability in the most general situation and to enlarge the applications of the method of moments, Markov began a systematic study of sequences of mutually dependent variables, and selected from among them an important class later named for him. A sequence $\{\xi;_n\}$ of random variables (or random phenomena of some other kind) is called a Markov chain if, given the value of the present variable ξ_n, the future ξ_{n+1} becomes independent of the past ξ_1, $\xi_2, \ldots, \xi_{n-1}$. If the conditional distribution of ξ_{n+1} given $\xi;_n$ (defined by transition probabilities at time n) does not depend on n, then the chain is called homogeneous. The possible values of $\xi;_n$ are the states of the chain. Such chains appeared for the first time in 1906 in Markov's paper "Rasprostranenie zakona bolshikh chisel na velichiny, zavisyashchie drug ot druga" ("The Extension of the Law of Large Numbers on Mutually Dependent Variables" *Izbrannye trudy*, pp. 339–361). Markov started with the statement that if the variance of the sum ($\xi;_1 + \ldots + \xi;_n$) grows more slowly than n^2, then the law of large numbers is true for the sequence $\{\xi;_n\}$, no matter how the random variables depend on each other. He also gave examples of dependent variables satisfying this condition, among them a homogeneous chain with a finite number of states. Markov obtained the necessary estimation of the variance from the convergence as → of the distribution of $\xi;_n$ to some final distribution independent of the values of $\xi;_1$ (the "ergodic" property of the chain).

In his next paper, "Issledovanie zamechatelnogo sluchaya zavisimykh ispytanii" ("Investigation of a Remarkable Case of Dependent Trials" in *Izvestiya Peterburgskoi akademii nauk*, 6th ser., **1** , no. 3 [1907], 61–80), Markov proved the central limit theorem for the sums $\xi + \ldots + \xi_n$, where $\{\xi;_n\}$ is a homogeneous chain with two states, 0 and 1. In 1908, in the article "Rasprostranenie predelnykh theorem ischislenia veroyatnostei na summu velichin, svyazannykh v tsep" ("The Extension of the Limit Theorems of Probability Calculus to Sums of Variables Connected in a Chain" *Izbrannye trudy*, pp. 365–397), he generalized this result to arbitrary homogeneous chains with finite numbers of states, whose transition probabilities satisfy some restrictions. The proof, as in all of Markov's works, was obtained by the method of moments. In "Issledovanie obshchego sluchaya ispytanii svyazannykh v tsep" ("Investigation of the General Case of Trials Connected in a Chain" *Izbrannye trudy* [1910], pp. 467–507), Markov demonstrated the central limit theorem for nonhomogeneous chains with two states under the condition that all four transition probabilities remain in a fixed interval (c_1, c_2) $(0 < c_1 < c_2 < 1)$. In other articles, published in 1911–1912, he studied various generalizations of his chains (compound chains where $\xi;_n$ depends on several previous variables, so-called Markov-Bruns chains, partly observed chains) and deduced for them the central limit theorem under some restrictions.

Markov arrived at his chains starting from the internal needs of probability theory, and he never wrote about their applications to physical science. For him the only real examples of the chains were literary texts, where the two states denoted the vowels and the consonants (in order to illustrate his results he statistically worked up the alternation of vowels and consonants in Pushkin's *Eugene Onegin (Ischislenie Ueroyatnostei*, 4th ed., pp. 566–577). Nevertheless, the mathematical scheme offered by Markov and extended later to families of random variables $\xi;_t$ depending on continuous time t (which are called Markov processes, as suggested by Khinchin) has proved very fruitful and has found many applications. The development of molecular and statistical physics, quantum theory, and genetics showed that a deterministic approach is insufficient in natural sciences, and forced physicists to turn to probabilistic concepts. Through this evolution of scientific views, the Markov principle of statistical independence of future from past if the present is known, appeared to be the necessary probabilistic generalization of Huygens' principle of "absence of after effect." The far-reaching importance of such a generalization is shown by the fact that although Markov was the first to study the chains as a new, independent mathematical object, a number of random phenomena providing examples of Markov chains or processes were considered by other scientists before his work or concurrently with it. In 1889 the biologist Francis Galton studied the problem of survival of a family by means of a model reducing to a Markov chain with a denumberable number of states. An example of a Markov chain was considered in 1907 by Paul and T. Ehrenfest as a model of diffusion. In 1912 Poincaré, in the second edition of his *Calcul des probabilité*; in connection with the problem of card shuffling, proved the ergodic property for a chain defined on a permutation group and mentioned the possibility of an analogous approach to problems of statistical physics. An important example of a continuous Markov process was studied on a heuristic level in 1900–1901 by L. Bachelier in the theory of speculation. The same process appeared in 1905–1907 in works of Einstein and M. Smoluchowski on Brownian motion.

Markov's studies on chains were continued by S. N. Bernstein, M. Fréchet, V. I. Romanovsky, A. N. Kolmogorov, W. Doeblin, and many others. The first rigorous treatment of a continuous Markov process, the process of Brownian motion, was provided in 1923 by Wiener. The foundations of the general theory of Markov processes were laid down in the 1930's by Kolmogorov. The modern aspect of the theory of Markov processes, which became an intensively developing autonomous branch of mathematics, resulted from work by W. Feller, P. Lévy, J. Doob, E. B. Dynkin, K. Ito., tnd other contemporary probabilists.

Markov also studied other topics in probability: the method of least squares, the coefficient of variance, and some urn schemes.

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