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(*b*. Alexotas, Russia [now Lithuanian S.S.R.], 22 June 1864; *d*. Göttingen, Germany, 12 January 1909)

mathematics.

Minkowski was born of German parents who returned to Germany and settled in Königsberg [now Kaliningrad, R.S.F.S.R.] when the boy was eight years old. His older brother Oskar became a famous pathologist. Except for three semesters at the University of Berlin, he received his higher education at KOuml;nigsberg, where he became a lifelong friend of both Hilbert, who was a fellow student, and the slightly older Hurwitz, who was beginning his professorial career. In 1881 the Paris Academy of Sciences had announced a competition for the Grand Prix des Sciences Mathématiques to be awarded in 1883, the subject being the number of representations of an integer as a sum of five squares of integers; Eisenstein had given formulas for that number but without proof. The Academy was unaware that in 1867 H. J. Smith had published an outline of such a proof, and Smith now sent a detailed memoir developing his methods. Without knowledge of Smith’s paper, the eighteen-year-old Minkowski, in a masterly manuscript of 140 pages, reconstructed the entire theory of quadratic forms in *n* variables with integral coefficients from Eisenstein’s sparse indications. He gave an even better formulation than Smith’s because he used a more natural and more general definition of the genus of a form, The Academy, unable to decide between two equally excellent, and substantially equivalent, works, awarded the Grand Prix to both Smith and Minkowski.

Minkowski received his doctorate in 1885 at KOuml;nigsberg: he taught at Bonn until 1894, then returned to KOuml;nigsberg for two years, In 1896 he went to Zurich, where he was Hurwitz’ colleague until 1902; Hilben then obtained the creation of a new professorship for him at GOuml;tlingen, where Minkowski taught until his death.

From his Grand Prix paper to his last work Minkowski never ceased to return to the arithmetic of quadratic forms in *n* variables (“*n*-ary forms”). Ever since Gauss’s pioneering work on binary quadratic forms at the beginning of the nineteenth century, the generalization of his results to *n*-ary forms had been the goal of many mathematicians, including Eisenstein, Hermite, Smith, Jordan, and Poincaré. Minkowski’s most important contributions to the theory were (1) for quadratic forms with rational coefficients, a characterization of equivalence of such forms under a linear transformation with rational coefficients, through a system of three invariants of the form and (2) in a paper of 1905, the completion of the theory of reduction for positive definite *n*-ary quadratic forms with real coefficients, begun by Hermite. The latter had defined a process yielding in each equivalence class (for transformations with integral coefficients) a finite set of “reduced” forms; but it was still possible for this set to consist in more than one form. Minkowski presented a new process of “reduction” giving a unique reduced form in each class. In the space of *n*-ary quadratic forms (of dimension $n(n + 1)/2$), the “fundamental domain” of all reduced forms proves to be a polyhedron; Minkowski made a detailed investigation of the relation of this domain to its neighbors and computed its volume, which enabled him to obtain asymptotic formulas for the number of equivalence classes of a given determinant, when the value of that determinant tends to infinity.

This 1905 paper was greatly influenced by the geometric outlook that Minkowski had developed fifteen years earlier—the “geometry of numbers,” as he called it, his most original achievement. He was led to it by the theory of ternary quadratic forms. Following brief indications given by Gauss, Dirichlet had developed a geometrical method of reduction of positive definite ternary forms; Minkowski’s brilliant idea was to use the concept of volume in conjunction with this geometric method, thus obtaining far better estimates than had been possible before. To make matters simpler, consider a binary positive definite quadratic form $F(x, y) = ax^2 + 2bxy + cy^2$. To say that F takes a value m when $x = p, y = q$ are integers, means, geometrically, that the ellipse E_m of equation $F(x, y)= m$ passes through the point (p, q). To find the minimum M of all such values m, obtained for p, q not both 0, Minkowski observed that for small α, certainly the ellipse E_α will not contain any such points; if one considers the ellipse $½E_\alpha$ and translates it by sending its center to every point (p, q) with integral coordinates, one obtains an infinite pattern of ellipses which do not touch each other. When α increases and reaches the value M, some of the corresponding ellipses will touch each other but no two will overlap. Now, if $A = ac - b^2$ is the area of the ellipse E_1, the ellipse $½E_M$ has area $AM^{2|4}$, $AM^{2|4}$ and the total area of the nonoverlapping ellipses which are translations of $½E_M$ and which have centers at the points (p, q) with $|p| \leq n$ and $|q| \leq n$ is $(2n + 1)^2(AM^{2|4})$. It is easy to see, however, that there is a constant $c > 0$ independent of n, such that all these ellipses are contained in a square of center 0 and of side $2n + 1 + c$, so that

$$(2n + 1)^2AM^{2|4}) \leq (2n + 1 + c)^2;$$

letting n grow to infinity gives the inequality

Not only can this argument be at once extended to spaces of arbitrary finite dimension, but Minkowski had a second highly original idea: He observed that in the preceding geometric argument, ellipses could be replaced by arbitrary convex symmetric curves (and, in higher-dimensional spaces, by symmetric convex bodies). By varying the nature of these convex bodies with extreme ingenuity (polyhedrons, cylinders), he immediately obtained far-reaching discoveries in many domains of number theory. For instance, by associating to an algebraic integer x in a field of algebraic numbers K of degree n over the rationals, the point in $n$dimensions having as coordinates the rational integers which are the coefficients of x with respect to a fixed basis, Minkowski gave lower bounds for the discriminant of K, which in particular proved that when n > 1, the discriminant may never be equal to 1 and thai there are only a finite number of fields of discriminants bounded by a given number.

Minkowski’s geometric methods also enabled him to reach a far better understanding of the theory of continued fractions and to generalize it into an algorithm which, at least theoretically, gives a criterion for a number to be algebraic. It was similar in principle to Lagrange’s well-known criterion that quadratic irrationals are characterized by periodic continued fractions; but Minkowski also showed that, for his criterion, periodicity occurs in only a small number of cases, which he characterized completely. Finally, if, for instance, one considers (as above, but in three dimensions) an ellipsoid $F(x, y, z) = 1$ in relation to the lattice L of points with integral coordinates, the largest possible number M will be obtained when the translated ellipsoids are “packed together” as closely as possible. If one makes a linear transformation of the space transforming the ellipsoid in a sphere, L is transformed into another lattice consisting of linear combinations with integral coefficients of three vectors. The problem of finding the largest M, then, is equivalent to the “closest packing of spheres” in space, when the centers are at the vertices of a

lattice L′ one has to find the lattice L′ that gives this closest packing. Minkowski began the study of that difficult problem (which extends to any n-dimensional space) and of corresponding problems when spheres are replaced by some other type of convex set (particularly polyhedrons); they have been the subject of fruitful research ever since.

The intensive use of the concept of convexity in his “geometry of numbers” led Minkowski to investigate systematically the geometrical properties of convex sets in n-dimensional space, a subject that had barely been considered before. He was the first to understand the importance of the notion of hyperplane of support (both geometrically and analytically), and he proved the existence of such hyperplanes at each point of the boundary of a convex body. Long before the modern conception of a metric space was invented, Minkowski realized that a symmetric convex body in an n-dimensional space defines a new notion of "distance" on that space and, hence, a corresponding "geometry." His ideas thus paved the way for the founders of the theory of normed spaces in the 1920's and became the basis for modern functional analysis.

The evaluation of volumes of convex bodies led Minkowski to the very original concept of "mixed volume" of several convex bodies: when K_1, K_2, K_3 are three convex bodies in ordinary space and t_1, t_2, t_3 are three real numbers ≥ 0, the points $t_1x_1 + t_2x_2 + t_3x_3$, when x_j varies in K_j for $j = 1, 2, 3$, fill a new convex body, written $t_1K_1 + t_2K_2 + t_3K_3$. When the volume of this new convex body is computed, it is seen to be a homogeneous polynomial in t_1, t_2, t_3 and the mixed volume $V(K_1, K_2, K_3)$ is the coefficient of $t_1t_2t_3$ in that polynomial. Minkowski discovered remarkable relations between these new quantities and more classical notions: if K_1 is a sphere of radius 1, then $V(K_1, K, K)$ is one third of the area of the convex surface bounding K; and $V(K_1, K_1, K)$ is one third of the mean value of the mean curvature of that surface. He also proved the inequality between mixed volumes $V(K_1, K_2, K_3)^2 \geq V(K_1, K_1, K_3)\ V(K_2, K_2, K_3)$, from which he derived a new and simple proof of the isoperimetric property of the sphere. As a beautiful application of his concepts of hyperplane of support and of mixed volumes, Minkowski showed that a convex polyhedron having a given number m of faces is determined entirely by the areas and directions of the faces, a theorem that he generalized to convex surfaces by a passage to the limit. He also determined all convex bodies having constant width.

Minkowski was always interested in mathematical physics but did not work in that field until the last years of his life, when he participated in the movement of ideas that led to the theory of relativity. He was the first to conceive that the relativity principle formulated by Lorentz and Einstein led to the abandonment of the concept of space and time as separate entities and to their replacement by a fourdimensional "space-time," of which he gave a precise definition and initiated the mathematical study; it became the frame of all later developments of the theory and led Einstein to his bolder conception of generalized relativity.

BIBLIOGRAPHY

Minkowski's writings were collected in *Gesammelte Abhandlungen*, D. Hilbert, ed., 2 vols. (Leipzig-Berlin, 1911). Among his books are *Geometrie der Zahlen* (Leipzig, 1896; 2nd ed., 1910); and *Diophantische Approximationen* (Leipzig, 1907; repr. New York, 1957).

On Minkowski's work, see Harris Hancock, *Development of the Minkowski Geometry of Numbers* (New York, 1939); and Frederick W. Lanehester, *Relativity. An Elementary Explanation of the Space-Time Relations As Established by Minkowski* (London, 1935).

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