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(*b*. Lugo, Italy, 12 January 1853; *d*. Bologna, Italy, 6 August 1925)

mathematics, mathematical physics.

Ricci-Curbastro[1] was the son of a noble family situated in the province of Ravenna. His father, Antonio Ricci-Curbastro, was a well-known engineer; his mother was Livia Vecchi. With his brother Domenico, Ricci received his elementary and secondary education from private teachers; he then, in 1869, entered the University of Rome to study philosophy and mathematics. After a year of study he returned home, and it was only in 1872 that he enrolled at the University of Bologna. The following year transferred to the Scuola Normale Superiore of Pisa, where he attended the courses of Betti, Dini, and Ernesto Padova. In 1875 Ricci defended a thesis entitled "On Fuches's Research Concerning Linear Differential Equations," for which he received the degree of doctor of physical and mathematical sciences. The following year—in conformity with the then existing requirements for teaching—he presented a paper "On a Generalization of Riemann's Problem Concerning Hypergeometric Functions."[2] Betti then asked Ricci to write a series of articles on electro- dynamics, particularly Maxwell's theory, for *Nuova Cimento*. Under the influence of Dini, Ricci took up Lagrange's problem of a linear differential eqution, on which he contributed a nineteen-page article to the *Giornale di matematiche di Battaglini*. Shortly afterward, having won a competition for a scholarship to study abroad, he spent a year (1877–1878) in Munich, where he attended the lectures of Felix Klein and A. Brill. Ricci greatly admired Klein, and his esteem was soon reciprocated; nevertheless, Ricci does not seem to have been decisively influenced by Klein's teaching. It was, rather, Riemann, Christoffel, and Lipschitz who inspired his future research. Indeed, their influence on him was even greater than that of his Italian teachers.

In 1879 Ricci worked as Dini's assistant in mathematics at Pisa. Then, on 1 December 1880, he was named professor of mathematical physics at the University of Padua, a position that he held without interruption for forty-five years. In 1891 he also began to teach higher algebra.

Ricci is best known for the invention of absolute differential calculus, which he elaborated over ten years of research (1884–1894). With this new calculus he was able to modify the usual procedures of the differential calculus in such a way that the formulas and results retain the same form whatever the system of variables used. This procedure requires the employment of systems of functions that behave, when a change of variables is made, like coefficients of expressions that are themselves independent (whether by nature or by convention) of the choice of variables. A further requirement is the introduction of an invariant element (called an absolute, from which the calculus takes its name), that is to say, an element that can also be used in dealing with other systems. The absolute that best lends itself to this operation is the quadratic differential form, which expresses, geometrically, the elementary distance between two points.

Ricci's attention was first drawn to the theory of the invariants of algebraic forms, which had been developed principally after Riemann wrote his thesis,[3] and to the works of Christoffel and of Lipschitz on the quadratic forms.[4] But it was essentially Christoffel's idea of covariant derivation that allowed Ricci to make the greatest progress. This operation, which possesses the characteristics of ordinary derivation, has the additional property of preserving, with respect to any change of variables, the invariance of the systems to which it is applied. Ricci realized that the methods introduced and utilized by these three authors required fuller development and were capable of being generalized. Their methods furnished the basis of Ricci's works on the quadratic differential forms (1884 and 1888) and on the parameters and the differential invariants of the quadratics (1886), which Ricci reduced to a problem of algebra. The method he used to demonstrate their invariance led him to the technique of absolute differential calculus, which he discussed in its entirety in four publications written between 1888 and 1892.

In 1893 Ricci revealed the first applications of his algorithm, to which he gave its specific name for the first time. Two years later, Klein urged him to make his methods more widely available in a complete exposition, but Ricci did not do so until five more years had passed. Meanwhile, he prepared a long paper on intrinsic geometry (published in the *Memorie dell' Accademia dei Lincei* in 1896), in which he examined the congruences of lines on an arbitrary Riemannian variety. He applied the absolute calculus to these problems by means of a special form given to the differential equations of the congruences, which appear with their covariant and contravariant systems and in this way arrived at the notion of a canonical orthogonal system of a given congruence. (In this case the coefficients of rolation replace the Christoffel symbols of absolute to find the contract tensor (today called Ricci's tensor) that plays a fundamental role in the general theory of relativity. He also discovered invariants that occur in the theory of the curvature of varieties.[5]

This intrinsic geometry complete one stage in the development of absolute calculus, and Ricci was now in a position to fulfill Klein's earlier request. In collaboration with Levi-Civita, he published a seventy-seven-page memoir entitled "Méthodes de calcul différentiel absolu et leurs applications." The following brief discussion of the paper is, of necessity, limited to the simplest expressions used by Ricci and Levi-Civita.[6]

Given a change of variables $x_1, \ldots, x_n$ into

$y_1, \cdots, y_n$:

$a_1 dx_1 + \cdots + a_n dx_n = b_1 + \cdots + b_n dy_n$,

one also has

The system a_j is then said to be covariant of order 1. This will be the case if the a_j are the derivatives of a function $\phi(x_1, \ldots, x_n)$. A system of arbitrary order m can then be generalized, from which a system

$Y_{r1r2} \cdots_{rm}$

may be obtained. The elements $dx_1, \ldots, dx_n$ form a contravariant system of order 1, which is written

From this expression a system of order m,

$Y(^{(r1\ r2\ \cdots rm)}$

may be derived.

Next, a quadratic form is selected. called the fundamental form,

this is an n-dimensional linear element of a variety V_n with the a_{rs} forming a covariant system of order 2. The a^{rs} is established and generalized to

$X^{(r1r2 \ldots rm)}$

which is called reciprocal to the covariant system $X_{s1s2} \cdots_{sm}$ with respect to ϕ. With the equalities established by Christoffel,[7] it is possible to find formulas for deriving, from any covariant gale:type="formula"system of order m, a covariant system of order $m + 1$. This is what Ricci called covariant derivation based on ϕ. The contravariant derivation of a contravariant system is then defined by passing to the reciprocal system, which is derived, and returning again to the reciprocal system.

A chapter on intrinsic geometry as an instrument of computation deals with normal congruences, geodesic lines, isothermal families of surfaces, the canonical system with respect to a given congruence, and the canonical forms of the systems associated with the fundamental form. With regard to the last problem, Ricci started with a system X_r to which he associated a congruence defined by the equations

whose covariant coordinated system will result from the elements $\lambda_{n|r} = X_r : \varrho$ with . The formulas $X_r = \varrho \cdot \lambda_{n|r}$ furnish the canonical expressions of the X_r.

The authors then show how to proceed in order to arrive at general rules. The succeeding chapters are devoted to analytical, geometric, mechanical, and physical applications.

Analytical applications include classification of the quadratic forms of differentials; absolute invariants and fundamental invariants of the form ϕ and differential parameters.

Geometric applications cover a study of two-dimensional varieties; remarks on surfaces of ordinary space; an extension of the theory of surfaces to linear spaces of n dimensions; groups of motions in an arbitrary variety; a complete study of the groups of motions of a three-dimensional variety; and comments on the relationship of this research with that done by Lie and Bianchi.

Mechanical applications include first integrals of the equations of dynamics. Here Ricci solved the Lagrange equations with respect to the second derivatives of the coordinates and found that

This is the form best suited to the question under examination. If, in seeking a function f of the x' and of the x'''s, it is desired that f = constant be a first integral of the equations, then certain conditions must be satisfied. The latter, applied to the case in which there are no forces, yield the homogeneous integrals of the geodesics of the variety V_n, whose length ds^2 is expressed by $2T\ dt^2$.

Linear integrals, the quadratics, and the conditions of existence are then considered. Finally, Ricci and Levi-Civita took up surfaces whose geodesics possess a quadratic integral and the transformation of the equations of dynamics.

In their treatment of physical applications Ricci and Levi-Civita first examined the problem of the reducibility to two variables of the equation $\Delta u = 0$ (binary potentials), then went on to consider vector fields, and finally, equations in general coordinates of electrodynamics, of the theory of heat, and of elasticity.

The authors set forth a general statement of their work in their preface:

The algorithm of absolute differential calculus, the *instrument matériel* of the methods,… can be found complete in a remark due to Christoffel. But the methods themselves and the advantages they offer have their *raison d'être* and their source in the intimate relationships that join them to the notion of an *n*-dimensional variety, which we owe to the brilliant minds of Gauss and Riemann… Being thus associated in an essential way with V_n, it is the natural instrument of all those studies that have as their subject such a variety, or in which one encounters as a characteristic element a positive quadratic form of the differentials of *n* variables or of their derivatives.

In mechanics this is the case for kinetic energy, and it later proved to be the case, in general relativity, for the elementary interval between two events in space-time. Meanwhile, however, Ricci's methods— which Beltrami judged important, while adopting a prudent and reserved attitude toward them—were not known beyond the restricted circle of his students, and the memoir in the *Annalen* did not evoke a particularly enthusiastic response.

In 1911 Ricci and Levi-Civita sent to the *Bulletin des sciences mathématiques* a detailed exposition of the absolute calculus. The editors of the journal published it in abridged form with the comment that "essentially, it is only a calculus of differential covariants for a quadratic form," while adding that it was "very interesting."

Ricci was now almost sixty, and more than twentyseven years had passed since he had begun his initial research. He probably was not aware that at the Zurich Polytechnikum, Marcel Grossmann, a colleague of Albert Einstein, had an intuition that only Ricci's methods could permit the expression of the quadrimensional metric of ds^2. And, indeed, it was by means of absolute differential calculus that Einstein was able to write his gravitational equations,[8] and on more than one occasion he paid tribute to the efficacy of this tool and to Ricci.[9]

In 1917 Levi-Civita, Ricci's brilliant student, introduced, with his new concept of parallel transport,[10] the geometric foundation of the algorithms of invariance, and Ricci's calculus gave rise to a series of developments and generalizations that confirmed its validity.

Ricci's other publications include a book on higher algebra (containing material from his course at Padua), a book on infinitesimal analysis, and papers on the theory of real numbers, an area in which he extended the research begun by Dedekind. Between 1900 and 1924 he published twenty-two items, most of which dealt with absolute differential calculus. His last work was a paper on the theory of Riemannian varieties,presented to the Intrnational Congress of Mathematics held at Toronto in August 1924.

Ricci was a member of the Istituto Veneto (admitted in 1892, president 1916–1918), the Reale Accademia of Turin (1918), the Società dei Quaranta (1921), the Reale Accademia of Bologna (1922), and the Accademia Pontificia (1925). He became a corresponding member of the Paduan Academy in 1905 and a full member in 1915. The Reale Accademia dei Lincei, which elec ed him a corresponding member in 1899 and a national associate member in 1916, published many of Ricci's works.

In addition to his activities in research and teaching, Ricci held a number of civic posts. He served as provincial councillor and assisted in public works projects, including water supply and swamp drainage, at Lugo. He was elected communal councillor of Padua, where he was concerned with public education and finance, although he declined the post of mayor. In 1884 he married Bianca Bianchi Azzarani, who died in 1914; they had two sons and one daughter.

NOTES

1. This is his complete name. It is also the way in which he signed all his works, except for the one he published with his former student Levi-Civita in 1900 in the *Mathematische Annalen*, where he kept only the first part of his name. This memoir, written in French, made its senior author famous under the simple name of Ricci, and we shall keep to this usage.

2. These first two works by Ricci have never been published.

3. B. Riemann, "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen," in *Gesammelte Werke*, 2nd ed. (Leipzig, 1892), 272–287.

4. Sec E. B. Christoffel, "Ueber die Transformation der homogenen Differentialausdrücke zweiten Grades," in *Journal für die reine und angewandte Mathematik*, **70** (1869), 46–70, 241–245; and R. Lipschitz, "Untersuchungen in Betreff der ganzen homogenen Funktionen von *n* Differentialen," *ibid*., 71–102.

5. Concerning the curvature of surfaces in hyperspaces, Ricci mentions, in his "Méthodes de calcul différentiel absolu" of 1900 (p. 156), a paper by Lipschitz that he considers fundamental: "Entwickelungen einiger Eigenschaften der quadratischen Formen von *n* Differentialen," in *Journal für die reine and angewandte Mathematik*, **71** (1870), 274–295. Compare also, for these questions of intrinsic geometry, F. Schur, "Ueber den Zusammenhang der Räume constanten Riemann'schen Krümmanugsmasses mit den projectiven Rä;uman," in *Mathematische Annalen*, **27** (1886), 537–567.

6. It should be noted that Ricci puts the upper indices (of the contravariants) in parentheses and that he always uses the sign σ for summations.

7.and

8. These are the well-known equations:

Ricci's theorem shows that the covariant derivation cancels the effects of the variation of the metric tensor (Ricci does not use the term "tensor") and operates intrinsically on geometric entities. With the aid of this theorem and of the rules of tensor contraction one can write:

where ∇_λ is the covariant derivation with respect to x^λ and is the Einstein tensor. This relationship, which is fundamental in general relativity, serves to express the principle of the conservation of energy.

9. Compare, for example, "Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation. I. Physikalischer Teil von Albert Einstein. II. Mathematischer Teil von Marcel Grossmann," in *Zeitschrift f`r Mathematik und Physik*, **62** (1913), 225–261.

10. T. Levi-Civita, "Zozione di parallelismo in una varietà qualunque," in *Rendiconti del Circolo matematico di Palermo*, **42** (1917), 173.

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Ricci laid the basis of absolute differential calculus in the following four articles: "Delle derivazioni covarianti e del loro uso nella analisi applicata," *in Studi editi dalla Università Padovana a commemorare l'ottavo centenario dalla origine della Università di Bologna*, III (Padua, 1888); "Sopra certi sistemi di funzioni," in *Atti dell' Accademia nazionale dei Lincei Rendiconti*, 4th ser., **5** (1889), 112–118; "Di un punto della teoria delle forme differenziali quadratiche," *ibid*., 643–651; and "Résumé de quelques travaux sur les systèmes variables de fonctions," in *Bulletin des sciences mathématiques*, **16** (1892), 167–189.

On the applications of absolute calculus, intrinsic geometry, varieties, and groups see "Di alcune applicazioni del calcolo differenziale assoluto alla teoria delle forme differenziali quadratiche e dei sistemi a due variabili," in *Atti del Istituto veneto di scienze, lettere ed arti*, 7th ser., **4** (1893),1336–1364; "Dei sistemi di coordinate atti a ridurre l'elemento lineare di una superficie alla forma $ds^2 = (U + V)(du^2 + dv^2)$," in *Atti dell' Accademia nazionale dei Lincei*. *Rendiconti*, 5th ser., **2** (1893), 73–81; "Sulla teoria delle linee geodetiche e dei sistemi isotermi di Liouville," in *Atti del Istituto veneto di scienze, lettere ed arti*, 7th ser., **5** (1894), 643–681; "Dei sistemi di congruenze ortogonali in una varietà qualunque," in *Atti dell' Accademia nazionale dei Lincei*. *Memorie*, 5th ser., **2** (1896), 275–322; and "Sur les groups continus de mouvements d'une variété quelconque à trois dimensions," in *Comptes rendus… de l' Académie des sciences de Paris*, **127** (1898), 344–346, 360–361.

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Ricci's memoir written with Levi-Civita, "Méthodes du calcul différentiel absolu et leurs applications," in *Mathematische Annalen*, **54** (1900), 125–201, appeared in a polish trans., in Praec *matematyczno-fizyczne*, **12** (1901), 11–94, and was reprinted in Collection de Monographies Scientifiques Étrangè(Paris, 1923).

Ricci's books are *Lezioni di algebra complementare* (Padua-Verona, 1900); and *Lezioni di analisi infinitesimale. Funzioni di una variabile* (Padua, 1926).

His last publications include "Sulla determinazione di varietà dotate di proprietà intrinseche date a priori," in *Atti dell' Accademia nazionale dei Lincei. Rendiconti*, 5th ser., **19** (1910), 181–187 (first semester), and 85–90 (second semester); "Di un metodo per la determinazione di un sistema completo di invarianti per un dato sistema di forme," in *Rendiconti del Circolo matematico di Palermo*, **33** (1912), 194–200; "Sulle varietà a tre dimensioni dotate di terne principali di congruenze geodetiche," in *Atti dell' Accademia nazionale dei Lincei. Rendiconti*, 5th ser., **27** 1918, 21–28, 75–87; "Riducibilità delle quadroche differenziali e ds^2 della statica einsteiniana," *ibid*., **31** (1922), 65–71; and "Di una proprietà delle congruenze di linee tracciate sulla sfera di raggio eguale ad l," *ibid*., **32** (1923), 265–267.

See also L'Unione Matematica Italiana, ed., *Opere de Ricci*, 2 vols. (1956–1957).

II. Secondary Literature. The first account of Ricci's life and works is the excellent one by Levi-Civita, "Commemorazione del socio nazionale prof. Gregorio RicciCurbastro, letta dal socio T. L.-C. nella seduta del 3 gennaio 1925," in *Atti dell' Accademia dei nazionle Lincei. Memorie*, 6th ser., **1** (1926), 555–567. Angelo Tonolo, another disciple of Ricci, "Commemorazione di Gregorio Ricci-Curbastro nel primo centenario della nascita," in *Rendiconti del Seminario matematico della Università di Padovra*, **23** (1954), 1–24, contains a beautiful portrait of Ricci and a partial bibliography. See also A. Natucci in *Giornale di matematiche di Battaglini*, 5th ser., **2** (1954), 437–442; and two articles in Enciclopedia *italiana*, on absolute differential calculus, XII, 796–798; and on Ricci's life and work, XXIX, 250.

Reports on almost all of Ricci's publications were published in *Bulletin des sciences mathématiques;* although sometimes very detailed, they often appeared only after a considerable delay. For the report on "Méthods de calcul différentiel absolu," see **35** (1911), 107–111.

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