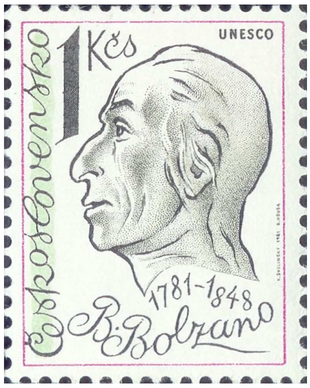
Bernard Bolzano (October 5, 1781 – December 18, 1848)



by Heinz Klaus Strick, Germany

Bernard Placidus Johann Nepomuk Bolzano was born in Prague, the fourth of twelve children, of whom only he and one brother survived to adulthood. His father, an art dealer, was born in northern Italy (whence the name Bolzano). His mother was the daughter of a Prague merchant. Bernard's parents raised him in a deeply religious household.

He attended a gymnasium run by the Church until at the age of 15 he began studies at Prague's *Charles University* in philosophy, mathematics, and physics. At 19, he commenced theological studies. A few days following the completion of his doctorate in mathematics, with a thesis on the question of what constitutes a rigorous mathematical proof, he was ordained a priest in the Catholic Church.

At that time, Prague was the capital city of the *Kingdom of Bohemia*, part of the Habsburg Empire. While Emperor Joseph II had granted his subjects freedom of religion in 1781, the ideas of freedom spawned by the French Revolution demanded much more: universal freedom of thought and national liberation. To fend off such movements for liberalisation, Emperor Francis I of Austria followed a conservative course, which was to have been supported by the creation in 1804 of a professorship in the philosophy of religion.

Bolzano applied for this professorship, and at the same time, applied for one in elementary mathematics, being equally qualified for both.

The authorities quickly realized that he had been appointed to the "wrong" professorship. As leader of the *Bohemian Enlightenment*, through his lectures he promulgated ideas of pacifism and socialism. In 1819, he was relieved of his position under the charge of spreading "false teachings" and charged with heresy. On being placed under house arrest, he turned to philosophical and mathematical questions.

In 1837 there appeared two of his most important works: *On the Best State* and *Theory of Science*. It was not until 1840 that he was allowed to publish nontheological works under the auspices of the *Royal Bohemian Scientific Society*.

The work *On the Best State* was a socialist Utopia, in which Bolzano advocated a wide-ranging principle of equality and in which he criticised property that is not obtained through work.

In his *Theory of Science*, he dealt with questions of judgment and truth, and he described the development of logic as a science, from Aristotle to Kant. In opposition to Kant and Hegel, he expressed the point of view that numbers, ideas, and "propositions in themselves" have an existence independent of the persons who "think" them.

During Bolzano's lifetime, his works failed to receive the recognition that they would otherwise have obtained had there not been a prohibition against publication, and also had many of his ideas not been so far ahead of their time. It thus came to pass that a number of his ideas were rediscovered only decades later.

Already in his early mathematical writings, Bolzano was concerned with making mathematical proofs and the chain of their argument more rigorous. In 1810, his *Contributions to a better-grounded presentation of mathematics* appeared. In 1816 there followed *The Binomial Theorem*, as a consequence of it the polynomial theorem and the series that serve for the calculation of the logarithmic and exponential functions, demonstrated more precisely than ever before.

In this work, BOLZANO criticised the brilliant yet insufficiently precise methods of LEONHARD EULER and JOSEPH LOUIS LAGRANGE. By the binomial theorem is meant here the binomial series that can be defined not only for natural number exponents but also for integer, rational, and even arbitrary real exponents *n*:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}\cdot x + \binom{n}{2}\cdot x^2 + \binom{n}{3}\cdot x^3 + \ldots = 1 + n\cdot x + n\cdot\frac{n-1}{2}\cdot x^2 + n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot x^3 + \ldots$$

BOLZANO wrote that the difference between $(1+x)^n$ and the *r*-th term of the series

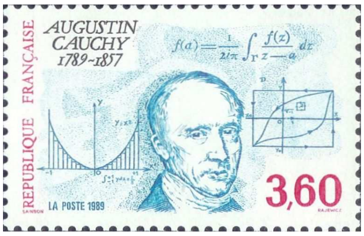
$$1 + n\cdot x + n\cdot\frac{n-1}{2}\cdot x^2 + n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot x^3 + \ldots + n\cdot\frac{n-1}{2}\cdot\ldots\cdot\frac{n-r+1}{r}\cdot x^r$$

“can be made less than any given quantity if one takes a sufficient number of terms of the series, and this makes sense only for $|x|<1$”.

In 1817, he wrote his *Pure analytical proof of the mean value theorem*, today known as

- **BOLZANO’s theorem**: If a function *f* is continuous on a closed interval $[a\,;b]$ ($a,b\in$ IR) and if furthermore, $f(a)\cdot f(b)<0$, then at least one zero of *f* lies in the interval $]a\,;b[$.

An important precondition for a proof of BOLZANO’s theorem is a precise definition of the notion of continuity; for BOLZANO, continuity in an interval meant that “when *x* is an arbitrary number [in the interval], the difference $f(x+\omega)-f(x)$ can be made less than any given value if one takes ω sufficiently small – a formulation that differs little from that used in contemporary mathematics.



In 1821 AUGUSTIN LOUIS CAUCHY published his

- **Convergence criterion**: A sequence $(a_n)_{n\in IN}$ converges if and only if for every $\varepsilon>0$, there exists a number n_0 such that for all *n*, *m* with $n\geq n_0$ one has $|a_n-a_m|<\varepsilon$.

Yet four years earlier, BOLZANO, in the above-mentioned paper, had given the same necessary and sufficient conditions for the convergence of an infinite series, but it went unnoticed:

- If a sequence of values $F_1(x), F_2(x), F_3(x), \ldots, F_n(x), \ldots, F_{n+r}(x)$ is such that the difference between its *n*'th term $F_n(x)$ and every subsequent term $F_{n+r}(x)$, no matter how far out in the series, remains smaller than any prescribed value whenever one has taken *n* sufficiently large, then there is always a certain definite quantity – and it is unique – that the terms of the sequence approach ever more closely and that come as close as desired if one continues the sequence sufficiently far.

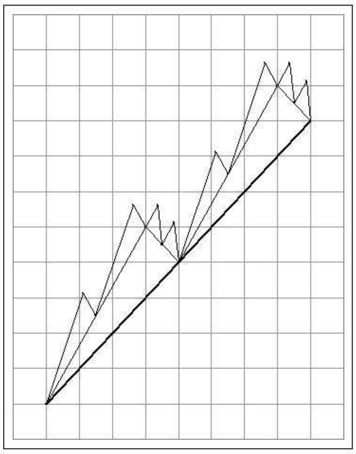
He wrote that in his opinion, there was “nothing impossible” about the existence of a limiting value (what he called “a certain definite quantity”), since “under this assumption, it is possible to determine this value as precisely as one might wish”.

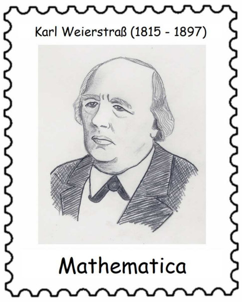
The mathematicians of the time did not realize that this was a case of circular reasoning, since before one can define a limit, it is first necessary to define what is meant by a real number.

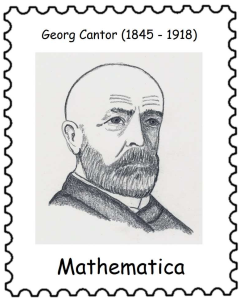
Around 1860, KARL WEIERSTRASS formulated a theorem that was only years later seen to have been the same theorem that BOLZANO had written down in 1817:

Bolzano-WEIERSTRASS theorem: Every bounded infinite sequence of numbers has at least one accumulation point.

(A real number is called an *accumulation point* of a sequence of real numbers if every open set containing the number, no matter how small, contains infinitely many terms of the sequence.)







(drawings: © Andreas Strick)

It was not until 1930 that it was discovered that BOLZANO (long before WEIERSTRASS) had succeeded in constructing a function that was everywhere continuous on an interval but nowhere differentiable. This function is the limit of a sequence of piecewise continuous linear functions. The figure shows the functions f_1, f_2, f_3 .

It is also in his last work, *Paradoxes of the infinite* (1847), in which one finds ideas that achieved recognition only many years later through the work of others, in this case by GEORG CANTOR (1845-1918):

“I assert the following: When two sets are both infinite, they can stand in such a relation to one another that:

(1) It is possible to pair each member of the first set with some member of the second in such wise that on the one hand, no member of either set fails to occur in one of the pairs, and on the other hand, not one of them occurs in two or more of the pairs; and at the same time it is yet possible that

(2) one of the two sets can comprise the other as a mere part of itself, in such a wise that the multiplicities to which they are reduced, when we regard all their members as interchangeable individuals, can stand in the most varied relationships to one another.”

Example: The mapping $x \to y = 2x$ associates with every point of the interval [0 ; 1] a point of the interval [0 ; 2] in an invertible one-to-one correspondence: even though the first interval is a proper subset of the second, both sets have the same “magnitude.”

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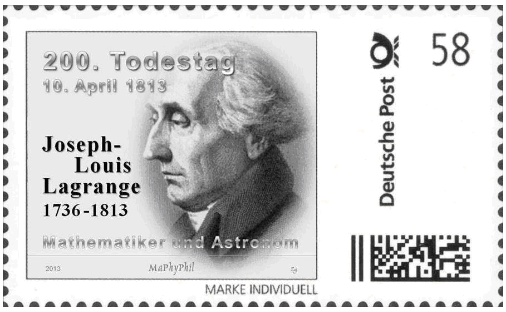
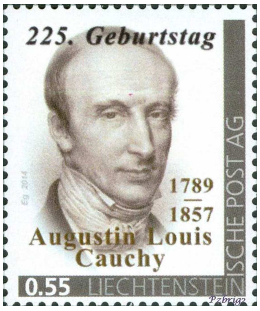
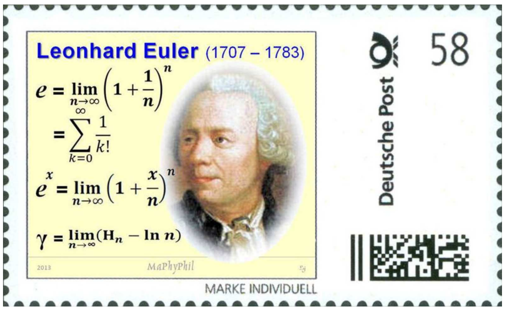
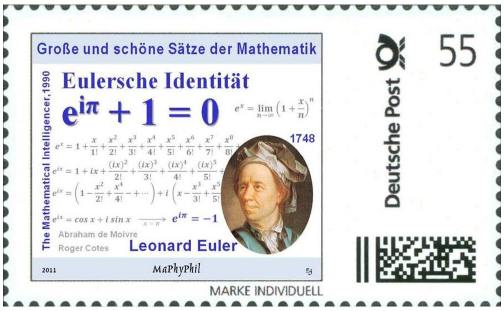
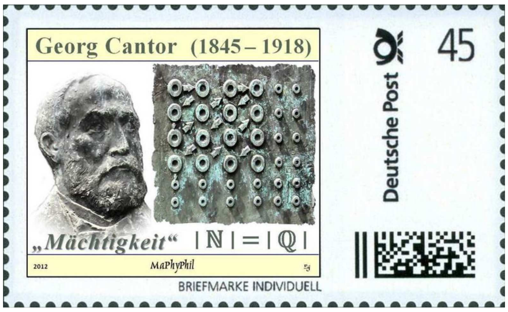
Here an important hint for philatelists who also like individual (not officially issued) stamps:

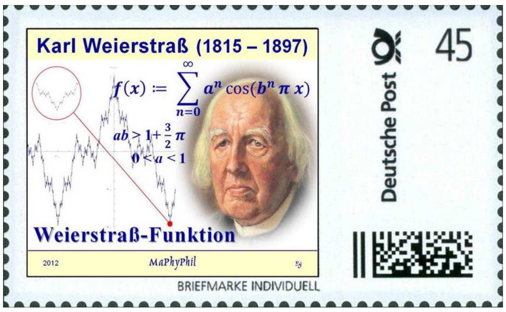












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