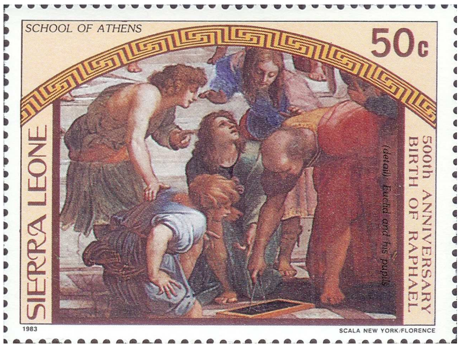
EUCLID OF ALEXANDRIA (ca. 300 B.C.E.)

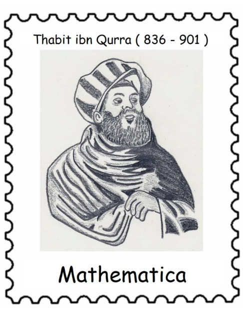
by Heinz Klaus Strick, Germany

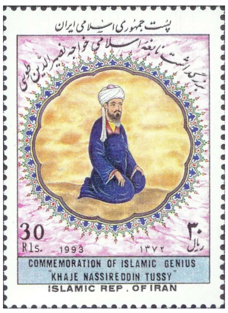


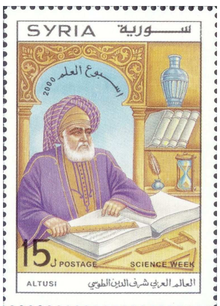
The pictured postage stamp from Sierra Leone shows a detail from the famous fresco *La scuola di Atene* (*the School of Athens*), painted by the Renaissance master RAPHAEL 500 years ago in the Stanza della Segnatura (the room containing the library of JULIUS II, formerly the location of the Supreme Tribunal of the Apostolic Signatura). The painting was meant to honour the philosophers and scientists who had had the greatest impact on the development of European culture. Those figures from ancient world bore the features of some of RAPHAEL's famous contemporaries. For instance, RAPHAEL painted the polymath LEONARDO DA VINCI as the representation of PLATO. For EUCLID's portrait, he chose the architect of St. Peter's basilica, DONATO BRAMANTE. The detail pictured here shows EUCLID bent over his slate in the process of completing a geometric construction. Opposite him is pictured PYTHAGORAS, engrossed in a book.

Little is known about the ancient Greek mathematician EUCLID. In many sources, 325 B.C.E. is given as the date when the *Elements* appeared. What is certain is that he lived and taught in Alexandria during the reign of PTOLEMY I SOTER, who had been a general under ALEXANDER THE GREAT. It is told that PTOLEMY once asked EUCLID whether there was perhaps an easier way to learn geometry than studying the Elements, to which the latter famously replied that there is no "*royal road*" to geometry/mathematics. Another anecdote points to EUCLID's attitude towards the acquisition of knowledge: A student of EUCLID asked him what he would have gained after learning all that the master was teaching. In reply, EUCLID ordered a slave to give the student three coins, since it was apparent that the student felt that what he learned was of value only if it increased his wealth.

The *Elements* was used as a textbook in mathematics until well into the nineteenth century. Next to the bible, it was the most widely disseminated book in the world; it is certainly the most influential work in the entire history of mathematics. That this compendium of the mathematical knowledge of antiquity survived is due to the translations by mathematicians in the Islamic world such as THABIT IBN QURRA (836-901) and NASIR AL-DIN AL-TUSI (1201-1274).







(drawing: © Andreas Strick)

In the late Middle Ages, Arabic texts were translated into Latin, and then into many other languages following the invention of printing.

It is noteworthy that the Jesuit MATTEO RICCI (1552-1610) used the Chinese translation of this book as a way of introducing Western mathematics into China.

The *Elements* comprises thirteen books (chapters): six on geometry in the plane; three on arithmetic, number theory, and the theory of proportions; one on incommensurable magnitudes; and three on solid geometry. It is quite possible that EUCLID himself discovered none of the theorems contained in the *Elements*. His significant contribution lay rather in his brilliant organization and presentation of the work of his forebears (above all that of the Pythagoreans as well as that of EUDOXUS OF CNIDUS and THEAETETUS). The rigorousness of his methods served as a model for the mathematicians and scientists who followed him.

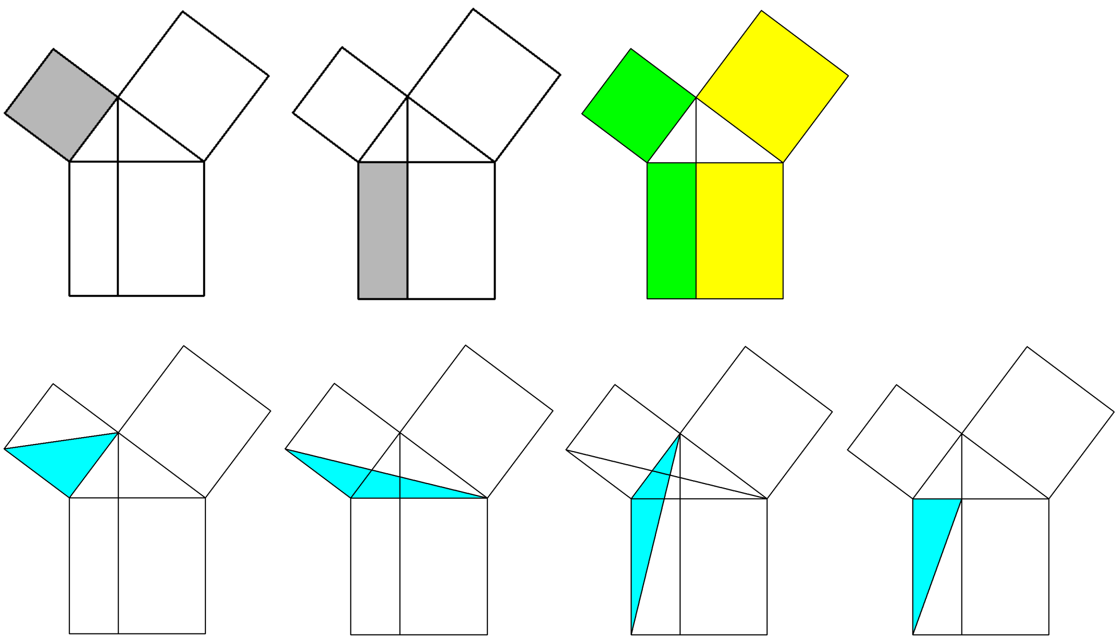
In terse prose, EUCLID derives, on the basis of 35 definitions of basic terms (such as "*a point is that which has no part*" and "*a line is breadthless length*") and five postulates (see below), as well as axioms of logic (such as "*Things which are equal to the same thing are also equal to one another*"), more than 450 mathematical propositions in strict logical sequence, one after another. The proofs of the propositions use only the definitions, postulates, axioms, and previously proved propositions. A proof should employ only statements that have been previously proven (or whose validity is guaranteed by the postulates and axioms).

In geometry in particular, there is the danger that one might attempt to draw on intuition in carrying out proofs, which is impermissible if a proof is to be rigorous.

Here are the five postulates of EUCLID:

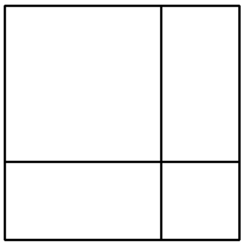
> *Let the following be postulated:*
> 1. *to draw a straight line from any point to any point,*
> 2. *to produce a finite straight line continuously in a straight line,*
> 3. *to describe a circle with any centre and distance,*
> 4. *that all right angles are equal to one another,*
> 5. *that, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

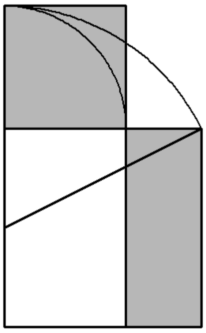
The first book of the *Elements* begins with proofs of certain constructions, such as the bisection of a line segment and the bisection of an angle. Then come theorems on angles (equality of the base angles in an isosceles triangle, equality of vertical and alternate angles, angle sum in a triangle), congruence theorems for triangles, and theorems on equal areas of triangles and parallelograms with equal bases and altitudes.
The finale is the proof of the *Pythagorean theorem* and its converse (Propositions 47 and 48). The proof of Proposition 47 is accomplished through use of the theorem that is known today as *EUCLID's theorem* or the *cathetus theorem*. The theorem is justly named, for before EUCLID, the theorem was proved with the aid of proportions, while here use is made of congruent triangles and the equal areas of triangles and half rectangles.



The second book, whose contents are described as geometric algebra, deals with the determination of areas, which, from today's point of view, may also be interpreted in such a way that they allow for the establishment of formulas and the solution of equations.

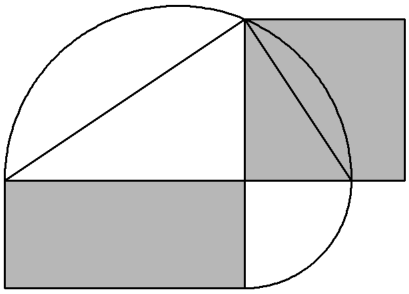
For example: "*If a straight line be cut at random, the square on the whole ist equal to the squares on the segments and twice the rectangle contained by the segments*". This proposition can be interpreted as the first *binomial formula*: $(a+b)^2 = a^2 + b^2 + 2ab$





Moreover, the quadratic equation (associated with what we today call the *golden section*) $a \cdot (a-x) = x^2$ is solvable by the transformation of areas (a line segment can be divided in such a way that the rectangle with sides the entire segment and one of the smaller segments is equal in area to the square with side the remaining smaller segment).

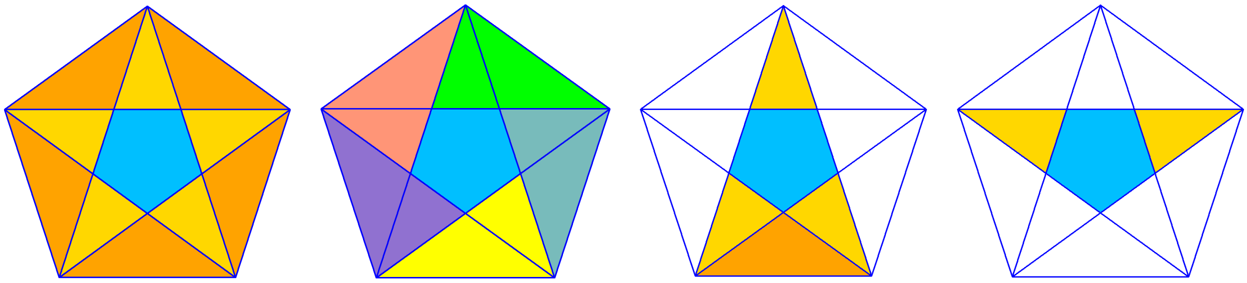
Finally, the second book contains a proof that every plane figure bounded by straight line segments can be transformed into a square of equal area; a special case of this, known today as EUCLID's *geometric mean theorem*, gives the transformation of a rectangle into a square of equal area.



The third book covers the geometry of the circle. It contains, among other things, the construction of a tangent to a circle, the theorems on central and inscribed angles (as a special case of THALES' theorem) and on angles in circumscribed quadrilaterals, as well as the theorem on intersecting chords: *If in a circle two straight lines cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other*.

In the fourth book, one encounters the constructions of the incircle and circumcircle of a triangle as well as the constructions of the regular pentagon, hexagon, and through a combination of constructions, the regular 15-gon as well as their incircles and circumcircles.

The fifth book, which harks back to the theory of proportions of EUDOXUS OF CNIDUS, deals with magnitudes and relationships. It represents an attempt to create a foundation for the calculation of incommensurable quantities. About 150 years prior to EUCLID, HIPPASUS OF METAPONTUM had discovered that there is no common measure for the side length of a regular pentagon and the length of a diagonal (thus if a side has length a rational number, the diagonal has length an irrational number).



The *similarity theory* of the sixth book contains, among other things, the theorem that an angle bisector in a triangle divides the opposite side in the ratio of the adjacent sides (*angle bisector theorem*), as well as the important property that the ratio of the areas of two similar figures is equal to the square of the similarity ratio of the sides. In addition, this book deals with the proportions $a : x = x : (a - x)$ and $a : x = x : b$ (cf. Book 2).

At the beginning of the seventh book there appear definitions of odd and even numbers, divisors and multiples, *prime numbers*, squares, and cubes, as well as *perfect numbers*. Proposition 2 introduces the method of what is today known as the *Euclidean algorithm* for determining the *greatest common divisor* (*gcd*) of two natural numbers:

Two given numbers are interpreted as the lengths of segments |AB| and |CD|. If the smaller segment CD does not measure AB (that is, if |AB| is not a multiple of |CD|) and if one by turns subtracts the smaller segment from the larger, then eventually, a segment must remain that measures the original segments.

Example: We seek the greatest common divisor of 6 and 15:
- Subtract the smaller from the larger: 15 – 6 = 9; the larger is now 9.
- Subtract the smaller from the larger: 9 – 6 = 3; the larger is now 6.
- Subtract the smaller from the larger: 6 – 3 = 3.
- 3 now measures both 6 and 15.

EUCLID recognized that in principle, every natural number can be decomposed into prime factors (Proposition 30: *If two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers*). The uniqueness of prime factorization was first proved only 2100 years later, by CARL FRIEDRICH GAUSS, in his *Disquisitiones Arithmeticae*.

At the end of the seventh book can be found a method for determining the *greatest common multiple* (*gcm*) of two numbers: *gcm*(*a*,*b*) = (*a* · *b*) / *gcd*(*a*,*b*).

The proof uses the property that computing the greatest common divisor of a and b yields the greatest common measure of the two associated line segments.

The eighth book of the *Elements* deals with continued proportions of natural numbers (in today's language, with *geometric sequences*). For example, the three numbers $a = 2^2$, $b = 2 \cdot 3$, $c = 3^2$ stand in proportion $a : b = b : c$. They form the beginning of a geometric sequence with 4 as the first term and with constant factor $q = 3/2$. Given two perfect squares *a* and *c*, there always exists a natural number *b* with the property that $a : b = b : c$.

This book also contains theorems like the following: *If a square measures a square, the side will also measure the side, and, if the side measure the side, the square will also measure the square*. Equivalently, a^2 divides b^2 if and only if *a* divides *b*. One can see at once from this example that for EUCLID, numbers are always to be interpreted geometrically: square numbers as geometric squares, products of two numbers as rectangles, and products of three numbers as rectangular parallelepipeds.

The ninth book opens with the study of properties of squares and cubes and the relationships between them, and then, without any transition, there follows Proposition 20, which today bears the name EUCLID's theorem:

- *Prime numbers are more than any assigned multitude of prime numbers.*

In his proof, EUCLID considers three prime numbers *a*, *b*, *c* and adds to the product of the three numbers the unit 1. Then this new number is either itself a prime number, or it is not. If it is a prime number, then with $a \cdot b \cdot c + 1$, we have found a further prime number (since it is larger than each of *a*, *b*, *c*). If, on the other hand, it is not prime, then there is some prime number *p* that divides it. It cannot be one of *a*, *b*, *c*, because then *p* would divide both the product $a \cdot b \cdot c$ and the number $a \cdot b \cdot c + 1$. But then *p* would also divide the difference, namely the unit 1, which is true for no prime number. Therefore, we have found a new prime number.

After a series of theorems about even and odd numbers (for example, Proposition 22: *If as many odd numbers as we please be added together, and their multitude be even, the whole will be even*), the book ends with Proposition 36, which has something to say about perfect numbers:

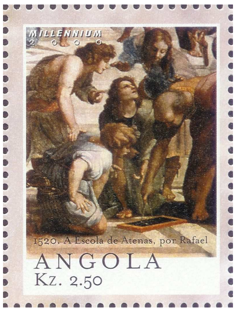
If as many numbers as we please beginning from an unit be set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect. (If the sum $1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1}$ is prime, then the number $\left(1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1}\right) \cdot 2^{n-1}$ is a perfect number.)

Note that a natural number is called perfect if it is the sum of its proper divisors. (The sum $1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1}$ can also be written as $2^n - 1$.)

Examples: For *n* = 2, we have that $1 + 2 = 3$ is prime. Therefore, $3 \cdot 2^1 = 6$ is a perfect number, and indeed, $1 + 2 + 3 = 6$. For *n* = 3, we see that $1 + 2 + 2^2 = 7$ is prime. Therefore, $7 \cdot 2^2 = 28$ is a perfect number: $1 + 2 + 4 + 7 + 14 = 28$. Perfect numbers are obtained as well for the cases n = 5 and n = 7.

Two thousand years later, LEONHARD EULER proved that all even perfect numbers can be generated by this method. (To date, "only" 48 perfect numbers have been found, and it is unknown whether there exist any odd perfect numbers. It is also unknown whether there are any perfect numbers other than those that have already been discovered, and in particular, whether their number is infinite or finite.)

Book 10 deals extensively with the problem of commensurable and incommensurable lengths and their associated constructions.



Book 11 begins with 28 definitions of concepts in the theory of three-dimensional space. It is conjectured that here, Euclid relied on sources older than those used for the earlier books on plane geometry, since no postulates are formulated for solid geometry.

From today's viewpoint, "Propositions" 2 and 3 are more along the lines of postulates for the introduction of planes. (*If two straight lines cut one another, they are in one plane, and every triangle is in one plane. If two planes cut one another, their common section is a straight line.*)

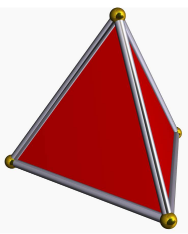
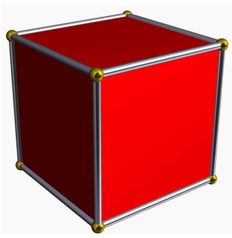
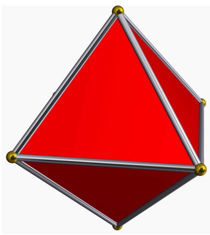
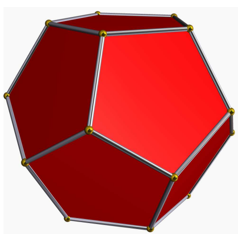
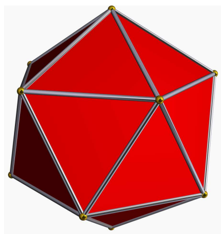
Following some theorems on parallel lines, the investigation turns to solid angles: *Any solid angle is contained by plane angles less than four right angles* (Proposition 21).

Statements are then given on the volumes of parallelepipeds: *Parallelepipedal solids which are on equal bases and of the same height are equal to one another* (Proposition 31).

Before Euclid, in Book 12, investigates pyramids, prisms, cylinders, cones, and spheres, he proves the following result: *Circles are to one another as the squares on the diameters* (Proposition 2). Then follows the fact that triangular prisms can always be decomposed into three pyramids of equal volume with triangular base (Proposition 7). From this, it follows (using the method of exhaustion) that the volume of a cylinder is equal to one-third the volume of a cone with the same area of the base and the same altitude (Proposition 10).

The book ends with Proposition 18: *Spheres are to one another in the triplicate ratio of their respective diameters*.

The Elements end with Book 13, containing propositions on the regular pentagon that form the basis for the construction of the sides of the five regular polyhedra (tetrahedron, hexahedron, octahedron, dodecahedron, icosahedron) inscribed in a sphere (Propositions 13–17). Proposition 18 is then given and proved: that these five solids are the only possible regular polyhedra.

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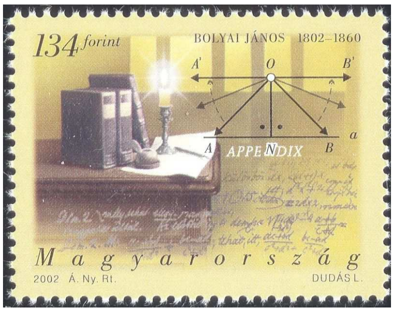
It is a remarkable fact that this 2300-year-old work remained a model over so many centuries for generations of mathematicians. Whenever a scientist described his own methods as *more geometrico* (in the manner of geometry), he was paying homage to the rigour of the methods presented in EUCLID's *Elements*.

Over the centuries, weak points in the Elements were discovered: holes in proofs were discovered and repaired, along with the discovery that some "proofs" were deeply flawed. A lack of comfort with the given postulates and axioms was articulated.

All of this – in particular the controversy over the fifth postulate (parallel postulate) – led in the nineteenth century to the discovery of *non-Euclidean geometries* by JÁNOS BOLYAI und NIKOLAI IVANOVICH LOBACHEVSKY and efforts to make the foundations of geometry more precise, which came to fulfilment with the work of that name (*Grundlagen der Geometrie*) by DAVID HILBERT in the year 1899.



EUCLID wrote other books that survived: *Data* (properties of geometric figures), *On Divisions of Figures* (constructions used in dividing figures into prescribed proportions), *Sectio canonis* (music theory), *Catoptrics* (the mathematical theory of mirrors), *Phaenomena* (spherical astronomy), and *Optics*. There are also books by EUCLID that have been lost, with only their names surviving, one of which was *Conics* (on conic sections).

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https://www.spektrum.de/wissen/euklid-um-300-v-chr/1024565

Translated by David Kramer

Quotations from Euclid's *Elements* translated
by Sir Thomas Little Heath, Cambridge University Press, 1908
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