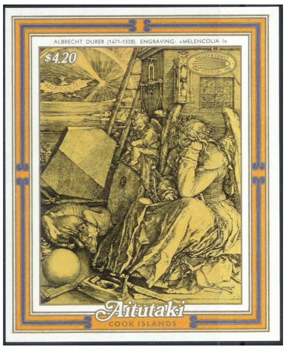
**LEONHARD EULER** (April 15, 1707 – September 18, 1783)

by HEINZ KLAUS STRICK, Germany


Without a doubt, LEONHARD EULER was the most productive mathematician of all time. He wrote numerous books and countless articles covering a vast range of topics in pure and applied mathematics, physics, astronomy, geodesy, cartography, music, and shipbuilding – in Latin, French, Russian, and German. It is not only that he produced an enormous body of work; with unbelievable creativity, he brought innovative ideas to every topic on which he wrote and indeed opened up several new areas of mathematics. Pictured on the Swiss postage stamp of 2007 next to the polyhedron from DÜRER's *Melencolia* and EULER's *polyhedral formula* is a portrait of EULER from the year 1753, in which one can see that he was already suffering from eye problems at a relatively young age.


EULER was born in Basel, the son of a pastor in the Reformed Church. His mother also came from a family of pastors. Since the local school was unable to provide an education commensurate with his son's abilities, EULER's father took over the boy's education. At the age of 14, EULER entered the University of Basel, where he studied philosophy. He completed his studies with a thesis comparing the philosophies of DESCARTES and NEWTON. At 16, at his father's wish, he took up theological studies, but he switched to mathematics after JOHANN BERNOULLI, a friend of his father's, convinced the latter that LEONHARD possessed an extraordinary mathematical talent.

At 19, EULER won second prize in a competition sponsored by the French *Academy of Sciences* with a contribution on the question of the optimal placement of a ship's masts (first prize was awarded to PIERRE BOUGUER, participant in an expedition of LA CONDAMINE to South America). One can assume that at the time, EULER had not yet seen an oceangoing vessel!




EULER went on to win this prestigious annual scientific competition a total of twelve times.

After an unsuccessful application for an open professorship in physics in Basel, EULER followed DANIEL and NICOLAUS II BERNOULLI in 1727 to St. Petersburg to join the Russian *Academy of Sciences*, which had been founded three years earlier by PETER THE GREAT. His appointment required him to give lectures on anatomy and physiology in the Faculty of Medicine, but in 1730 he became a professor of physics, and in 1733 he took over the chair in mathematics vacated by DANIEL BERNOULLI.


In 1734, he married KATHARINA GSELL, the daughter of a Swiss painter at the Petersburg Academy. It was a happy marriage, producing thirteen children, of whom only five survived to adulthood.

Internal political turmoil in Russia prompted EULER to move from St. Petersburg to Berlin, where he became director of the mathematical division of the Prussian Academy of Sciences.


The first president of the Academy was PIERRE LOUIS MOREAU DE MAUPERTUIS and EULER supported him intensively in his work.  In the twenty-five years that he spent in Berlin, not only did he write 380 scientific articles, he also looked after the observatory, as well as the botanical gardens and the water supply for the gardens at Sanssouci, King FREDERICK THE GREAT's summer residence in Potsdam.


With his *Lettres à une princesse d'Allemagne sur quelques sujets de Physique et Philosophie* (letters to a German princess on some topics in physics and philosophy), he produced texts for a lay readership, similar to his *Complete Introduction to Algebra* of 1770, the text of which he gave to a tailor to review to ensure a suitable level of understandability. He provided a stimulus to physics with the method that he developed for the solution of differential equations (equations in which the unknown function appears along with one or more derivatives).

During his twenty-five years in Berlin, EULER maintained contact with St. Petersburg, corresponding intensively with his friend CHRISTIAN GOLDBACH (1690–1764), most of whose questions he was able to answer. Most, but not all.

For even today, the GOLDBACH conjecture of 1742 remains unsolved:

- *Can every even number greater than 2 be expressed as the sum of two prime numbers?*

In 1766, Euler returned to St. Petersburg. In Berlin, his place was assumed by Joseph-Louis Lagrange (1736–1813).


Although he became completely blind shortly after his return to St. Petersburg, he produced more than four hundred additional works, which he dictated to his son and son-in-law. On his death in 1783, his oeuvre comprised 866 works (available at http://math.dartmout.deu/ ~euler/). Pierre Simon Laplace (1749–1827) commented on this abundance with the following words:

*Read Euler, read Euler; he is the master of us all.*

With his textbooks on the analysis of the infinite (*Introductio in analysin infinitorum*), on differential calculus (*Institutiones calculi differentialis*), and integral calculus (*Institutiones calculi integralis*), Euler contributed greatly to the systemization of mathematical analysis. He introduced a great number of mathematical concepts and notations, including the concept of a function, modern notation for the trigonometric functions, the notation $f(x)$, the summation symbol $\Sigma$, and the notation $i$ for the imaginary unit. Euler also introduced the notation $e$ for the base of the natural logarithm. Today, the number is often referred to as *Euler's number*, although it was discovered long before Euler by John Napier (1550–1617).

In 1734, Euler astounded the scientific world with a sensational discovery: That the harmonic series $\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)_{n\in IN}$ diverges had already been proved by a variety of methods. It was conjectured that the analogous series of reciprocals of squares converged. Indeed, Johann Bernoulli had found a majorant for the series $\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}\right)_{n\in IN}$ that converged to the value 2. However, he was unable to determine the limiting value, just as before him Leibniz, Stirling, de Moivre, and others had also failed.

Euler exhibited a rare virtuosity in dealing with the power series developments of functions. The sine function can be written as follows in a power series: $\sin(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$

From this, Euler wrote down the representation: $f(x)=\frac{\sin(x)}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots$

This function takes on the value 1 at $x = 0$, and the zeros are located at $\pm\pi, \pm 2\pi, \pm 3\pi, \ldots$ .

The function can also be written as an infite product of linear factors:

$$f(x)=\frac{\sin(x)}{x}=(1-\frac{x}{\pi})(1+\frac{x}{\pi})(1-\frac{x}{2\pi})(1+\frac{x}{2\pi})(1-\frac{x}{3\pi})(1+\frac{x}{3\pi})\cdot\ldots=(1-\frac{x^2}{\pi^2})(1-\frac{x^2}{4\pi^2})(1-\frac{x^2}{9\pi^2})\cdot\ldots$$

If we now multiply out the product of linear factors and compare coefficients of $x^2$, we obtain

$\frac{1}{3!}=\frac{1}{6}=\frac{1}{\pi^2}+\frac{1}{4\pi^2}+\frac{1}{9\pi^2}+\frac{1}{16\pi^2}+\ldots$ that is,

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots=\frac{\pi^2}{6}=1{,}6449340668\ldots$$

If one considers only the even powers of this infinite series, one sees that

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\frac{1}{64}+\ldots=\frac{1}{4}\cdot(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots)=\frac{\pi^2}{24}=0{,}4112335167\ldots$$

For the odd powers, the result is

$$1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\ldots=\frac{\pi^2}{6}-\frac{\pi^2}{24}=\frac{\pi^2}{8}=1{,}233700550\ldots$$

Moreover, one obtains

$$f(\frac{\pi}{2})=\frac{1}{\pi/2}=(1-\frac{1}{4})(1-\frac{1}{16})(1-\frac{1}{36})\cdot\ldots=\frac{3}{4}\cdot\frac{15}{16}\cdot\frac{35}{36}\cdot\ldots=\frac{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot\ldots}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot\ldots}$$

a result that JOHN WALLIS (1616–1703) had obtained in 1650 by a completely different method.

In the seventeenth century, we should mention, it was still common practice to work with infinite sequences and their limits intuitively, without worrying about proofs of convergence.

Using the same method of comparing coefficients, EULER derived the result that the sum of the reciprocals of fourth powers of the natural numbers converges to $\pi^4/90$ and that the analogous sum for sixth powers converges to $\pi^6/945$. He continued these derivations up to the 26th power.

In contrast, even today, no method has yet been found to determine a closed-form expression for the limits of the sums of reciprocal *odd* powers.

EULER generalized the problem of sums of reciprocal powers by introducing the zeta function:

$\zeta(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\ldots$ with

$\zeta(s)=\frac{1}{1-\frac{1}{2^s}}\cdot\frac{1}{1-\frac{1}{3^s}}\cdot\frac{1}{1-\frac{1}{5^s}}\cdot\ldots$ , where only prime powers appear.

EULER discovered that while the sequence of sums of fractions with 1 in the numerator (the harmonic series) diverges, it does so very slowly, and that one can compare the values of this sequence with the natural logarithmic function. The sequence

$$c_n=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}-\log(n)$$

converges to $\gamma$= 0.577 215 664 . . . .

In 1781, EULER calculated this limit value to 16 decimal places. LORENZO MASCHERONI (1750–1800) did the calculation to 32 places. Today, the number $\gamma$ is called the EULER-MASCHERONI constant.


Simultaneously with JEAN D'ALEMBERT (1717–1783), EULER developed a formal system of calculation with complex numbers and investigated functions of complex variables; however, complex numbers had no real significance for EULER.

In 1743, in studying power series with complex arguments, he discovered the following simple relationship between the trigonometric functions and the exponential function:

$$\cos(z)+i\cdot\sin(z)=\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\dots\right)+i\cdot\left(\frac{z}{1!}-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\dots\right)=1+\frac{iz}{1!}+\frac{(iz)^2}{2!}+\frac{(iz)^3}{3!}+\frac{(iz)^4}{4!}+\dots=e^{iz}$$

(EULER's formula). For $z=\pi$, one obtains the remarkable relationship $e^{i\pi}+1=0$, while for $z=\frac{1}{2}\cdot\pi$ the result is $e^{i\cdot\frac{1}{2}\pi}=\cos(\frac{1}{2}\pi)+i\cdot\sin(\frac{1}{2}\pi)=i$, from which it follows that $i^i=(e^{i\cdot\frac{1}{2}\pi})^i=e^{-\frac{1}{2}\pi}$.


From the relationships $\int_0^\infty x^0e^{-x}dx=1$, $\int_0^\infty x^1e^{-x}dx=1$, $\int_0^\infty x^2e^{-x}dx=2$, $\int_0^\infty x^3e^{-x}dx=6$, ...

Euler got the idea of defining $\Gamma(z)=\int_0^\infty e^{-x}x^{z-1}dx$ (called the gamma function) to interpolate

between the values of the factorial function and extend the definition to arbitrary $z\in\mathbb{R}$. Namely, one has in general, $\Gamma(z+1)=z\cdot\Gamma(z)$ and $\Gamma(1)=1$. Therefore, for $n\in\mathbb{N}$, we have $\Gamma(n+1)=n!$.

In the field of number theory, EULER supplied three different proofs of the *Little FERMAT theorem*, which PIERRE DE FERMAT had formulated in 1634 but been unable to prove. FERMAT's conjecture that all integers of the form $2^{2^n}+1$ are prime was disproved by Euler when he factored $2^{32}+1=4\,294\,967\,297$ into the product 641 · 6 700 417.

In looking for this factorization, EULER was able to use FERMAT's little theorem and a theorem about the unique representation of prime numbers of the form 4*k* + 1, namely that every prime number that on division by 4 gives remainder 1 can be represented uniquely (that is, in only one way) as the sum of two squares (and conversely).

This theorem was also discovered by FERMAT, but it was only with EULER that a proof was found. Fermat's conjecture that the equation $x^n+y^n=z^n$ has no solutions for $n > 2$ was proved by Euler in the case $n = 3$.



In connection with his third proof of FERMAT's little theorem, EULER defined in 1763 the function $\varphi(n)$, which is now called EULER's function in his honor. EULER's function associates with every natural number $n$ the number of natural numbers $k$ with $k \le n$ that are relatively prime to $n$. The function has the multiplicative property that $\varphi(a\cdot b)=\varphi(a)\cdot\varphi(b)$ whenever $a$ and $b$ are relatively prime. Remarkably, this function can be written as a product whose factors involve the reciprocals of the prime divisors of $n$, and yet the product of these factors is always an integer:

$$\varphi(n) = n \cdot (1-\frac{1}{p_1}) \cdot (1-\frac{1}{p_2}) \cdot (1-\frac{1}{p_3}) \cdot \ldots \cdot (1-\frac{1}{p_r})$$

For example, 10 is relatively prime to 1, 3, 7, 9, and we have

$$\varphi(10) = 10 \cdot (1-\frac{1}{2}) \cdot (1-\frac{1}{5}) = 10 \cdot \frac{1}{2} \cdot \frac{4}{5} = 4$$

Greek and Islamic mathematicians were fascinated by the so-called *amicable numbers* 220 and 284, which have the property that the sum of the proper divisors of each of the numbers is equal to the other:

220 = 1 + 2 + 4 + 71 + 142, 284 = 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110.

Around 1300, the Moroccan mathematician IBN AL-BANNA AL-MARRAKUSHI AL-AZDI (1256–1321) discovered that the numbers 17296 and 18416 also possess this remarkable property. In 1638, DESCARTES discovered yet another pair: 9 363 584 and 9 437 056. Between 1747 and 1750, EULER discovered an additional 58 pairs of numbers with this property because he was able to give a prescription for forming such pairs. Today, it is known that there also exist pairs of amicable numbers that cannot be found through EULER's method.

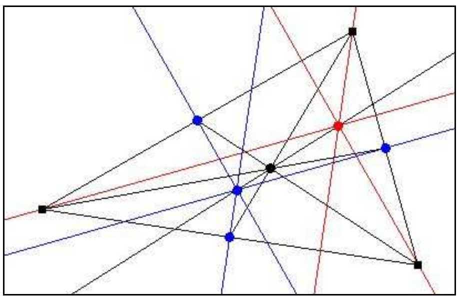
EULER derived a technique for determining the number of possible decompositions of a natural number as a sum of distinct natural numbers, and he proved that this number is the same as the number of all possible decompositions of a natural number (that is, with not necessarily distinct summands) into odd summands. His proof also made use of infinite sums and products.

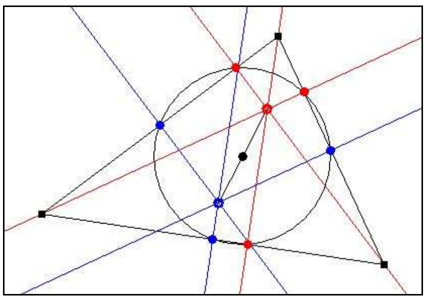
For example, the number 7 can be decomposed by each method in four ways:

7 = 1+6 = 2+5 = 3+4 = 1+2+4 und 7 = 5+1+1 = 3+3+1 = 3+1+1+1+1 = 1+1+1+1+1+1+1

EULER also devoted considerable effort to the study of geometric problems. For instance, he discovered that in every triangle, the point of intersection $S$ of the medians (*centroid* = the triangle's center of mass), the point of intersection $H$ of the altitudes (*orthocenter*), and the point of intersection $M$ of the perpendicular bisectors of the sides (*circumcenter* = the center of the triangle's circumcircle) are collinear; the line on which they lie is called the *EULER line*, and the segment $MH$ is divided by the point $S$ in the proportion 1 to 2.

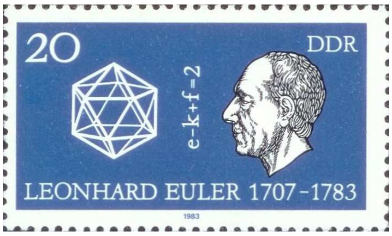
Furthermore, one can draw a circle, known as *EULER's circle* or the *nine-point circle*, with center at the midpoint of the segment $MH$ and passing through the feet of the three altitudes and the midpoints of the three sides.





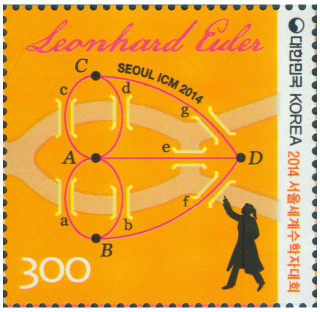

Around 1750, EULER discovered that for convex polyhedra, the numbers $V$ of vertices, $E$ of edges, and $F$ of faces are related by the simple formula

$V - E + F = 2$, known as EULER's *formula for polyhedra*.




In 1736, Euler solved a problem posed by the mayor of Danzig, which became famous as the problem of the *seven bridges of Königsberg*:

- *Is it possible to take a walk through the city of Königsberg in such a way that one crosses each of the seven bridges over the Pregel River connecting the island with the surrounding parts of the city exactly once before returning to one's starting point?*




In his thirteen-page solution, Euler explained why the proposed walk is impossible, and he formulated a general criterion for the existence of such a circuit. He was not satisfied with the mathematical content of his investigation: *You see, then . . . that the nature of this solution can scarcely be called mathematical, and I fail to understand why it should have been sought from a mathematician rather than from anyone else, since the solution is based purely on common sense, and its discovery did not require the invocation of any mathematical principles.*

However, this paper marked the beginning of a whole new area of mathematics: graph theory. Today, we call a graph *Eulerian* if it contains a closed path that traverses each edge exactly once. Euler's general criterion mentioned in the preceding paragraph is quite simple: a graph is Eulerian if and only if at every vertex, the number of edges that meet there is an *even* number.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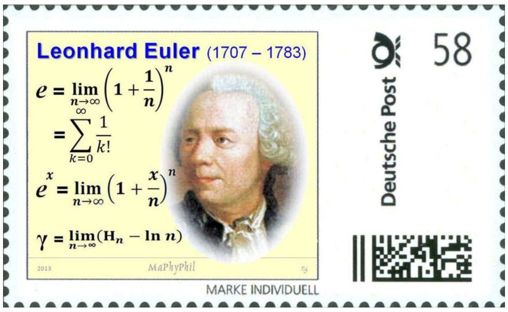


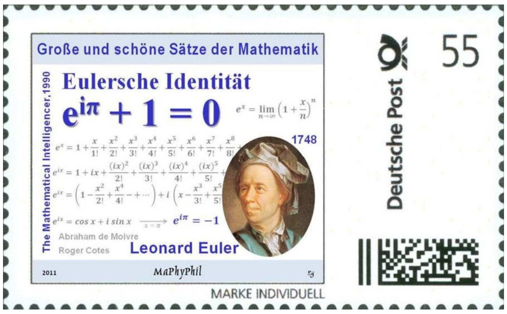


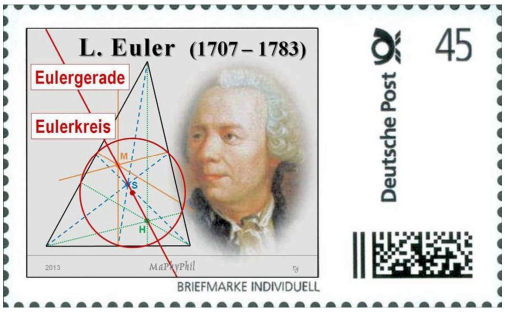


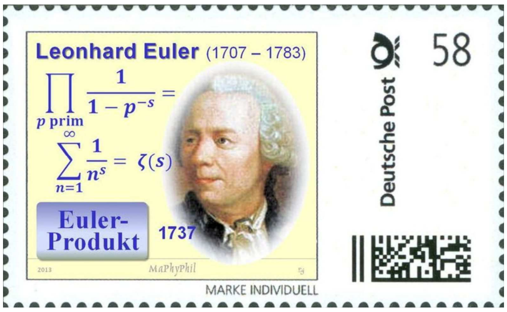


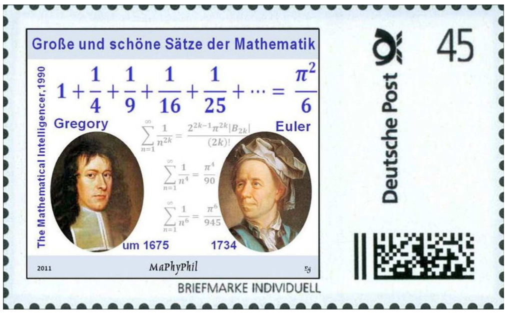


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