CARL FRIEDRICH GAUSS (April 30, 1777 – February 23, 1855)

by HEINZ KLAUS STRICK, Germany

Even during his lifetime, the Braunschweig (Brunswick) native CARL FRIEDRICH GAUSS was called *princeps mathematicorum*, the prince of mathematics. The number of his important mathematical discoveries is truly astounding.

His unusual talent was recognized when he was still in elementary school. It is told that the nine-year-old GAUSS completed almost instantly what should have been a lengthy computational exercise.

The teacher, one Herr Büttner, had presented to the class the addition exercise

$1 + 2 + 3 + \cdots + 100$.

GAUSS's trick in arriving at the sum 5050 was this: Working from outside to inside, he calculated the sums of the biggest and smallest numbers, $1 + 100$, $2 + 99$, $3 + 98$, . . . , $50 + 51$, which gives fifty times 101.

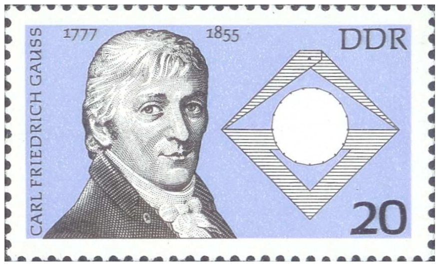
Herr BÜTTNER realized that there was not much he could offer the boy, and so he gave him a textbook on arithmetic, which GAUSS worked through on his own. Together with his assistant, MARTIN BARTELS (1769-1836), BÜTTNER convinced the boy's parents, for whom such abilities were outside their ken (the father worked as a bricklayer and butcher; the mother was practically illiterate), that their son absolutely had to be placed in a more advanced school.

From age 11, GAUSS attended the *Catherineum high school*, and at 14, he was presented to Duke CARL WILHELM FERDINAND VON BRAUNSCHWEIG, who granted him a stipend that made it possible for him to take up studies at the *Collegium Carolinum* (today the University of Braunschweig).

So beginning in 1795, GAUSS studied mathematics, physics, and classical philology at the University of Göttingen, which boasted a more extensive library. His physics professor, GEORG CHRISTOPH LICHTENBERG (1742–1799), awakened in GAUSS a lifelong interest in experimentation. After a period of uncertainty as to what subject he should concentrate in – indeed, GAUSS was also quite talented linguistically – a brilliant idea for the solution of a millennia-old geometric problem finally, on March 30, 1796, tipped the balance in favor of mathematics.

GAUSS's insight was the realization of just which regular polygons could be constructed with straightedge and compass: a regular n-gon is constructible with straightedge and compass if and only if n has as its divisors only powers of 2 and FERMAT primes, that is, primes of the form

$p = 2^{2^m} + 1$ ($m \in \mathbb{N}_0$).



Since GAUSS's discovery no further progress on this problem has been made (namely, only five FERMAT primes are known, and it is unknown whether the number of such primes is finite). In particular, GAUSS was able to give a construction for $p = 2^{2^2} + 1$, that is, for the regular 17-gon.

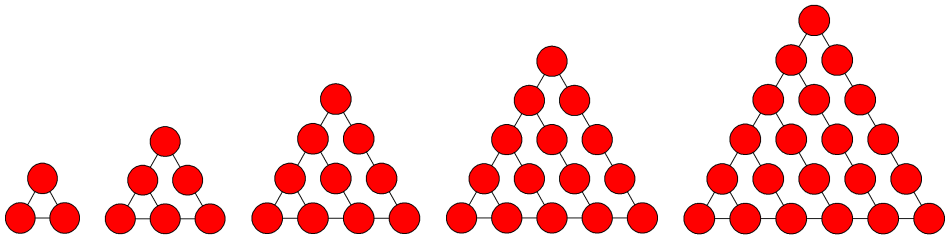
It is with this result that GAUSS began a diary, written in Latin, in which by 1814 he had written down 146 "discoveries". This diary was discovered only years after his death and published much later. Many priority disputes were thereby resolved in GAUSS's favor.

For example, one knows from the diary entries that even in his adolescence, GAUSS had worked on problems of the distribution of prime numbers. Perhaps the continual use of tables of logarithms as a calculational aid, and particularly calculation with the number $\ln(10) \approx 2.3$, led GAUSS at age 15 to the conjecture that $\pi(x)$, the number of primes less than x, is approximately proportional to $x/\ln(x)$, a statement whose precise proof was achieved only one hundred years later.

x	10	100	1.000	10.000	100.000	1.000.000	10.000.000
$\pi(x)$	4	25	168	1229	9592	78498	664579
$x/\pi(x)$	2,5	4,0	6,0	8,1	10,4	12,7	15,0

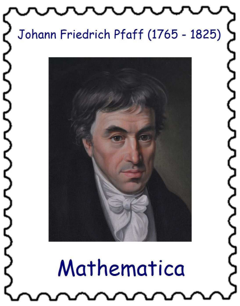
increment +2,0 +2,1 +2,3 +2,3 +2,3

One also finds the word *Ευρηκα*, ("eureka!", I have found it!) in a diary entry from 1796 together with num = Δ + Δ + Δ. GAUSS had discovered a proof of a conjecture made by PIERRE DE FERMAT (1608-1665): Every positive integer can be represented as the sum of at most three triangular numbers, where the sequence of triangular numbers is given by 1, 3, 6, 10, 15, 21. . . .



At the University of Göttingen, GAUSS became friends with FARKAS BOLYAI (1775–1856), with whom he maintained contact throughout his life.

He ended his studies without taking any examinations; however, his sponsor and sovereign insisted that he took his doctoral degree from the "domestic" University of Helmstedt. His dissertation, written in Latin, of 1799 is dedicated with enormous gratitude to his sovereign "Serenissimo PRINCIPI AC DOMINO CAROLO GUILIELMO FERDINANDO". His dissertation adviser was the most respected German mathematician of the time, JOHANN FRIEDRICH PFAFF (1765-1825).



In his dissertation, Gauss takes a critical look at the proof of the fundamental theorem of algebra published by JEAN LE ROND D'ALEMBERT (1717-1783), and then gives a rigorous proof, avoiding, however, the use of complex numbers, which, in his opinion, still had no "right of citizenship" in mathematics. In the course of his life, he produced additional three proofs of this theorem (but now using calculation with complex numbers). The *fundamental theorem of algebra* states that every polynomial equation of degree n has precisely n solutions in the set of complex numbers.

On receiving his doctorate, GAUSS was granted an allowance by the duke while he worked on his book *Disquisitiones arithmeticae* (*Arithmetic Investigations*).

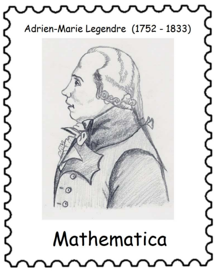
As an aside, in 1800 he published an algorithm for computing the date on which Easter falls:

From the year x one first calculates several auxiliary quantities: $a \equiv x \bmod 19$; $b \equiv x \bmod 4$; $c \equiv x \bmod 7$; $k \equiv x \; div \; 100$; $p \equiv (8k+13) \; div \; 4$; $q \equiv k \; div \; 4$; $M \equiv (15+k-p-q) \bmod 30$; $N \equiv (4+k-q) \bmod 7$; $d \equiv (19a+M) \bmod 30$; $e \equiv (2b+4c+6d+N) \bmod 7$.

Then Easter falls on the (22 + d + e)th day of March. If this number is greater than 31, then one must subtract 31 to obtain the date of Easter in April. The formula has the following exceptions: If $d+e = 35$, then Easter falls on April 19; if $d = 28$, $e = 6$, and $a > 10$, then Easter falls on April 18.

We have reproduced here the version of the algorithm from 1816; in the intervening years, GAUSS had introduced computation with congruences, in particular the "modulo" notation: $a \equiv b \bmod n$ (read "a is congruent to b modulo n") means that a and b have the same remainder on division by n; $a \; div \, b$ means the quotient $a \div b$ with the remainder being ignored.

The *Disquisitiones* appeared after a long delay in 1801. Each of the seven chapters, taken for itself alone, aroused international interest in mathematical circles. In his foreword, GAUSS expressly acknowledged his precursors PIERRE DE FERMAT, LEONHARD EULER, JOSEPH-LOUIS LAGRANGE (1736-1813), and above all ADRIEN-MARIE LEGENDRE (1752-1833), whose book *Essai sur la théorie des nombres* of 1797 had an unfortunate temporal overlap with the *Disquisitiones*.



(drawing: © Andreas Strick)

The individual chapters deal with the theory of arithmetic and congruences, with the proof of LEONHARD EULER's (1707-1783) conjecture on quadratic reciprocity, with the theory of quadratic forms (the solution of equations of the form $ax^2 + 2bxy + cy^2 = m$), with continued fractions and primality tests, and with the solution of equations of the form $x^n = 1$ ($n \in$ IN), and $x^m \equiv 1 \bmod p$.

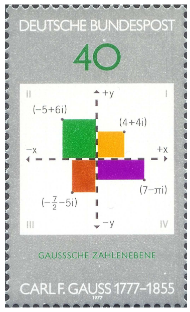
The law of quadratic reciprocity describes the conditions under which quadratic congruence equations are solvable.

- If p and q are prime numbers, then the two congruence equations $x^2 \equiv p \bmod q$ and $x^2 \equiv q \bmod p$ are either both solvable or both not solvable unless p and q each have remainder 3 on division by 4, in which case one equation is solvable, while the other is not.

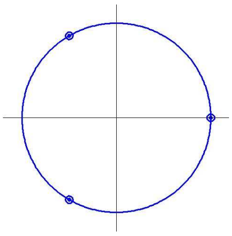
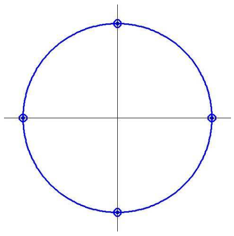
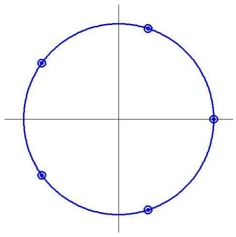
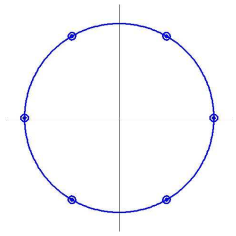
Here are a few examples:

The equation $x^2 \equiv 5 \bmod 7$ is not solvable; that is, there are no perfect squares in the sequence 5, 12, 19, 26, 33, 40, ..., since the "reciprocal" equation $x^2 \equiv 7 \bmod 5 \equiv 2 \bmod 5$ has no solution. There is no square whose final digit is 2 or 7. The equation $x^2 \equiv 5 \bmod 11$ has a solution, since it is clear that perfect squares appear in the sequence of numbers 5, 16, 27, 38, 49, 60, ... But then the reciprocal equation $x^2 \equiv 11 \bmod 5 \equiv 1 \bmod 5$ must have at least one solution (there exist squares with 1 or 6 as final digit). The equation $x^2 \equiv 3 \bmod 11$ is solvable, since in the sequence 3, 14, 25, 36, 47, 58, ... there are some perfect squares. But then the reciprocal equation $x^2 \equiv 11 \bmod 3 \equiv 2 \bmod 3$ has no solution, that is, in the sequence 2, 5, 8, 11, 14, 17, 20, 23, ... are to be found no perfect squares.

An equation of the form $x^n = 1$ is called a *cyclotomic equation* (from the Greek *kyklos* = circle, *temnein* = to cut), since one can write down the n solutions in the form $x_k = \cos\left(\frac{k \cdot 2\pi}{n}\right) + i \cdot \sin\left(\frac{k \cdot 2\pi}{n}\right)$ with $k = 0, 1, 2 \ldots, n-1$.



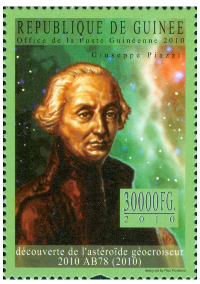
If one then draws these points, as was GAUSS's practice after 1820, in the complex plane (also known as the *GAUSSian plane*), in which the number 1 is drawn to the right of the origin and the number i above it, then these points form the vertices of a regular n-gon on the unit circle.

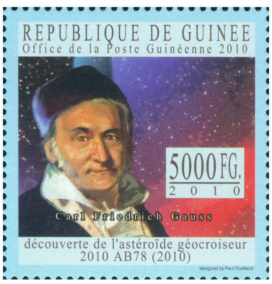
			
solutions of $x^3 = 1$:	solutions of $x^4 = 1$:	solutions of $x^5 = 1$:	solutions of $x^6 = 1$:
$x = 1 \vee x = -\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}$ $\vee x = -\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2}$	$x = +1 \vee x = +i \vee$ $x = -1 \vee x = -i$	$x = 1 \vee x = \frac{\sqrt{5}-1}{4} + i \cdot \frac{\sqrt{2\sqrt{5}+10}}{4}$ $\vee x = \frac{-\sqrt{5}-1}{4} + i \cdot \frac{\sqrt{-2\sqrt{5}+10}}{4}$ $\vee x = \frac{-\sqrt{5}-1}{4} - i \cdot \frac{\sqrt{-2\sqrt{5}+10}}{4}$ $\vee x = \frac{\sqrt{5}-1}{4} - i \cdot \frac{\sqrt{2\sqrt{5}+10}}{4}$	$x = 1 \vee x = \frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \vee$ $x = -\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2} \vee x = -1$ $\vee x = -\frac{1}{2} - i \cdot \frac{\sqrt{3}}{2}$ $\vee x = \frac{1}{2} - i \cdot \frac{\sqrt{3}}{2}$

GAUSS had suddenly become famous. As an expression of gratitude to the duke, he turned down an invitation to take up residence in Saint Petersburg, hoping that his sovereign would build him an observatory in Braunschweig. His particular interest in astronomy was strengthened by yet another sensational accomplishment, one that made his name known among nonmathematicians as well.

On January 1, 1801, the Italian astronomer GIUSEPPE PIAZZI discovered the asteroid Ceres, but then after a couple of days lost all trace of it as the asteroid vanished behind the sun. Using PIAZZI's data from his sightings, GAUSS calculated the asteroid's orbit using the method of least squares, making possible the rediscovery of Ceres the following year by HEINRICH OLBERS.

GAUSS had come up with the method of minimizing errors by considering the squares of deviations from a given model when he was only 17 years old. He presented the theory behind the method together with a description of its practical application to astronomy in 1809, in the paper *Motus corporum coelestium in sectionibus conicis solem ambientium* (the motion of the celestial bodies that orbit the sun in conic sections). Again here there is a priority dispute, since LEGENDRE proposed the same method independently of GAUSS in 1806.

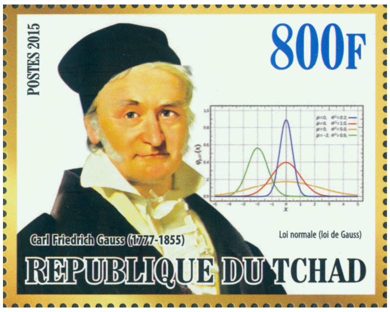




After the duke of Braunschweig was mortally wounded in 1807 in the Battle of Jena and Auerstedt, GAUSS had to move to Göttingen to assume a professorship of astronomy in order to provide for himself and his new family. He was not pleased that the post included teaching duties, and he made every effort to keep them to a minimum.

His happy marriage to JOHANNA OSTHOFF lasted only four years. She died giving birth to their third child. To provide care for his children, he married a friend of JOHANNA's, MINNA WALDECK, with whom he had an additional three children. When his second wife died in 1831, his youngest daughter, THERESE, took care of the household, in which GAUSS's mother had resided since the death of her husband; she died in 1839, at the age of 95.

As director of the observatory in Göttingen, GAUSS worked on improving the design of telescopes and investigated the question of constructing optics with minimal distortion. In this capacity he was in frequent contact with the director of the Königsberg observatory, FRIEDRICH WILHELM BESSEL (1784-1846).



In connection with his astronomical investigations, GAUSS discovered that random observational errors are normally distributed. The graph of the associated function (the *GAUSSian distribution*) has the shape of a *bell curve*, pictured on Germany's last ten-mark note (valid from 1989 to 2001).

In 1797, Gauss had his first experience as a surveyor, mapping the Kingdom of Westphalia as ordered by its ruler, King JÉRÔME, a brother of NAPOLEON. In 1818, not only did he undertake the direction of a survey of the Kingdom of Hanover, but he carried out the actual work over a period of 14 years, without considering the associated physical exertion. In the process, he improved the methods of geodetic surveying in many respects: He invented the *heliotrope*, a device that uses mirrors to direct sunlight in the direction of the observer, who can be up to one hundred kilometres away. He developed the mathematics of error calculation further, devised a method for the systematic solution of linear equations (*GAUSSian elimination*), and invented methods for a minimally distorted representation of the curved surface of the earth using cartographic projection (*GAUSSian coordinates*).

As an example of the use of *GAUSSian elimination*, the following system of three equations in three unknowns can be solved by bringing it step by step into triangular form and then solving the equations from bottom to top:

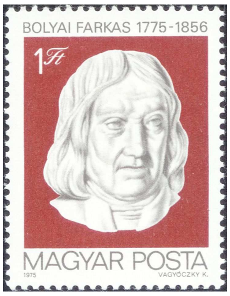
$$\left|\begin{array}{r} 1x+2y+1z=+8 \\ 1x-1y+3z=+8 \\ 1x-2y-1z=-6 \end{array}\right| \Leftrightarrow \left|\begin{array}{r} 1x+2y+1z=+8 \\ -3y+2z=+0 \\ -1y-4z=-14 \end{array}\right| \Leftrightarrow \left|\begin{array}{r} 1x+2y+1z=+8 \\ -3y+2z=+0 \\ -14z=-42 \end{array}\right| \Leftrightarrow \left|\begin{array}{l} x=+1 \\ y=+2 \\ z=+3 \end{array}\right|$$

Spurred on by his practical geodetic work, GAUSS worked intensively on questions of geodetic surveying and differential geometry, that is, the curvature of surfaces in three-dimensional space (one measure of curvature is called *GAUSSian curvature*) as well as the determination of the volume of curved surfaces, which can be computed with the aid of the *GAUSS integral theorem*.

Already during his geometry classes in school, Gauss began to doubt whether *Euclidean geometry* was the only possible geometric system. Do there exist, for example, geometries in which the angle sum of a triangle is not always equal to 180°?

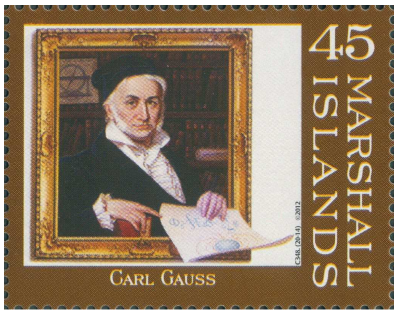
Euclid developed his geometry on the basis of systematic deduction from a set of five *axioms* or *postulates*. The fifth, known as the *parallel postulate* (*if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles*), appears to be less elementary than the other four, and for that reason, geometers searched over and over, and always in vain, to derive it from the others.

Around 1817, Gauss realized that such a derivation was impossible and instead investigated the question what type of geometry one might be able to construct if one threw out the fifth postulate. He discussed his methods with his friend Farkas Bolyai, but he decided against publishing his ideas. After all, it was only a few years previously that the philosopher Immanuel Kant had authoritatively declared in his *Critique of Pure Reason* that the geometry of Euclid was logically necessary and hence incontrovertible. Independently of Gauss and of each other, the Russian mathematician Nikolai Ivanovich Lobachevsky (1797-1856) and János Bolyai (1802-1860), the son of Farkas Bolyai, developed a “new” geometry without the parallel postulate and have since then been considered the discoverers of non-Euclidean geometry (the term is due to Gauss).



János was the first to publish, in 1823. Lobachevsky, who had studied mathematics in Russian Kazan under Martin Bartels (the one-time assistant in the Braunschweig elementary school), gave a lecture in 1826 on an “imaginary” geometry and wrote a number of articles on the subject. These became known in Europe only after 1837, when a paper appeared in French translation. Gauss is reported to have said regarding Bolyai’s discovery, “to praise him would simply be to praise myself,” and on Lobachevsky’s publication, that he had obtained the same insights 54 years earlier. However, he was so impressed with Lobachevsky’s work that he saw to it that he was named a corresponding member of the *Royal Academy of Sciences*. Having learned Russian on his own, Gauss was able to read Lobachevsky’s work in the original.

In 1828, at a conference in Berlin, Gauss met the young physicist Wilhelm Eduard Weber (1804-1891). He was able to bring him to Göttingen as professor of physics. Together, they developed a theory of magnetism, invented the magnetometer, and used the deviation of a compass needle in an electrically generated magnetic field to transmit information between the observatory and the physics institute. However, they did not develop the economic possibilities of this promising invention of the telegraph.



In 1836, together with the respected naturalist Alexander von Humboldt (1769-1859), they founded the *Göttingen Magnetic Union*, the first international research society, which set as its goal the worldwide investigation of the temporal and spatial changes in the Earth’s magnetic field.

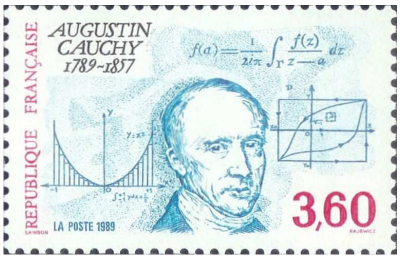
The centimetre gram second system (CGS) of units developed by Gauss and Weber was officially adopted in 1881 at a scientific congress in Paris; there the unit of magnetic induction, the gauss, was defined (1 gauss = 10^{-4} tesla). Today, one of the methods for determining the Earth's magnetic field is called *Gauss's law*.

In 1837, Weber had to leave Göttingen for political reasons (seven professors from Göttingen, including Gauss's son-in-law Heinrich Ewald and the brothers Jacob and Wilhelm Grimm, were dismissed because they had protested against the abolition of the constitution of the Kingdom of Hanover by the new king). Gauss could not bring himself to make a public statement on the matter, perhaps because of his conservative and monarchistic leanings.

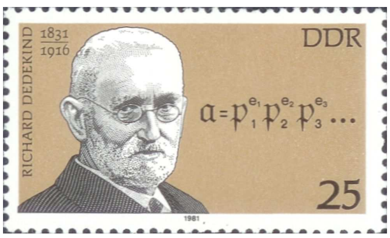
In 1839, Gauss drafted a general theory of attractive and repulsive forces (*potential theory*). In his later years, he wrote a report for the University of Göttingen's widow's bank, which included some of the first calculations of the amounts that should be contributed to a retirement account based on actuarial tables and probabilistic considerations.

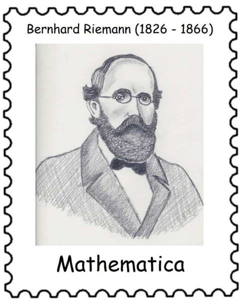
Gauss died in 1855, leaving behind a colossal lifetime achievement in many areas of mathematics, physics, and astronomy. In going over his papers and diaries, it was discovered that his apparent contempt for the pioneering work of several young mathematicians such as Niels Henrik Abel (1802-1829) and János Bolyai can be explained by the fact that he had made the same discoveries many years earlier but not published his results because they seemed to him incomplete – in accord with his motto, *pauca sed matura* (little but ripe).

For example, in 1811 he did not consider ripe a theorem on complex functions, the main theorem of complex analysis developed fourteen years later by Augustin Cauchy (1789-1857) or in 1819 his discovery of the noncommutative multiplication of four-dimensional objects, that is, the quaternions discovered by William Rowan Hamilton (1805-1865) in 1843.



Nevertheless, he supported some of the students who wrote their doctoral dissertations under his supervision, the last of whom were Richard Dedekind (1831-1916, *On the Elements of the Theory of Eulerian Integrals*) and Bernhard Riemann (1826-1866, *Hypotheses of the Foundations of Geometry*).





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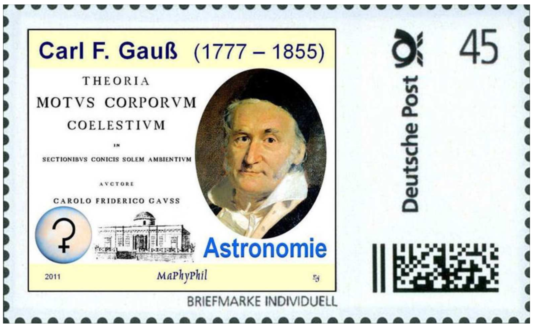

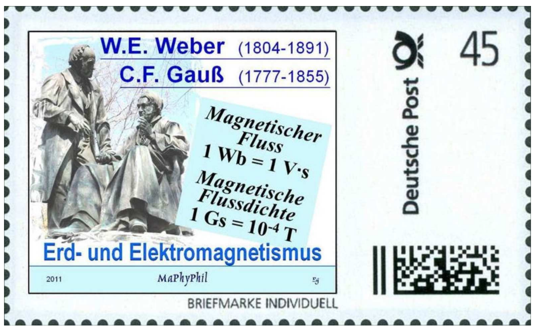

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