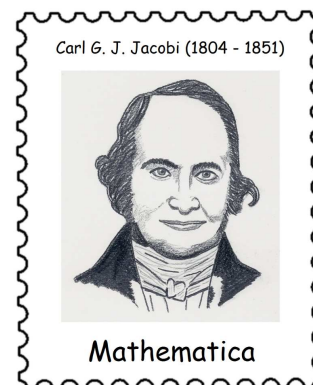


## CARL GUSTAV JACOB JACOBI (December 10, 1804 – February, 18, 1851)

by HEINZ KLAUS STRICK, Germany

JACQUES SIMON JACOBI, the second son of the wealthy Jewish banker SIMON JACOBI, was tutored privately by his uncle until he reached the age of 10. At that point, he entered a gymnasium; on the basis of his knowledge and abilities, he was placed in the top level (*prima*). But since he was too young at the end of the school year to be admitted to a university, he was compelled to repeat the same class for four years. His mathematics teacher soon noticed that there was no point in wasting the lad's time on the usual material.



(drawing: © Andreas Strick)

While his fellow pupils were working on the basics of algebra, young JACQUES was reading on his own EULER'S *Introductio in analysin infinitorum* and attempting to find a general solution to equations of the fifth degree by means of radicals (that is, using the four basic arithmetic operations and extraction of roots).



When he reached the age of 16, he was finally allowed to leave the gymnasium, which he did, having obtained the highest grades in mathematics, classical languages, and history. At university in Berlin, he attended lectures on philosophy, classical languages, and mathematics. In mathematics, which after two years he chose as his major subject, there were few courses at a sufficiently advanced level, so that here, too, he had to

inform himself on his own about the current state of research. At the age of 19, he submitted his dissertation (on the decomposition of quotients of algebraic numbers and on the reordering of infinite series). At the same time, he passed a series of examinations that qualified him to teach mathematics, Greek, and Latin at a gymnasium.

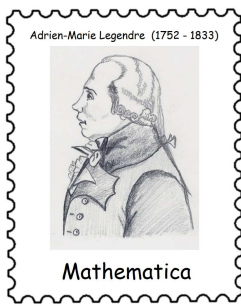
As a Jew, he in fact had little hope of finding a position as a high-school teacher. Nevertheless, word had gotten around about the remarkable abilities that resided in this young man. And so it happened that the Joachimsthal gymnasium in Berlin offered him a position. However, JACOBI was by now already working on his habilitation thesis.

Through the STEIN-HARDENBERG reforms of 1812, the king of Prussia had granted to the Jews freedom of movement and the rights of citizenship, including the right to pursue an academic career. However, these rights were revoked in 1822, and all Jews were dismissed from the civil service.

Like many other educated Jews, JACOBI decided to resort to an expedient: around the year 1825, he converted to Christianity (connected with the change from JACQUES SIMON to CARL GUSTAV JAKOB). Now the way stood open for him to pursue a career as a university professor. As a privatdozent, he gave his first lecture at the University of Berlin on the theory of curves and surfaces in space.

Motivated by GAUSS'S research on quadratic and biquadratic residues, JACOBI studied cubic residues and informed GAUSS of his discoveries. The latter was impressed, and he contacted his friend the Königsberg astronomer FRIEDRICH WILHELM BESSEL in order to learn more about the 21-year-old, who in the meantime had accepted a vacant privatdozent position at the University of Königsberg.





JACOBI also wrote to ADRIEN-MARIE LEGENDRE (1752–1833), the leading expert on elliptic functions, since he had developed some new ideas on that topic. LEGENDRE, too, realized that JACOBI had made some decisive contributions.

But it was not only JACOBI who had gone beyond LEGENDRE’s methods. The Norwegian mathematician NIELS HENRIK ABEL – independently of JACOBI – had further developed the theory using an approach similar to JACOBI’s.



The arc length of an ellipse with semiaxes  $a$ ,  $b$  and eccentricity  $\varepsilon$  can be expressed by the following integral:

$$f(x) = a \cdot \int_0^x \sqrt{1 - \varepsilon^2 \cos^2(t)} dt$$

Each central angle  $x$  is associated with the arc length  $f(x)$ . Such an integral cannot be expressed in terms of elementary functions; that is, there is no expression in elementary functions whose derivative is equal to the integrand.

Nevertheless, the values of the function  $f$  can be calculated numerically. ABEL and JACOBI investigated the properties of the associated inverse function and thereby initiated the theory of elliptic functions.

Both mathematicians submitted papers on this subject to the newly founded *Journal für die reine und angewandte Mathematik* (Journal for pure and applied mathematics, called “Crelle’s journal”). The new ideas presented in the work of the one stimulated new ideas in the work of the other, and within several months, the theory of elliptic functions had undergone a substantial development.

In a letter, LEGENDRE gave the following assessment: "They are making such rapid progress, my good sirs, in their speculations, that it is scarcely possible to follow them, in particular for an old man ... I congratulate myself that I have lived long enough to witness this good-natured competition between two young, equally strong athletes who are directing all their efforts to the benefit of our science to push back ever farther the boundaries [of knowledge]."

This “good-natured competition” came to an abrupt end with the death of ABEL in April 1829. That same year, JACOBI published his compendium *Fundamenta nova theoriae functionum ellipticarum*.

The positive evaluations of GAUSS and LEGENDRE had the result that shortly after his 23rd birthday, JACOBI was named assistant professor, and full professor at the age of 27. The enormous reputation that JACOBI’s lectures enjoyed led to ever greater numbers of students enrolling at the University of Königsberg.

He made a habit of including the most recent result of mathematical research in his lectures. He was the first to institute mathematical seminars, based on the model of the classics.

In 1842, BESSEL and JACOBI attended, as representatives of Prussia, a conference in Manchester hosted by the British Association for the Advancement of Science. There, they met the Irish astronomer and mathematician WILLIAM ROWAN HAMILTON. This meeting would result in a series of articles by JACOBI on the mathematics of the solar system.



On his return trip by way of Paris, JACOBI gave a lecture to the *Académie des Sciences*. He had arrived at the peak of his renown.

When in 1843, JACOBI was diagnosed with diabetes, his doctor recommended that he recuperate in a milder climate. However, JACOBI was unable to afford such a trip – the fortune he had inherited had been lost in a bank crash.



ALEXANDER VON HUMBOLDT, with whom JACOBI had maintained a correspondence since 1828, supported his application to the king of Prussia, and so JACOBI was able to journey to Italy, accompanied by GUSTAV LEJEUNE DIRICHLET, with whom he had become friendly. In the mild climate, his health improved noticeably, and he began to work again on his mathematical research. In Rome, he studied DIOPHANTUS's *Arithmetica*, which was housed in the Vatican Library.

JACOBI, for whom the change in climate had obviously done a great deal of good, had no desire to return to Königsberg. In recognition of his service, the king of Prussia agreed to a transfer to Berlin (accompanied by an increase in salary justified by the increased cost of living in the capital as well as a supplement to offset the expected costs associated with his illness). As a member of the *Prussian Academy of Sciences*, he was entitled, but not obliged, to give lectures.

During the Revolution of 1848, JACOBI gave a political speech at the Berlin *Constitutional Club* of the Liberal Party, a speech that simultaneously alienated both the monarchists and the republicans. Following the suppression of the revolutionary movements, his application to be formally transferred to the *University of Berlin* was denied. In addition, because he had shown himself insufficiently grateful to his king, the supplement to his salary was withdrawn. JACOBI could no longer afford to live in Berlin, and so a friend took him and his family into his house in Gotha (Duchy of Saxe-Coburg and Gotha).

It was only after JACOBI was offered a professorship at the University of Vienna that the Prussian regime realized that the most famous mathematician in Europe after GAUSS was prepared to leave the country. To avoid such a humiliation, he was offered a permanent position in Berlin after all. However, he was able to visit his wife and seven children only in the semester vacations, since they continued to live in Gotha.

In January 1851, he became ill with a flulike infection, and before he could recover, he became infected with smallpox. He died a few days later, at the age of 46.



At the memorial service at the Academy of Sciences, DIRICHLET praised the life work of the deceased and compared it with that of EULER:

*To be sure, JACOBI's 180 publications could not match the over 800 papers of EULER, but just think what might still have been expected from JACOBI were he to have lived as long as EULER (1707-1783)!*

Like EULER, JACOBI produced significant work in disparate areas of mathematics: he obtained important results in the development of the theory of differential equations, in the calculus of variations (one can make deductions about the extrema of functions from a study of the effect of small variations), in differential geometry (e.g., determining geodesic curves on ellipsoids), and in analytical mechanics.

JACOBI's name appears in numerous mathematics terms, such as the JACOBIAN matrix (determination of area and volume in higher-dimensional spaces) and the JACOBI method (further development of GAUSS's algorithm for solving systems of linear equations).

A series of publications in number theory began with JACOBI's investigation of cubic residues, and was then continued with extensions to the four-squares theorem of Joseph Louis LAGRANGE, who in 1770 had proven that every natural number can be expressed as the sum of four perfect squares.

PIERRE DE FERMAT had determined that every prime number  $p$  that on division by 4 leaves a remainder of 1 (written  $p \equiv 1 \pmod{4}$ ) can be expressed as the sum of two squares.



From this, JACOBI derived the following theorem:

- A natural number  $n$  can be expressed as the sum of two squares if and only if the prime factors  $p$  of  $n$  that leave remainder 3 on division by 4 appear an even number of times.

LEGENDRE had found a criterion for when a natural number can be expressed as a sum of three squares. Among the natural numbers that cannot be expressed as a sum of fewer than four squares (that is, those numbers that can be expressed as a sum of four squares but not as a sum of fewer than four squares), there are those that admit more than one representation.

For example, 28 can be written both as  $4^2 + 2^2 + 2^2 + 2^2$  and as  $3^2 + 3^2 + 3^2 + 1^2$ ; the number 55 can be expressed in three ways:  $7^2 + 2^2 + 1^2 + 1^2$ ;  $5^2 + 5^2 + 2^2 + 1^2$ ;  $6^2 + 3^2 + 3^2 + 1^2$ ; and for 87, there are four possible such expressions:  $9^2 + 2^2 + 1^2 + 1^2$ ;  $7^2 + 6^2 + 1^2 + 1^2$ ;  $7^2 + 5^2 + 3^2 + 2^2$ ;  $6^2 + 5^2 + 5^2 + 1^2$ .

Using elliptic functions, JACOBI proved a formula that allows one to determine the number  $r(n)$  of representations of a natural number  $n$  as a sum of four squares (including all possible orderings of the numbers and allowing both plus and minus signs):

- If  $n$  is divisible by 4, then  $r(n) = 8 \cdot \sigma(n) - 32 \cdot \sigma(\frac{1}{4} \cdot n)$ . If  $n$  is not divisible by 4, then  $r(n) = 8 \cdot \sigma(n)$ , where  $\sigma(n)$  is equal to the sum of all divisors of  $n$ . (JACOBI's theorem)

While EULER was equally at home with problems in pure and applied mathematics, JACOBI's main interest was decidedly in the realm of pure mathematics – according to the first thesis of his Königsberg inaugural address:

*Mathesis est scientia eorum, quae per se clara sunt.*

(Mathematics is the science of those things that are clear in and of themselves.)

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<https://www.spektrum.de/wissen/carl-g-j-jacobi-1804-1851/1200879>

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