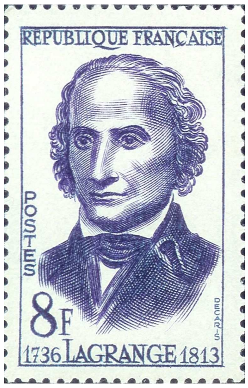
JOSEPH LOUIS LAGRANGE (January 25, 1736 – April 10, 1813)

by HEINZ KLAUS STRICK, Germany



The French mathematician JOSEPH-LOUIS LAGRANGE, of Italian origin (the entry in the baptismal record reads GIUSEPPE LODOVICO LAGRANGIA), was born and raised in Turin (at the time, part of the Kingdom of Sardinia), the oldest of eleven children of a military bursar. The boy's father was planning on a career as a solicitor for his son, but after suffering losses from financial speculation, he agreed to a change in his son's course of study.

After reading a book by EDMOND HALLEY on algebraic applications in optics, the 17-year-old Lagrange decided to devote himself to mathematics and physics. At the age of 18, he wrote (in Latin) a letter to LEONHARD EULER in Berlin to point out some analogies between the binomial theorem and higher derivatives of the product of two functions. For the product of two functions *f* and *g* , one has

$$(f \cdot g)' = 1 \cdot f' \cdot g + 1 \cdot f \cdot g' ,\ (f \cdot g)'' = 1 \cdot f'' \cdot g + 2 \cdot f' \cdot g' + 1 \cdot f \cdot g'' ,\ (f \cdot g)''' = 1 \cdot f''' \cdot g + 3 \cdot f'' \cdot g' + 3 \cdot f' \cdot g'' + 1 \cdot f \cdot g'''$$

and in general,

$$(f \cdot g)^{(n)} = \binom{n}{0} \cdot f^{(n)} \cdot g^{(0)} + \binom{n}{1} \cdot f^{(n-1)} \cdot g^{(1)} + \binom{n}{2} \cdot f^{(n-2)} \cdot g^{(2)} + \ldots + \binom{n}{n} \cdot f^{(0)} \cdot g^{(n)}$$

where $f^{(n)}$ denotes the *n*th derivative, with $f^{(0)}$ the function *f* itself.

LAGRANGE was very disappointed to learn that GOTTFRIED WILHELM LEIBNIZ and JOHANN BERNOULLI had discovered this fact before him.

At the age of 19, LAGRANGE was appointed professor of mathematics at the artillery school in Turin. There he published his first articles on *differential equations* (equations that relate functions with its derivatives) and on the *calculus of variations* (extreme value problems in several variables).

EULER was so impressed by these publications that he proposed to the president of the Berlin *Academy of Sciences*, PIERRE LOUIS MOREAU DE MAUPERTUIS, that LAGRANGE be offered a position in Berlin. LAGRANGE was too modest to accept the appointment, but he became a corresponding member of the *Academy*.

MAUPERTUIS had become famous in 1736 when he acted as chief of the *French Geodesic Mission* sent by King LOUIS XV to Lapland to measure the length of a degree of arc of the meridian. At the same time there was also an equatorial mission which was led by the French astronomer CHARLES MARIE DE LA CONDAMINE.

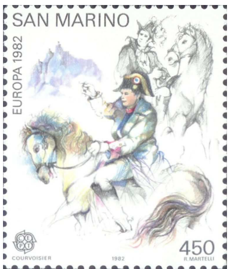
At the age of 21, he founded the *Royal Academy of Sciences* in Turin and became an editor of the journal *Mélanges de Turin*, which published articles in French and Latin. Among these could be found numerous articles by LAGRANGE himself: investigations into the theory of oscillations of strings and papers on fluid mechanics, the orbits of Jupiter and Saturn, and the theory of probability.

In 1763, he made his first trip abroad. His destination was London, but he became ill en route and had to break his journey in Paris. There he met JEAN-BAPTISTE LE ROND D'ALEMBERT, who offered him a more suitable position than the one he held in Turin. LAGRANGE won a number of prizes of the Paris *Académie des Sciences* (for work on the *motion of the moon* as well as the *3-body-problem*).

In 1766, LAGRANGE was encouraged by D'ALEMBERT to move to Berlin; however, he accepted the offer of the Prussian king FREDERICK II only after it became clear that EULER would be returning to St. Petersburg. Thus LAGRANGE became EULER's successor as director of the *Mathematics Division of the Prussian Academy of Sciences*.

During his twenty-years' stay in Berlin, LAGRANGE further developed methods for dealing with functions of several variables and for the solution of differential equations. After the death of Frederick II in 1786, LAGRANGE received numerous offers, and in the end, he decided to accept a position in Paris, where he was able to complete work on his *Mécanique analytique* (analytic mechanics) with the support of ADRIEN MARIE LEGENDRE; in this work, numerous problems were solved, using exclusively mathematical methods.

During the French Revolution, he was appointed to the Commission for the Reform of Weights and Measures, which had the task of introducing a decimal system of weights and measures, as well as to the *Bureau des Longitudes*. When a law was promulgated requiring all foreign residents to leave the country, ANTOINE LAURENT DE LAVOISIER intervened and obtained an explicit exemption for LAGRANGE. A few months later, Lavoisier would become a victim of the guillotine during the Reign of Terror.



Under NAPOLEON, LAGRANGE was again rehabilitated and loaded with titles and honours.

His last works provided a decisive impetus for the further development of differential geometry (the study of curves and surfaces in space) and complex analysis (the study of complex-valued functions of a complex variable).

Today, many theorems and concepts recall in their names the achievements of LAGRANGE:

- The *Four squares theorem* of LAGRANGE states that every natural number can be written as the sum of at most four perfect squares.
- *WILSON's theorem*: LAGRANGE proved that a natural number n is a prime number if and only if $(n-1)!+1$ is divisible by n.

- Moreover, he showed that the continued fraction developments of square roots are always infinite and periodic (EULER had proved the converse):

$$\sqrt{2}=\sqrt{2}=1+(\sqrt{2}-1)=1+\frac{(\sqrt{2}-1)\cdot(\sqrt{2}+1)}{\sqrt{2}+1}=1+\frac{1}{1+\sqrt{2}}=1+\frac{1}{1+\left(1+\frac{1}{1+\sqrt{2}}\right)}$$

$$=1+\frac{1}{2+\frac{1}{2+\frac{1}{1+\sqrt{2}}}}=...=[1;\overline{2}]$$

$$\sqrt{3}=1+\frac{2}{1+\sqrt{3}}=1+\frac{2}{1+\left(1+\frac{2}{1+\sqrt{3}}\right)}=1+\frac{2}{2+\frac{2}{1+\sqrt{3}}}=1+\frac{1}{1+\frac{1}{1+\sqrt{3}}}$$

$$=1+\frac{1}{1+\frac{1}{1+1+\frac{1}{1+\frac{1}{1+\sqrt{3}}}}}=...=[1;\overline{1,2}]$$

$$\sqrt{5}=2+(\sqrt{5}-2)=2+\frac{(\sqrt{5}-2)\cdot(\sqrt{5}+2)}{\sqrt{5}+2}=2+\frac{1}{2+\sqrt{5}}$$

$$=2+\frac{1}{2+\left(2+\frac{1}{2+\sqrt{5}}\right)}=2+\frac{1}{4+\frac{1}{2+\left(2+\frac{1}{2+\sqrt{5}}\right)}}=...=[2;\overline{4}]$$

- *PELL's equation*: LAGRANGE proved that the equation $x^2-n\cdot y^2=1$ can be solved in the set of integers if *n* is a natural number that is not a perfect square, and he gave a procedure to generate the infinite collection of solution pairs (*x ; y*). To find the "smallest" solution you have to consider the part of the continued fraction expansion of $\sqrt{n}$ up to the penultimate place.

Example: In the case $x^2-11\cdot y^2=1$ consider the part of the continued fraction expansion of $\sqrt{11}$

$\sqrt{11}=3+\frac{1}{3+\frac{1}{6+...}}=[3;\overline{3,6}]$ up to the penultimate place, before the digits begin to repeat,

that is $3+\frac{1}{3}=\frac{10}{3}$. The smallest solution is then the pair (10 ; 3); indeed, we have $10^2-11\cdot 3^2=1$.

Additional solutions are obtained by taking powers $(10-3\cdot\sqrt{11})^k$ of this solution:

$(10-3\cdot\sqrt{11})^2=100-60\cdot\sqrt{11}+99=199-60\cdot\sqrt{11}$, that is, (199 ; 60);

the next solution pair is $(10-3\cdot\sqrt{11})^3=1000-900\cdot\sqrt{11}+2970-297\cdot\sqrt{11}=3970-1197\cdot\sqrt{11}$, that is, (3970 ; 1197), and so on.

- *LAGRANGE remainder*: LAGRANGE also estimated the error that is incurred if one truncates the *TAYLOR expansion* of a function after *n* steps:

$$f(x)=f(a)+(x-a)\cdot f'(a)+\tfrac{1}{2!}\cdot(x-a)^2\cdot f''(a)+...+\tfrac{1}{n!}\cdot(x-a)^n\cdot f^{(n)}(a)+R_n$$

with $R_n=\frac{1}{(n+1)!}\cdot(x-a)^{n+1}\cdot f^{(n+1)}(a+\vartheta(x-a))$

- *Lagrange interpolation formula*: He gave a general term for an nth-degree polynomial whose graph passes through $n - 1$ given points.

Examples (n = 2, n = 3):

$$L_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \cdot y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \cdot y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \cdot y_2$$

$$L_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \cdot y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

- There are also a number of formulas named for Lagrange:

Lagrange identities:

$$(a_1^2 + a_2^2) \cdot (b_1^2 + b_2^2) = (a_1b_1 + a_2b_2)^2 + (a_1b_2 - a_2b_1)^2$$

$$(a_1^2 + a_2^2 + a_3^2) \cdot (b_1^2 + b_2^2 + b_3^2) = (a_1b_1 + a_2b_2 + a_3b_3)^2 + (a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + (a_2b_3 - a_3b_2)^2$$

Lagrange vector equation: $(\vec{a} \times \vec{b}) * (\vec{c} \times \vec{d}) = (\vec{a} * \vec{c}) \cdot (\vec{b} * \vec{d}) - (\vec{b} * \vec{c}) \cdot (\vec{a} * \vec{d})$

Formula for calculating the volume V of a three-sided pyramid with vertices $O(0|0|0)$, $A(a_1|a_2|a_3)$, $B(b_1|b_2|b_3)$, $C(c_1|c_2|c_3)$:

$$V = \tfrac{1}{6} \cdot [c_1 \cdot (a_2b_3 - a_3b_2) + c_2 \cdot (a_3b_1 - a_1b_3) + c_3 \cdot (a_1b_2 - a_2b_1)]$$

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www.spektrum.de/wissen/joseph-louis-lagrange-1736-1813/862789

Translated by David Kramer

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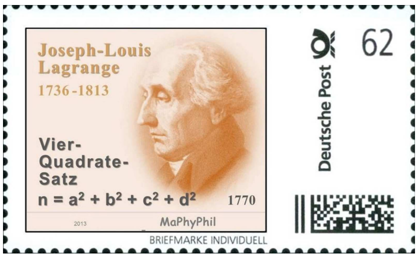

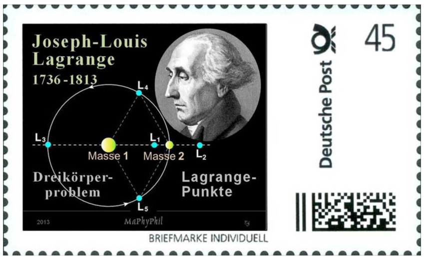

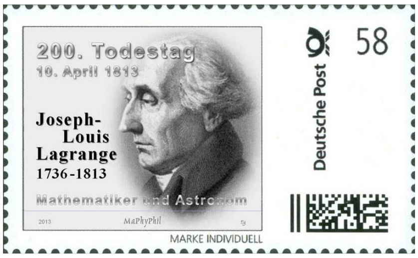

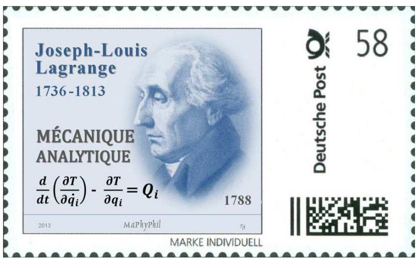

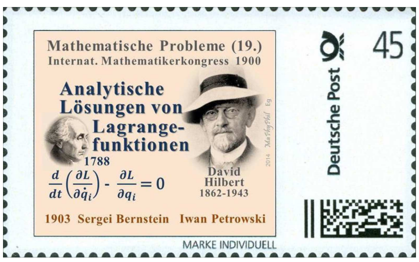

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