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(*b*. Kāshān, Iran; *d*. Samarkand [now in Uzbek, U.S.S.R], 22 June 1429)

astronomy, mathematics.

The biographical data on al-Kāshī are scattered and sometimes contradictory. His birthplace was a part of the vast empire of the conqueror Tamerlane and then of his son Shāh Rukh. The first known date concerning al-Kāshī is 2 June 1406 (12 Dhūʾl-Hijja, A.H. 808), when, as we know from his *Khaqānī zīj*, he observed a lunar eclipse in his native town. [1] According to Suter, al-Kāshī died about 1436; but Kennedy, on the basis of a note made on the title page of the India Office copy of the *Khaqānāi zīj*, gives 19 Ramaḍān A.H. 832, or 22 June 1429[2]. The chronological order of al-Kāshī's works written in Persian or in Arabic is not known completely, but sometimes he gives the exact date and place of their completion. For instance, the *Sullam al-samāʾ* ("The Stairway of Heaven"), a treatise on the distances and sizes of heavenly bodies, dedicated to a vizier designated only as Kamāl al-Dīn Mahmūd, was completed in Kāshān on 1 March 1407.[3] In 1410–1411 al-Kāshī wrote the *Mukhtaṣar dar ilm-ihayʾat* ("Compendium of the Science of Astronomy") for Sultan Iskandar, as is indicated in the British Museum copy of this work. D. G. Voronovski identifies Iskandar with a member of the Tīmūrid dynasty and cousin of Ulugh Bēg, who ruled Fars and Iṣfahān and was executed in 1414.[4] In 1413–1414 al-Kāshī finished the *Khaqānī zīj*. Bartold assumes that the prince to whom this *zīj* is dedicated was Shāh Rukh, who patronized the sciences in his capital, Herat;[5] but Kennedy established that it was Shāh Rukh's son and ruler of Samarkand, Ulugh Bēg. According to Kennedy, in the introduction to this work al-Kāshī complains that he had been working on astronomical problems for a long time, living in poverty in the towns of Iraq (doubtless Persian Iraq) and mostly in Kāshān. Having undertaken the composition of a *zīj*, he would not be able to finish it without the support of Ulugh Bēg, to whom he dedicated the completed work.[6] In January 1416 al-Kāshī composed the short *Risāla dar sharḥ-i ālāt-i raṣd* ("Treatise on . . . Observational Instuments"), dedicated to Sultan Iskandar, whom Bartold and Kennedy identify with a member of the Kārā Koyunlū, or Turkoman dynasty of the Black Sheep.[7] Shishkin mistakenly identifies him with the above-mentioned cousin of Ulugh Bēg.[8] At almost the same time, on 10 February 1416, al-Kāshī completed in Kāshān *Nuzha al-ḥadāiq* ("The Garden Excursion"), in which he described the "Plate of Heavens," an astronomical instrument he invented. In June 1426, at Samarkand, he made some additions to this work.

Dedicating his scientific treatises to sovereigns or magnates, al-Kāshī, like many scientists of the Middle Ages, tried to provide himself with financial protection. Although al-Kāshī had a second profession—that of a physician—he longed to work in astronomy and mathematics. After a long period of penury and wandering, al-Kāshī finally obtained a secure and honorable position at Samarkand, the residence of the learned and generous protector of science and art, Sultan Ulugh Bēg, himself a great scientist.

In 1417–1420 Ulugh Bēg founded in Samarkand a *madrasa*—a school for advanced study in theology and science—which is still one of the most beautiful buildings in Central Asia. According to a nineteenth century author, Abū Tāhir Khwāja, "four years after the foundation of the *madrasa*," Ulugh Bēg commenced construction of an observatory; its remains were excavated from 1908 to 1948.[9] For work in the *madrasa* and observatory Ulugh Bēg took many scientists, including al-Kāshī, into his service. During the quarter century until the assassination of Ulugh Bēg in 1449 and the beginning of the political and ideological reaction, Samarkand was the most important scientific center in the East. The exact time of al-Kāshī's move to Samarkand is unknown. Abū Ṭāhir Khawāja states that in 1424 Ulugh Bēg discussed with al-Kāshī, Qāḍī Zāde al-Rūmī, and another scientist from Kāshān, Muʿin al-Dīn, the project of the observatory.[10]

In Samarkand, al-Kāshī actively continued his mathematical and astronomical studies and took a great part in the organization of the observatory, its provision with the best equipment, and in the preparation of Ulugh Bēg's *Zij*, which was completed after his (al-Kāshī's) death. Al-Kāshī occupied the most prominent place on the scientific staff of Ulugh Bēg. In his account of the erection of the Samarkand observatory the fifteenth-century historian Mirkhwānd mentions, besides Ulugh Bēg, only al-Kāshi, calling him "the support of astronomical science"and "the second Ptolemy."[11] The eighteenth-century historian Sayyid Raqīm, enumerating the main founders of the observatory and calling each of them *maulanā* ("our master," a usual title of scientists in Arabic), calls al-Kāshī *maulanā-i ālam* (*maulanā* of the world).[12]

Al-Kāshī himself gives a vivid record of Samarkand scientific life in an undated letter to his father, which was written while the observatory was being built. Al-Kāshī highly prized the erudition and mathematical capacity of Ulugh Bēg, particularly his ability to perform very difficult mental computations; he described the prince's scientific activity and once called him a director of the observatory.[13] Therefore Suter's opinion that the first director of the Samarkand observatory was al-Kāshī, who was succeeded by Qāḍī Zāde, must be considered very dibious.[14] On the other hand, al-Kāshī spoke with disdain of Ulugh Bēg's nearly sixty scientific collaborators, although he qualified Qādi Zāde as "the most learned of all."[15]Telling of frequent scientific meetings directed by the sultan, al-Kāshīgave several examples of astronomical problems propounded there. These problems, too difficult for others, were solved easily by al-Kāshī. In two cases he surpassed Qāḍī Zāde, who misinterpreted one proof in al-Bīrūnī's *al-Qānūn al-Masʿūdī* and who was unable to solve one difficulty connected with the problem of determining whether a given surface is truly plane or not. Nevertheless his relations with Qāḍi Zāde were amicable. With great satisfaction al-Kāshī told his father of Ulugh Bēg'spraise, related to him by some of his friends. He emphasized the atmosphere of free scientific discussion in the presence of the sovereign. the letter included interesting information on the construction of the observatory building and the instruments. This letter and other sources characterize al-Kāshīas the closest collaborator and consultant of Ulugh Bēg, who tolerated al-Kāshī's ignorance of court etiquette and lack of good manners.[16] In the introduction to his own *Zij* Ulugh Bēg mentions the death of al-Kāshī and calls him "a remarkable scientist, one of the most famous in the world, who had a perfect command of he science of the ancients, who contributed to its development, and who could solve the most difficult problems."[17]

Al-Kāshī wrote his most important works in Samarkand. In July 1424 he completed *Risāla al-muḥiṭiyya* ("The Treatise on the Circumference"), masterpiece of computational technique resulting in a the determination of 2π to sixteen decimal places. On 2 March 1427 he finished the textbook *Miftāḥ al-ḥisāb* ("The Key of Arithmetic"), dedicated to Ulugh Bēg. It is not known when he completed his third chef d'oeuvre, *Risāla al-water waʾl-jaib* ("The Treatise on the Chord and Sine"), in which he calculated the sine of 1° with the same precision as he had calculated π.Apparently he worked on this shortly before his death; some sources indicate that the manuscript was incomplete when he died and that it was finished by Qāḍī Zāde.[18] Apparently al-Kāshī had developed his method of calculation of the sine of 1° before he completed *Miftāḥ al-ḥisāb,* for in the introduction to this book, listing his previous works, he mentions *Risāla al-watar waʾl-jaib*.

As was mentioned above, al-Kāshī took part in the composition of Ulugh Bīg's *Zīj*. We cannot say exactly what he did, but doubtless his participation was considerable. The introductory theoretical part of the *Zīj* was completed during al-Kāshī's lifetime, and he translated it from Persian into Arabic.[19]

Mathematics. Al-Kāshī's best-known work is *Miftāḥ al-ḥisāb* (1427), a veritable encyclopedia of elementary mathematics intended for an extensive range of students; it also considers the requirements of calculators—astronomers. land surveyors, architects, clerks, and merchants. In the richness of its contents and in the application of arithmetical and algebraic methods to the solution of various problems, including several geometric ones, and in the clarity and elegance of exposition, this voluminous textbook is one of the best in the whole of medieval literature; it attests to both the author's erudition and his pedagogic ability.[20] Because of its high quality the *Miftāḥ al-ḥisāb* was often recopied and served as a manual for hundreds of years; a compendium of it was also used. The book's title indicates that arithmetic was viewed as the key to the solution of every kind of problem which can be reduced to calculation, and al-Kāshī defined arithmetic as the "science of rules of finding numerical unknowns with the aid of corresponding known quantities."[21] The *Miftāḥ al-ḥisāb* is divided into five books preceded by an introduction: "On the Arithmetic of Integers," "On the Arithmetic of Fractions," "On the 'Computation of the Astronomers'"(on sexagesimal arithmetic), "On the Measurement of Plane Figures and Bodies," and "On the Solution of Problems by Means of Algebra [linear and quadratic equations] and of the Rule of Two False Assumptions, etc." The work comprises many interesting problems and carefully analyzed numerical examples.

In the first book of the *Miftāḥ*, al-Kāshī describes in detail a general method of extracting roots of integers. The integer part of the root is obtained by means of what is now called the Ruffini—Horner method. If the root is irrational, (a and r are integers), the fractional part of the root is calculated according to the approximate formula [22]Al-Kāshī himself expressed all rules of computation in words, and his algebra is always purely "rhetorical." In this connection he gives the general rule for raising a binomial to any natural power and the additive rule for the successive determination of binomial coefficients; and he constructs the so-called Pascal's triangle (for $n = 9$). The same methods were presented earlier in the *Jāmiʿal-ḥisāb biʾl takht waʾl-tuzāb* ("Arithmetic by Means of Board and Dust") of Naṣīr al-Din al-Ṭūsī (1265). The origin of these methods is unknown. It is possible that they were at least partly developed by al-Khayyāmī the influence of Chinese algebra is also quite plausible.[23]

Noteworthy in the second and the third book is the doctrine of decimal fractions, used previously by al-Kāshī in his *Risāla al-muhītīyya*. It was not the first time that decimal fractions appeared in an Arabic mathematical work; they are in the *Kitāb al-fusūl fiʾl-hisāb al-Hindi* ("Treatise of Arithmetic") of al Ulīdisī (mid-tenth century) and were used occasionally also by Chinese scientists.[24] But only al-Kāshī introduced the decimal fractions methodically, with a view to establishing a system of fractions in which (as in the sexagesimal system) all operations would be carried out in the same manner as with integers. It was based on the commonly used decimal numeration, however, and therefore accessible to those who were not familiar with the sexagesimal arithmetic of the astronomers. Operations with finite decimal fractions are explained in detail, but al-Kāshī does not mention the phenomenon of periodicity. To denote decimal fractions, written on the same line with the integer, he sometimes separated the integer by a vertical line or wrote in the orders above the figures; but generally he named only the lowest power that determined all the others. In the second half of the fifteenth century and in the sixteenth century al-Kāshī's decimal fractions found a certain circulation in Turkey, possibly through ʿAlī Qūshjī, who had worked with him at Samarkand and who sometime after the assassination of Ulugh Bēg and the fall of the Byzantine empire settled in Constantinople. They also appear occasionally in an anonymous Byzantine collection of problems from the fifteenth century which was brought to Vienna in 1562.[25] It is also possible that al-Kāshī's ideas had some influence on the propagation of decimal fractions in Europe.

In the fifth book al-Kāshī mentions in passing that for the fourth-degree equations he had discovered "the method for the determination of unknowns in. . . seventy problems which had not been touched upon by either ancients or contemporaries."[26] He also expressed his intention to devote a separate work to this subject, but it seems that he did not complete this research.

Al-Kāshī's theory should be analogous to the geometrical theory of cubic equations developed much earlier by Abu'l-Jũd Muhammad ibn Laith, al-Khayyāmī (eleventh century), and their followers: the positive roots of fourth-degree equations were constructed and investigated as coordinates of points of intersection of the suitable pairs of conics. It must be added that actually there are only sixty-five (not seventy) types of fourth-degree equations reducible to the forms considered by Muslim mathematicians, that is, the forms having terms with positive coefficients on both sides of the equation. Only a few cases of fourth-degree equations were studied before al-Kāshī.

Al-Kāshī's greatest mathematical achievements are *Risāla al-muhitiyya and Risāla al-watar waʾl-jaib,* both written in direct connection with astronomical researches and especially in connection with the increased demands for more precise trigonometrical tables.

At the beginning of the *Risāla al-muḥīṭīyya* al-Kāshī points out that all approximate values of the ratio of the circumference of a circle to its diameter, that is, of π, calculated by his predecessors gave a very great (absolute) error in the circumference and even greater errors in the computation of the areas of large circles, Al-Kāshī tackled the problem of a more accurate computation of this ratio, which he considered to be irrational, with an accuracy surpassing the practical needs of astronomy, in terms of the then-usual standard of the size of the visible universe or of the "sphere of fixed stars."[27] For that purpose he assumed, as had the Iranian astronomer Qutb al-Din al-Shīrāzī (thirteenth-fourteenth centuries), that the radius of this sphere is 70,073.5 times the diameter of the earth. Concretely, al-KĀshī posed the problem of calculating the said ratio with such precision that the error in the circumference whose diameter is equal to 600,000 diameters of the earth will be smaller than the thickness of a horse's hair. Al-Kāshī used the following old Iranian units of measurement: I parasang (about 6 kilometers) = 12,000 cubits, 1 cubit = 24 inches (or fingers), 1 inch = 6 widths of a medium-size grain of barley, and I width of a barley grain = 6 thicknesses of a horse's hair. The great-circle circumference of the earth is considered to be about 8,000 parasangs, so al-Kāshī's requirement is equivalent to the computation of π with an error no greater than $0.5 \cdot 10^{-17}$. This computation was accomplished by means of elementary operations, including the extraction of square roots, and the technique of reckoning is elaborated with the greatest care.

Al-Kāshī's measurement of the circumference is based on a computation of the perimeters of regular inscribed and circumscribed polygons, as had been done by Archimedes, but it follows a somewhat different procedure. All calculations are performed in sexagesimal numeration for a circle with a radius of 60. Al-Kāshī's fundamental theorem—in modern notation—is as follows: In a circle with radius r,

where $crd\ \alpha°$ is the chord of the arc $\alpha°$ and $\alpha° < 180°$. Thus al-Kāshī applied here the "trigonometry of chords" and not the trigonometric lines themselves. If $\alpha = 2\phi°$ and $d = 2$, then al-Kāshī's theorem may be written trigonometrically as

which is found in the work of J. H. Lambert (1770). The chord of 60° is equal to r, and so it is possible by means of this theorem to calculate successively the chords $c_1, c_2, c_3. . . .$ of the arcs 120°, 150°, 165°, in general the value of the chord c_n of the arc will be . The chord c_n being known, we may, according to Pythagorean theorem, find the side of the regular inscribed $3 \cdot 2^n$-sided polygon, for this side a_n is also the chord of the supplement of the arc $\alpha_n°$ up to 180°. The side b_n of a similar circumscribed polygon is determined by the proportion b_n: $a_n = r: h$, where h is the apothem of the inscribed polygon. In the third section of his treatise al-Kāshī ascertains that the required accuracy will be attained in the case of the regular polygon with $3 \cdot 2^{28} = 805, 306, 368$ sides.

He resumes the computation of the chords in twenty-eight extensive tables; he verifies the extraction of the roots by squaring and also by checking by 59 (analogous to the checking by 9 in decimal numeration); and he establishes the number of sexagesimal places to which the values used must be taken. We can concisely express the chords c_n and the sides a_n by formulas

and

where the number of radicals is equal to the index n. In the sixth section, by multiplying a_{28} by $3 \cdot 2^{28}$, one obtains the perimeter p_{28} of the inscribed $3 \cdot 2^{28}$-sided polygon and then calculates the perimeter p_{28} of the corresponding similar circumscribed polygon. Finally the best approximation for $2\pi\ r$ is accepted as the arithmetic mean whose sexagesimal value for $r = 1$ is $6\ 16^{I}\ 59^{II}\ 28^{III}\ 1^{IV}\ 34^{V}\ 51^{VI}\ 46^{VIII}\ 50^{IX}$, where all places are correct. In the eighth section al-Kāshī translates this value into the decimal fraction $2\pi = 6.2831853071795865$, correct to sixteen decimal places. This superb result far surpassed all previous determinations of π. The decimal approximation $\pi \approx 3.14$ corresponds to the famous boundary values found by Archimedes, Ptolemy used the sexagesimal value $3\ 8^{I}\ 30^{II}$ (≈ 3.14166), and the results of al-Kāshī's predecessors in the Islamic countries were not much better. The most accurate value of π obtained before al-Kāshī by the Chinese scholar Tsu Chʾung-chih (fifth century) was correct to six decimal places. In Europe in 1597 A. van Roomen approached al-Kāshī's result by calculating π to fifteen decimal places; later Ludolf van Ceulen calculated π to twenty and then to thirty-two places (published 1615).

In his *Risāla al-walar waʾl-jaib* al-Kāshī again calculates the value of sin 1° to ten correct sexagesimal places; the best previous approximations, correct to four places, were obtained in the tenth century by Abuʾl-Wafāʾ and Ibn Yũnus. Al-Kāshī derived the equation for the trisection of an angle, which is a cubic equation of the type $px = q + x^3$—or, as the Arabic mathematicians would say, "Things are equal to the cube and the number." The trisection equation had been known in the Islamic countries since the eleventh century; one equation of this type was solved approximately by al-Bīũnī to determine the side of a regular nonagon, but this method remains unknown to us. Al-Kāshī proposed an original iterative method of approximate solution, which can be summed up as follows: Assume that the equation

possesses a very small positive root x; for the first approximation, take ; for the second approximation, ; for the third, , and generally $x_0 = 0$.

It may be proved that this process is convergent in the neighborhood of values of . Al-Kāshī used a somewhat different procedure: he obtained x_1 by dividing q by p as the first sexagesimal place of the desired root, then calculated not the approximations $x_2, x_3, \ldots$ themselves but the corresponding corrections, that is, the successive sexagesimal places of x. The starting point of al-Kāshī's computation was the value of sin 3°, which can be calculated by elementary operations from the chord of 72° (the side of a regular inscribed pentagon) and the chord of 60°. The sin 1° for a radius of 60 is obtained as a root of the equation

The sexagesimal value of sin 1° for a radius of 60 is $1\ 2^{\mathrm{I}}\ 49^{\mathrm{II}}\ 43^{\mathrm{III}}\ 11^{\mathrm{IV}}\ 14^{\mathrm{V}}\ 44^{\mathrm{VI}}\ 16^{\mathrm{VII}}\ 26^{\mathrm{VIII}}\ 17^{\mathrm{IX}}$; and the corresponding decimal fraction for a radius of 1 is 0.017452406437283571. All figures in both cases are correct.

Al-Kāshī's method of numerical solution of the trisection equation, whose variants were also presented by Ulugh Bēg, Qāḍi Zāde, and his grandson Maḥmūd ibn Muḥammad Mīrīm Chelebī (who worked in Turkey),[28] requires a relatively small number of operations and shows the exactness of the approximation at each stage of the computation. Doubtless it was one of the best achievements in medieval algebra. H. Hankel has written that this method "concedes nothing in subtlety or elegance to any of the methods of approximation discovered in the West after Viéte."[29] But all these discoveries of al-Kāshīs's were long unknown in Europe and were studied only in the nineteenth and twentieth centuries by such historians of science as Sédillot, Hankel, Luckey, Kary-Niyazov, and Kennedy.

Astronomy. Until now only three astronomical works by al-Kāshī have been studied. His *Khāqānī Zij*, as its title shows, was the revision of the *īlkhānī Zij* of Naṣīr al-Dīn al-Ṭūsī. In the introduction to al-Kāshī's *Zij* there is a detailed description of the method of determining the mean and anomalistic motion of the moon based on al-Kāshī's three observations of lunar eclipses made in Kāshān and on Ptolemy's three observations of lunar eclipses described in the *Almagest*. In the chronological section of these tables there are detailed descriptions of the lunar Muslim (Hijra) calendar, of the Persian solar (Yazdegerd) and Greek-Syrian (Seleucid) calendars, of al-Khayyāmī's calendar reform (Malikī) of the Chinese-Uigur calendar, and of the calendar used in the II-Khan empire, where Naṣīr al-Dīn al-Ṭūsī had been working. In the mathematical section there are tables of sines and tangents to four sexagesimal places for each minute of arc. In the spherical astronomy section there are tables of transformations of ecliptic coordinates of points of the celestial sphere to equatorial coordinates and tables of other spherical astronomical functions.

There are also detailed tables of the longitudinal motion of the sun, the moon, and the planets, and of the latitudinal motion of the moon and the the planets. Al-Kāshī also gives the tables of the longitudinal and latitudinal parallaxes for certain geographic latitudes, tables of eclipses, and tables of the visibility of the moon. In the geographical section there are tables of geographical latitudes and longitudes of 516 points. There are also tables of the fixed stars, the ecliptic latitudes and longitudes, the magnitudes and "temperaments" of the 84 brightest fixed stars, the relative distances of the planets from the center of the earth, and certain astrological tables. In comparing the tables with Ulugh Bēg's *Zij*, it will be noted that the last tables in the geographical section contain coordinates of 240 points, but the star catalog contains coordinates of 1,018 fixed stars.

In his *Miftāḥ al-ḥisāb* al-Kāshi mentions his *Zij al-tashilāt* ("Zij of Simplifications") and says that the also composed some other tables.[30] His *Sullam alsamā*ʾ, scarcely studied as yet, deals with the determination of the distances and sizes of the planets.

In his *Risāla dar sharḥ-i ālāt-i raṣd* ("Treatise on the Explanation of Observational Instruments") al-Kāshi briefly describes the construction of eight astronomical instruments: triquetrum, armillary sphere, equinoctial ring, double ring, Fakhrī sextant, an instrument "having azimuth and altitude," an instrument "having the sine and arrow," and a small armillary sphere. Triquetra and armillary spheres were used by Ptolemy; the latter is a model of the celestial sphere, the fixed and mobile great circles of which are represented, respectively, by fixed and mobile rings. Therefore the armillary sphere can represent positions of these circles for any moment; one ring has diopters for measurement of the altitude of a star, and the direction of the plane of the ring determines the azimuth. The third and seventh instruments consist of several rings of armillary spheres. The equinoctial ring (the circle in the plane of the celestial equator), used for observation of the transit of the sun through the equinoctial points, was invented by astronomers who worked in the tenth century in Shīrāz, at the court of the Buyid sultan ʿAḍūd al-Dawla. The Fakhrī sextant, one-sixth of a circle in the plane of the celestial meridian, used for measuring the altitudes of stars in this plane, was invented about 1000 by al-Khujandī in Rayy, at the court of the Buyid sultan Fakhr al-Dawla. The fifth instrument was used in the Mrāgha observatory directed by Naṣ al-Din al-Ṭūsā. The sixth instrument, al-Kāshī say, did not exist in earlier observatories; it is used for determination of sines and "arrows" (versed sines) of arcs.

In *Nuzha al-ḥadāiq* al-Kāshī describes two instruments he had invented: the "plate of heavens" and the "plate of conjunctions." The first is a planetary equatorium and is used for the determination of the ecliptic latitudes and longitudes of planets, their distances from the earth, and their stations and retro-gradations; like the astrolabe, which it resembles in shape, it was used for measurements and for graphical solutions of problems of planetary motion by means of a kind of nomograms. The second instrument is a simple device for performing a linear interpolation.

NOTES

1. See E. S. Kennedy, *The Planetary Equatorium . . .*, p. 1.

2. H. Suter, *Die Mathematiker und Astronomen . . .*, pp. 173–174; Kennedy, *op. cit.,* p. 7.

3. See M. Krause, “Stambuler Handschriften . . .,” p. 50; M. Ṭabāṭabāʾi,” “Jamshīd Ghiyāth al-Dīn Kāshānī,” p. 23.

4. D. G. Voronovski, “Astronomuy Sredney Sredney Azii ot Muhammeda al-Havarazmi do Ulugbeka i ego shkoly (IX-XVI vv.),” pp. 127, 164.

5. See V. V. Bartold, *Ulugbek i ego uremya*, p. 108.

6. Kennedy, *op. cit.*, pp. 1-2.

7. Bartold, *op. cit.*, p. 108; Kennedy, *op. cit.*, p. 2.

8. V. A. Shishkin, “Observatoriya Ulugbeka i ee issledovanie,”p. 10.

9. See T. N. Kary-Niyazov, *Astronomicheskaya shkola Ulugbeka,* 2nd ed., p. 107; see also Shishkin, *op. cit.*

10. See Kary-Niyazov, *loc. cit.*

11. See Bartold, *op. cit.* p. 88.

12. *Ibid.*, pp. 88–89.

13. E. S. Kenedy, “A Letter of Jamshī al-Kāshī to His Father,” p. 200.

14. Suter, *op, cit.*, pp. 173, 175; E. S. Kennedy, “A Survery of Islamic Astronomical Tables,” p. 127.

15. Kennedy, “A Letter...,” p. 194.

16. See Bartold, *op, cit.*, p. 108.

17. See *Zij-i Ulughbeg*, French trans., p. 5.

18. See Kennedy, *The Planetary Equatorium...*, p. 6.

19. See *Taʿrīb al-zij;* Kary-Niyazov, *op. cit.*, 2nd ed., ppl. 141–142.

20. See P. Luckey, *Die Rechenkunst...* A. P. Youschkevitch; *Geschichte der Mathematik im Mittelalter* p. 237 ff.

21. al-Kāshī, *Klyuch arifinetiki...*, p. 13.

22. See P. Luckey, “Die Auszichung des *n*-ten Wurzel...”

23. P. Luckey, “Die Ausziehung des *n*-ten Wurzel...”; Juschkewitsch, *op. cit.*, pp., 240–248.

24. See A. Saiden, “The Earliest Extant Arabic Arithmetic...”; Juschkewitsch, *op. cit.*, pp. 21–23.

25. H. Hunger and K. Vogel, *Ein byzantinisches Rechenbuch des* 15. *Jahrhunderts*, p. 104.

26. al-Kāshi, *Klyuch arifmetiki...*, p. 192.

27. *Ibid.*, p. 126.

28. Kary-Niyazov, *op. cit.*, 2nd ed, p. 199; Qāḍī Zāde, *Risāla fī istikhraāj jaib daraja wāhida; Mīrīm Chelebī Dastūr al-ʿamal wa taṣḥīḥ al-jadwal.*

29. H. Hankel, *Zur Geschichte der Mathematik*..., p. 292.

30. al-Kāhī*Klyuch arifmetiki*..., p. 9.

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His individual works are the following:

1. *Sullam al-samāʿ fi ḥall ishkāl waqaʿa liʾl-muqaddimī fiʾl-abʿād waāl-ajrām* ("The Stairway of Heven, on Resolution of Difficulties Met by Predecessors in the Determination of Distances and Sizes"; 1407). Arabic MSS in London, India Office 755; and Oxford, Bodlye 888/4.

2. *Mukhtaṣar dar ʿlim-i hayʾ at* ("Compendium on the Science of Astronomy") or *Risāla dar hayʾ* at ("Treatise on Astronomy"; 1410–1411). Persian MSS in London and Yezd.

3. *Zij-i Khaqāni fī takmīl-i Zij-i Īlkhānī* ("Khaqāni Zij— perfection of īlkhānī Zij" 1413–1414). Persian MSS in London, Istanbul, Teharan, Yezd, Meshed, and Hyderabad-Deccan, the most important being London, India Office 2232, which is described in E. S. Kennedy, "A Survey of Islamic Astronomical Tables," pp. 164–166.

4. *Risāla dar sharḥ-i ālāt-i raṣd* ("Treatise on the Explanation of Observational Instruments"; 1416). Persian MSS in Leiden and Teharan, the more important being Leiden, Univ. 327/12, which has been pub. as a supp. to V. V. Bartold, *Ulugbek i ego uremya;* and E. S. Kennedy," Al-Kāshi's Treatise on Astronomical Observation Instruments," pp. 99, 101, 103. There isd an English trans. in Kennedy, "Al-Kāshī's Treatise...," pp. 98–104; and a Russian trans. in V. A. Shishkin, "Observatoriya Ulugebeka i ee issledovanie," pp. 91–94.

5. *Nuzha al-ḥadāiq fi kayfiyya ṣanʿsa al-āla al-musammā bi ṭabaq al-manāṭiq* ("The Garden Excursion,; on the Method of Construction of the Instrument Called Plate of Heavens"; 1416). Arabic MSS are in London, Dublin, and Bombay, the moist important being London, India Office Ross 210. There is a litho. ed. of another MS as a Supp. to the Teheran ed. of *Miftāḥ al-ḥisā* see also *Risāla fiʿl-ʾamal bi ashal āla min qabl al-nujūm;* G. D. Jala-lov, "Otlichie 'Zij Guragani' ot drugikh podobnykh zijey" and "K voprosu o sostavelnii planetnykh tablits samarKandskoy observatorii"; T. N. Kary-Niyazov, *Astronomicheskaya shkola Ulugbeka;* and E. S. Kennedy, "Al-Kāshī's 'Plate of Conjunctions.'"

6. *Risāal-muḥīṭīyya* ("Treatise on the Circumference"; 1424). Arabic MSS are in Istanbul, Teheran, and Meshed, the most important being Istanbul, Ask. müze. 756. There is an ed. of another MS in *Majmūʾ* and one of the Istanbul MS with German trans. in P. Luckey, *Der Lehrbrief über den Kreisumfang von Gamšīd b. Masʿūd al-Kāši*. Russian trans. are in "Matematicheskie trakaty," *pp*.327–379; and in *Klyuch arifmetiki*, pp. 263–308, with photorepro . of Istanbul MS pp. 338–426.

7. *Ilkaḥāt an-Nuzha* ("Supplement to the Excursion" 1427). There is an ed. of a MS in Majmūʾ.

8. *Miftāḥ al-ḥisāb* ("The Key of Arithmetic") or *Miftāḥ al-ḥussāb fi 'ilm al-ḥisāb* ("The Key of Reckoners in the Science of Arithmetic"). Arabic MSS in Leningrad, Berlin, Paris, Leiden, London, Istanbul, Teheran, Meshed, Patna, Peshawar, and Rampur, the most important being Leningrad, Publ. Bibl. 131; Leiden, Univ. 185; Berlin, Preuss. Bibl. 5992 and 2992a, and Inst. Gesch. Med. Natur. 1.2; Paris, BN 5020; and London, BM 419 and India Office 756. There is a litho. ed. of another MS (Teheran, 1889). Russian trans. are in "Matematicheskie traktaty," pp. 13–326; and *Klyuch arifmetiki*, pp. 7–262, with photorepro. of Leiden MS on pp. 428–568, There is an ed. of the Leiden MS with commentaries (Cairo, 1968). See also P. Luckey, "Die Ausziehung dos *n*-ten Wurzel..." and "Die Rechenkunst bei Ğamšid b. Masʿud al-Kāašsī..."

9. *Talkhīis al-Miftāah* ("Compendium of the Key"). Arabic MSS in London, Tashkent, Istanbul, Baghdad, Mosul, Teheran, Tabriz, and Patna, the most important being London, India Office 75; and Tashkent, Inst. vost. 2245.

10. *Risāla al-watar waʾl-jaib* ("Treatise on the Chord and Sine"). There is an ed. of a MS in *Majmūʾ*.

11. *Taʿrib al-zij* ("The Arabization of the Zīj"), an Arabic trans. of the intro. to Ulugh Bēg's *Zīj*. MSS are in Leiden and Tashkent.

12. *Wujūuh al-ʿamal al-ḍarb fiʿl-takht waʿl-turāb* ("Ways of Multiplying by Means of Board and Dust"). There is an ed. of an Arabic MS in *Majmūʾ*.

13. *Natāʿij al-ḥaqāʾiq* (“Results of Verities”). There is an ed. of an Arabic MS in *Majmūʾ*.

14. *Miftāḥ al-asbāb fi ʿilm al-zij* (“The Key of Causes in the Science of Astronomical Tables”). There is an Arabic MS in Mosul.

15. *Risāla dar sakht-i asṭurlāb* (“Treatise on the Construction of the Astrolabe”). There is a Persian MS in Meshed.

16. *Risāla fi maʾrifa samt al-qibla min dāira hindiyya maʾrūfa* (“Treatise on the Determination of Azimuth of the Qibla by Means of a Circle Known as Indian”). There is an Arabic MS at Meshed.

17. Al-Kāshī’s letter to his father exists in 2 Persian MSS in Teheran. There is an ed. of them in M. Ṭabāṭabāʿī, “Nāma-yi pisar bi pidar,” in *Amūzish wa parwarish,***10** , no. 3 (1940), 9–16, 59–62. An English trans. is E S. Kennedy, “A Letter of Jamshīd al-Kāshī to His Father” English and Turkish trans. are in A. Sayili, “Ghiyāth al-Dīn al-Kāshī’s Letter on Ulugh Bēg and the Scientific Activity in Samarkand,” in *Türk tarih kurumu yayinlarinden*, 7th ser., no. 39 (1960).

II. Secondary Literature. See the following: V. V. Bartold, *Ulugbek i ego uremya* (“Ulugh Bēeg and His Time”; Petrograd, 1918), 2nd ed. in his *Sochinenia* (“Works”), II, pt. 2 (Moscow, 1964), 23–196, trans into German as “Ulug Beg und Seine Zeit,” in *Abhandlungen für die Kunde des Morgenlandes,***21** no, 1 (1935); L. S. Bretanitzki and B. A. Rosenfeld, “Arkhitekturnaya glava traktata ‘Klyuch arifmetiki’ Giyas ad-Dina Kashi” (“An Architectural Chapter of the Treatise ‘ The Key of Arithmetic’ by Ghiyāth al-Dīn Kāshī”), in *Iskusstvo Azerbayjana*, **5** (1956), 87–130; C. Brockelmann, *Geschichte der arabischen literature* 2nd ed., II (Leiden, 1944), 273 and supp. II (Leiden, 1942), 295; Mīrīm Chelebī, *Dastūr al-ʿamal was taṣḥīh al-jadwal* (“Rules of the Operation and Correction of the Tables”; 1498), Arabic commentaries to Ulugh Bēg’s Zīj, contains an exposition of al-Kāshī’s. *Risāla al-watar waʾ l-jaib*—Arabic MSS are in Paris, Berlin, Istanbul, and Cairo, the most important being Pairs, BN 163 (a French trans. of the exposition is in L. A. Sédillot, “De lʾalgèbre chez les Arabes,” in *Journal asiatique*, 5th ser., **2** [1853], 323–350; a Russian trans. is in *Klyuch arifmetiki*, pp. 311–319); A. Dakhel, *The Extraction of the n-th Root in the Sexagesimal Notation. A Study of Chapter* 5, *Treatise* 3 *of Miftāḥ al Ḥisāb*, W. A. Hijab and E. S. Kennedy, eds.(Beirut, 1960); H. Hankel, *Zur Geschichte der Mathematik im Altertum und Mittelalter* (Leipzig, 1874); and H. Hunger and K. Vogel, *Ein byzantinisches Rechenbuch des 15. Jahrhunderts* (Vienna, 1963), text, trans., and commentary.

See also G. D. Jalalov,“Otlichie ‘Zij Guragani’ ot drugikh podobnykh zijey” (“The Difference of ‘Gurgani Zij’ from Other Zījes”), in *Istoriko-astronomicheskie issledovaniya*, **1** (1955), 85–100; “K voprosu o sostavlenii planetnykh tablits samarkandskoy observatorii” (“On the Question of the Composition of the Planetary Tables of the Samarkand Observatory”), *ibid*., 101–118; and “Giyas ad-Din Chusti (Kashi)—krupneyshy astronom i matematik XV veka” (“Ghiyāth al-Dīn Chūstī [Kāshī]— the Greatest Astronomer and Mathematician of the XV Century”), in *Uchenye zapiski Tashkentskogo gosudarstvennogo pedagogicheskogo instituta*, **7** (1957), 141–157; T. N. KaryNiyazov, *Astronomicheskaya shkola Ulugeka* (Moscow-Leningrad, 1950), 2nd ed. in his *Izbrannye trudy* (“Selected Works”), VI (Tashkent, 1967); and “Ulugbek i Savoy Jay Singh, “in *Fiziko-matematicheskie nauki v stranah Vostoka*, **1** (1966), 247–256; E. S. Kennedy, “Al-Kāshī’s Plate of Conjunctions,” on *Isis*, **38** , no. 2 (1947), 56–59; “A Fifteenth-Century Lunar Eclipse Computer,” in *Scripta mathematica*, **17** , no. 1–2 (1951), 91–97; “An Islamic Computer for Planetary Latitudes,” in *Journal of the American Oriental Society*, **71** (1951), 13–21; “A Survey of Islamic Astronomical Tables,” in *Transactions of the American philosophical Society*, n.s. **46** no. 2 (1956), 123–177; “Parallax Theory in Islamic Astronomy,” in *Isis*, **47** , no. 1 (1956), 33–53; *The Planetary Equatorium of Jamshid Ghiyāth al-Din al-Kāshi* (Princeton, 1960); “A Letter of Jamshid al-Kāshi to His Father. Scientific Research and Personalities of a Fifteenth Century Court,” in *Commentarii periodici pontifici Instituti biblici, Orientalia*, n.s. **29** , fasc. 29 (1960), 191–213; “Al-Kāshi’s Treatise on Astronomical Observation Instruments,” in *Journal of Near Eastern Studies*, **20** , no. 2 (1961), 98–108; “A Medieval Interpolation Scheme Using Second-Order Differences,” in *A Locust’s Leg. Studies in Honour of S. H. Tegi-zadeh* (London, 1962), pp. 117–120; and “The Chinese-Uighur Calendar as Described in the Islamic Sources,” in *Isis*, **55** , no. 4 (1964), 435–443; M. Krause, “Stambuler Handschriften islamischer Mathematiker,” in *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik*, Abt. B, 3 (1936), 437–532; P. Luckey, “Die Ausziehung des *n*-ten Wurzel and der binomische Lehrsatz in der islamischen Mathematik,” in *Mathematische Annalen*, **120** (1948), 244–254; “Die Rechenkunst bei Ǧamšid b. Mas’ūd al-Kāši mit Rückblicken auf die ältere Geschichte des Rechnens,” in *Abhandlungen für die Kunde des Morgenlandes*, 31 (Wiesbaden, 1951); and *Der Lehrbrief uber den Kreisumfang von Ǧamšid b. Mas’ūd al-Kăsi*, A. Siggel, ed. (Berlin, 1953); *Risāla fi ʿl-ʾamal bi ashal āla min qabl alnujūm* (“Treatise on the Operation With the Easiest Instrument for the Planets”), a Persian exposition of al-Kāshi’s *Nuzha*—available in MS as Princeton, Univ. 75; and in English trans. with photorepro. in E. S. Kennedy, *The Planetary Equatorium;* B. A. Rosenfeld and A. P. Youschkevitch, “O traktate Qāḍī-Zāde ar-Rūmi ob opredelenii sinusa odnogo gradusa” (“On Qāḍi-Zāde al-Rūmi’s Treatise on the Determination of the Sine of One Degree”), in *Istoriko-matematicheskie issledovaniya*, **13** (1960), 533–556; and Mūsā Qāḍi Zāde al-ūmi, *Risāla fī istikhrāj jaib daraja wāhida* (“Treatise on Determination of the Sine of One Degree”), an Arabic revision of al-Kāshi’s *Risāla al-watar wa’l-jaib*—MSS are Cairo, Nat. Bibl. 210 (ascribed by Suter, p. 174, to al-Kāshi himself) and Berlin, Inst. Gesch. Med. Naturw. 1.1; Russian trans. in B. A. Rosenfeld and A. P. Youschkevitch, “O traktate Qāḍī-Zāde...” and descriptions in G. D. Jalalov, “Giyas ad-Dīn Chusti (Kashi)...” and in Ṣālih Zakī Effendī, *Athār bāqiyya, I.*

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