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(*b*. second half of third century b.c.; *d*. early second century b.c.)

mathematical sciences.

Very little is known of the life of Apollonius. The surviving references from antiquity are meager and in part untrustworthy. He is said to have been born at Perga (Greek Πέϱγη), a small Greek city in southern Asia Minor, when Ptolemy Euergetes was king of Egypt (i.e., between 246 and 221 b.c.)[1] and to have become famous for his astronomical studies in the time of Ptolemy Philopator, who reigned from 221 to 205 b.c.[2] Little credence can be attached to the statement in Pappus that he studied for a long time with the pupils of Euclid in Alexandria.[3] The best evidence for his life is contained in his own prefaces to the various books of his *Conics*. From these it is clear that he was for some time domiciled at Alexandria and that he visited Pergamun and Ephesus.

The prefaces of the first three books are addressed to one Eudemus of Pergamum. Since the Preface to Book II states that he is sending the book by the hands of his son Apollonius, he must have been of mature age at the time of its composition.[4] We are told in the Preface to Book IV that Eudemus is now dead;[5] this and the remaining books are addressed to one Attalus. The latter is commonly identified with King Attalus I of Pergamum (reigned 241–197 b.c.); but it is highly unlikely that Apollonius would have negelected current etiquette so grossly as to omit the title of “King” (*βασιλ*ϵ*ύς*) when addressing the monarch, and Attalus was a common name among those of Macedonian descent. However, a chronological inference can be made from a passage in the Preface to Book II, where Apollonius says,“... and if Philonides the geometer, whom I introduced to you in Ephesus, should happen to visit the neighborhood of Pergamum, give him a copy [of this book].”[6] Philonides, as we learn from a fragmentary biography preserved on a papyrus and from two inscriptions, was an Epicurean mathematician and philosopher who was personally known to the Seleucid kings Antiochus IV Epiphanes (reigned 175–163 b.c.) and Demetrius I Soter (162–150 b.c.). Eudemus was the first teacher of Philonides. Thus the introduction of the young Philonides to Eudemus probably took place early in the second century b.c. The *Conics* were composed about the same time. Since Apollonius was then old enough to have a grown son, it is reasonable to accept the birth date given by Eutocius and to place the period of Apollonius’ activity in the late third and early second centuries b.c. This first well with the internal evidence which his works provide on his relationship to Archimedes (who died an old man in 212–211 b.c.); Apollonius appears at times to be developing and improving on ideas that were originally conceived by Archimedes (for examples see p. 189). It is true that Apollonius does, however, refer to Conon, an older (?) contemporary and correspondent of Archimedes as a predecessor in the theory of conic sections.[7]

Of Apollonius’ numerous works in a number of different mathematical fields, only two survive, although we have a good idea of the content of several others from the account of them in the encyclopedic work of Pappus (fourth century a.d.). But it is impossible to establish any kind of relative chronology for his works or to trace the development of his ideas. The sole chronological datum is that already established, that the *Conics* in the form that we have them are the work of his mature years. Thus the order in which his works are treated here is an arbitrary one.

The work on which Apollonius’ modern fame rests, the *Conics* (*ϰωνιϰά*), was originally in eight books. Books I–IV survive in the orginal Greek, Books V–VII only in Arabic translation. Book VIII is lost, but some idea of its contents can be gained from the lemmas to it given by Pappus.[8] Apollonius recounts the genesis of his *Conics* in the Preface to Book I[9]: he had originally composed a treatise on conic sections in eight books at the instance of one Naucrates, a geometer, who was visiting him in Alexandria; this had been composed rather hurriedly because Naucrates was about to sail. Apollonius now takes the opportunity to write a revised version. It is this revised version that constitutes the *Conics* as we know it.

In order to estimate properly Apollonius’ achievement in the *Conics*, it is necessary to know what stage the study of the subject had reached before him. Unfortunately, since his work became the classic textbook on the subject, its predecessors failed to survive the Byzantine era. We know of them only from the scattered reports of later writers. It is certain, however, that investigation into the mathematical properties of conic sections had begun in the Greek world at least as early as the middle of the fourth century b.c., and that by 300 b.c. or soon after, textbooks on the subject had been written (we hear of such by Aristaeus and by Euclid). Our best evidence for the content of these textbooks comes from the works of Archimedes. Many of these are concerned with problems involving conic sections, mostly of a very specialized nature; but Archimedes makes use of a number of more elementary propositions in the theory of conics, which he states without proof. We may assume that these propositions were already well known. On occasion Archimedes actually states that such and such a proposition is proved “in the Elements of Conics” (*ʾεν τοις ϰωνιϰοις στοιϰείοις*).[10] Let us leave aside the question of what work(s) he is referring to by this title; it is clear that in his time there was already in existence a corpus of elementary theorems on conic sections. Drawing mainly on the works of Archimedes, we can characterize the approach to the theory of conics before Apollonius as follows.

The three curves now known as parabola, hyperbola, and ellipse were obtained by cutting a right circular cone by a plane at right angles to a generator of the cone. According to whether the cone has a right angle, an obtuse angle, or an acute angle at its vertex, the resultant section is respectively a parabola, a hyperbola, or an ellipse. These sections were therefore named by the earlier Greek investigators "section of a right-angled cone," "section of an obtuse-angled cone," and "section of an acute-angled cone," respectively; those appellations are still given to them by Archimedes (although we know that he was well aware that they can be generated by methods other than the above). With the above method of generation, it is possible to characterize each of the curves by what is known in Greek as a σύμπτωμα, i.e., a constant relationship between certain magnitudes which vary according to the position of an arbitrary point taken on the curve (this corresponds to the equation of the curve in modern terms). For the parabola (see Figure 1), for an arbitrary point K, $KL^2 = 2\ AZ \bullet ZL$ (for suggested proofs of this and the *σύμπτωμα* of hyperbola and ellipse, see Dijksterhuis, *Archimedes*, pp.57–59, whom I follow closely here). In algebraic notation, if $KL = y$, $ZL = x$, $2\ AZ = p$, we get the characteristic equation of the parabola $y^2 = px$. Archimedes frequently uses this relationship in the parabola and calls the parameter p "the double of the distance to the axis" (*ʽα διπλασία τας μέχρι τού ʼάξονος*)[11] exactly describing 2 *ZA* in Figure 1 ("axis" refers to the axis of the *cone*). For the hyperbola and ellipse the following *σύμπτωμα* can be derived (see Figures 2 and 3):

in algebraic notation, if $KL = y$, $ZL = x_1$, $PL = x_2$ $2\ ZF = p$, $PZ = a$,

This is found in Archimedes in the form equivalent to

It is to be noted that in this system *ZL* always lies on the axis of the section and that *KL* is always at tight angles to it. In other words, it is a system of "orthogonal conjection."

Apollonius' approuch is radically differnt. He generates all three curves from the double oblique circular cone, as follows: in Figures 4,5, and 6 *ZDE* is the cutting plane. We now cut the cone with another plane orthogonal to the first and passing through the axis of the cone; the is known as the axial triangle (*ABG*); the latter must intersect the base of the cone in a diameter (*BG*) orthogonal to the line in which

the cutting plane intersects it (or its extension); it intersects the cutting plane in a straight line *ZH*. Then, if we neglect the trivial cases where the cutting plane generates a circle, a straight line, a pair of straight lines, or a point, there are three possibilities:

(a) The line *ZH* in which the cutting plane intersects the axial triangle intersects only one of the two sides of the axial triangle, *AB*, *AG*; i.e., it is parallel to the other side (Figure 4).

(b) *ZH* intersects one side of the axial triangle below the vertex *A* and the other (extended) above it (Figure 5).

(c) *ZH* intersects both sides of the axial triangle below *A* (Figure 6).

In all three cases, for an arbitrary point K on the curve.

Furthermore, in case (a),

If we now construct a line length Θ such that $\Theta = BG^2 \bullet AZ/AB \bullet AG$, it follows from (1) and (2) that $KL^2 = LZ \bullet \Theta$. Since none of its constituent parts is dependent on the position of K, Θ is a constant. In algebraic terms, if $KL = y$, $LZ = x$, and $\Theta = p$, then $y^2 = px$. In cases (b) and (c),

If we now construct a line length Ξ such that $\Xi = BS \bullet SG \bullet ZT/AS^2$, it follows from (1) and (3) that

Thus

and

In algebraic terms, if $KL = y$, $LZ = x$, $\Xi = p$, and $ZT = a$,

The advantage of such formulation of the *σύμπτώματα* of the curves from the point of view of classical Greek

geometry is that now all three curves can be determined by the method of "application of areas," which is the Euclidean way of geometrically formulating problems that we usually express algebraically by equations of second degree. For instance, Euclid (VI, 28) propounds the problem "To a given straight line to apply a parallelogram equal to a given area and falling short of it by a parallelogrammic figure similar to a given one." (See Figure 7, where for simplicity rectangles have been substituted for parallelograms.) Then the problem is to apply to a line of given length b a rectangle of given area *A* and side *x* such that the rectangle falls short of the rectangle *bx* by a rectangle similar to another with sides *c*, *d*. This is equivalent to solving the equation

Compare the similar problem Euclid VI, 29: “To a given straight line to apply a parallelogram equal to a given area and exceeding it by a parallelogrammic figure similar to a given one.”

This method is used by Apollonius to express the *σύμπτώματα* of the three curves, as follows (see Figures 8–10).

For case (a) (Figure 8), a rectangle of side *x* (equal to the abscissa) is applied (*παραβάλλεται*) to the line-length *p* (defined as above): this rectangle is equal to the square on the ordinate y. The section is accordingly called parabola (*παραβολη*, meaning “exact application”).

For case (*b*) (Figure 9), there is applied to *p* a rectangle, of side *x*, equal to y^2 and exceeding

ύπερβάλλον) by a rectangle similar to p/a. The section is accordingly named hyperdola (*ύπερβολη*), meaning “excess”).

For case (c) (Figure 10) there is applied to *p* a rectangle of side x. equal to y^2 anf falling short (*'ελλείπον*) of *p* by a rectangle similar to p/a. The section is accordingly named ellipe (*'ελλειψς*, meaning “falling short”).

This approach has several advantages over the older one. First, all three curves can be represented by the method of “application of areas” favored by classical Greek geometry (it has been appropriately termed “geometrical algebra” in recent times); the older approuch allowed this to be done only for the parabola. In modern terms, Apollonius refers the equation of all three curves to a coordinate system of which one axis isd a given diameter of the diameter. This brings us to a second advantage: Apllonius’ method of generating the curves immediately produces

oblique conjugation, whereas the older method produces orthogonal conjection. As we shall see, oblique conjugation was not entirely unknown to earlier geometers; but it is typical of Apollonius’ approach that he immediately develops the most general formulation. It is therefore a logical step, given this approuch, for Apollonius to prove (1,50 and the preceding propositions) that a σύμτπωμα equivalent to those derived above casn be established for any diameter of a conic and its ordicates: in modern terms, the coordinated of the curves can be transposed to any diameter and its tangent.

We cannot doubt that Apollonius’ approach to the generation and basic definition of the conic sections, as outlined above, was radically new. It is not easy to determine how much of the *content* of the *Conics* is new. It is likely that a good deal of the nomenclature that his work made standard was introduced by him; in particular, the terms “parabola,” “hyperbola,” and “ellipse” make sense only in terms of Apollonius’ method. To the parameter which we have called *p* he gave the name *'ορθία*(“orthogonal side” [of a rectangle]), referring to its use in the “application”: this term survives in the modern *latus rectum*. He defines “diameter” as *any* line bisecting a system of parallel chords in a conic, in accordance with the new generality of his coordinate system: this differs from the old meaning of “diameter” of a conic section (exemplified in Archimedes), which is (in Apollonian and modern terminology) the axis. But though this new terminology reflects the new approach, it does not in itself exclude the possibility that many of Apollonius’ results in the *Conics* were already known to his predecessors. That this is true at least for the first four books is suggested by his own Preface to Book I. He says there:[13]

The first four books constitue an elementary introduction. The first contains the methods of generating the three sections and their basic properties (*ανμπτώματα*) developed more fully and more generally (*καθόλνμάλλον*) than in the writings of others; the second contains the properties of the diameter and axes of the sections, the asymptotes and other things...; the third contains many surprising theorems useful for the syntheses of solid loci and for determinations of the possibilities of solutions (*διοριομούς*); of the latter the greater part and the most beautiful are new. It was the discovery of these that made me aware that Euclid has not worked out the whole of the locus for three and four lines,[14] but only a fortuitous part of it, and that not very happily; for it was not possible to complete the synthesis without my additional discoveries. The fourth book deals with how many ways the conic sections can meet one another and the circumference of the circle, and other additional matters, neither of which has been treated by my predecessors, namely in how many points a conic section or circumference of a circle can meet another. The remaining books are particular extensions (*περιονσοιαστικώτερα*); one of them [V] deals somewhat fully with minima and maxima, another [VI] with equal and similar conic sections, another [VII] with theorems concerning determinations (*διοριστικών*), another [VIII] with determinate conic problems.

From this one gets the impression that Books I-IV, apart from the subjects specifically singled out as original, are merely reworkings of the results of Apollonius’ predecessors. This is confirmed by the statement of Pappus, who says that Apollonius supplemented the four books of Euclid’s *Conics* (which Pappus *May* have known) and added four more books.[15]

Apollonius also claims to have worked out the methods of generating the sections and setting out their *ανμπτώματα* “more fully and more generally” than his predecessors. The description “more generally” is eminently justified by our comparison of the two methods. However, it is not clear to what “more fully” (*'επὶ πλεον*) refers. Neugebauer suggests that Apollonius meant his introduction of conjugate hyperboals (conjugate diameters in ellipse and hyperbola are dealt with in I, 15–16).[16] At least as probable is a more radical alternative, rejected by Neugebauer, that Apollonius is referring to his treatment of the two branches of the hyperbola as a unit (exemplified in I, 16 and frequently later). It is true that Apollonius applies the name “hyperhbola” only to a single branch of the curve (he refers to the two branches as the “opposite *sections”* [*τομαὶ 'αντικείμεναι*]); it is also true that in his own Preface to Book IV[17] he reveals that at least Nicoteles among his predecessors had considered the two branches together; but Apollonius’ very definition of a conic surface[18] as the surface *on both sides* of the vertex is significant in this context; and it is unlikely that any of his predecessors had *systematically* developed the theory of both branches of the hyperbola. Here again, then, we may reasonably regard Apollonius as an innovator in his *method*. But we are not justified in assuming that any of the

results stated in the first four books were unknown before Apollonius, except where he specifically states this, In this part of the work we must see him rather as organizing the results of his predecessors, consisting in part of haphazard and disconnected sets of theorems, into an exposition ordered rationally according to his own very general method. His mastery is such that it seems impossible to separate difffernt sources (as one can, for instance, in the comparable work of Euclid on elementary geometry).

Nevertheless, we may suspect that Archimedes, could he have read Books I-IV of the *Conies*, would have found few results in them that were not already familiar to him (although he might well have been surprised by the order and manual connection of the theorems). The predecessors of Archimedes were already aware that section could be generated by methods other than that described on p. 180. Euclid states that an ellips can be produced by cutting a cylinder by a plane not parallel to the base. [19] Archimedes himself certainly knew that there were many different ways of generating the sections from a cone. The bewst proof of this is *De Conoidibus et sphaerodibus* VII-IX, in which it is shown that for any ellipse it is possible to find an *oblique* circular cone from which that ellipse can be generated. Furthermore, it is certain that the essential properties of the oblique conjugation of at least the parabola were known to Apollonius' predecessors; for that is the essence of propositions I-III of Archimedes' *Quadrature of the Parabola*, which he states are proved "in the Elements of Conics."[20] In *De Conoidibus et sphaeroidibus* III, Archimedes states "If two tangents be drawn from the same point to *any* conic, and two chords be drawn inside the section parallel to the two tangents and intersecting one another, the product of the two parts of each chord [formed by the intersection] will have the same ratio to one another as the squares on the tangents … this is proved in the Elements of Conics."[21] It is plausible to interpret this, with Dijksterhuis, as treatment of all three sections in oblique conjugation.[22]

It is probable then that much of the contents of Books I-IV was already known before Apollonius. Conversely, Apollonius did not include in the "elementary introduction" of Books I-IV al theorems on conies known to his predecessors. For example, in a parabola the subnormal to any tangent formed on the diameter (*HZ* in Figure 11) is constant and equal to half the parameter *p*. This is assumed without proof by Archimedes[23] and by Diocles (perhaps a contemporary of Apollonius) in his proof of the focal property of the parabola in his work *On Burning Mirrors*.[24] We can therefore be sure that it was a well–known theorem in the *Elements of conics*. Yet in Apollonius it can be found only by combining the results of propositions 13 and 27 of Book V, one of the "particular extensions."

If Apollonius omitted some of his predecessors' results from the elementary section, we must not be surprised if he omitted altogether some results with which he was perfectly familiar: his aim was not to compile an encyclopedia of all possible theorems on conic sections, but to write a systematic textbook on the "elements" and to add some more advanced theory which he happened to have elaborated. The question has often been raised in modern times why there is no mention in the *Conics* of the focus of the parabola. The focal properties of hyperbola and ellipse are treated in III, 45–52: Apollonius proves, *inter alia*, that the focal distances at any point make equal angles with the tangent at that point and that their sum (for the ellipse) or difference (for the hyperbola) is constant. There is no mention of directrix, and from the *Conics* we might conclude that Apollonius was totally ignorant of the focus-directrix property of conic sections. However, it happens that Pappus proves at length that if a point moves in such a way that the ratio of its distance from a fixed point and its orthogonal distance from a fixed straight line is constant, then the locus of that point is a conic section; and that according as the ratio is equal to, greater than, or less than unity, the section will be respectively a parabola, a hyperbola, or an ellipse.[25] This amounts to the generation of the sections from focus and directrix. Pappus gives this proof as a lemma to Euclid's (lost) book On Surface Loci; hence, it has been plausibly concluded that the proposition was there stated without proof by Euclid.[26] If that is so, here is a whole topic in the theory of conics that must have been completely familiar to Apollonius, yet which he omits altogether. Thus the lack of any mention of the focus of the parabola in the *Conics* is not an argument for Apollonius' ignorance of it. I agree with those who argue on a priori grounds that he must have known of it.[27] Since he very probably dealt with it in his work(s) on burning mirrors (see p.189) there was all the more justification for omitting it from the *Conics*. In any case, we now have a proof of it, by Diocles, very close to the time of Apollonius. Since Diocles further informs us that a parabolic burning mirror was constructed by Dositheus, who corresponded with Archimedes, it is highly probable that the focal property of the parabola was well known before Apollonius.

For a detailed summary of the contents of the *Conics*, the reader is referred to the works of Zeuthen and Heath listed in the Bibliography. Here we will only supplement Apollonius' own description quoted above by noting that Book III deals with theorems on the rectangles contained by the segments of intersecting chords of a conic (an extension to conics of that proved by Euclid for chords in a circle), with the harmonic properties of pole and polar (to use the modern terms: there are no equivalent ancient ones), with focal properties (discussed above), and finally with propositions relevant to the locus for three and four lines (see n.14). Of Books V-VII, which are, to judge from Apollonius' own account, largely original, Book V is that which has particularly evoked the admiration of modern mathematicians: it deals with normals to conics, when drawn as maximum and minimum straight lines from particular points or sets of points to the curve. Apollonius finally proves, in effect, that there exists on either side of the axis of a conic a series of points from which one can draw only one normal to the opposite side of the curve, and shows how to construct such points: these points form the curve known, in modern terms, as the *evolute* of the conic in question. Book VII is concerned mainly with propositions about inequalities between various functions of conjugate diameters. Book VIII is lost, but an attempt at restoration from Pappus' lemmas to it was made by Halley in his edition of the *Conics*. If he is right, it contained problems concerning conjugate diameters whose functions (as "determined" in Book VII) have given values.

For a modern reader, the *Conics* is among the most difficult mathematical works of antiquity. Both form and content are far from tractable. The author's rigorous rhetorical exposition is wearing for those used to modern symbolism. Unlike the works of Archimedes, the treatise does not immediately impress the reader with its mathematical brilliance. Apollonius has, in a way, suffered from his own success: his treatise became canonical and eliminated its predecessors, so that we cannot judge by direct comparison its superiority to them in mathematical rigor, consistency, and generality. But the work amply repays closer study; and the attention paid to it by some of the most eminent mathematicians of the seventeenth century (one need mention only Fermat, Newton, and Halley) reinforces the verdict of Apollonius' contemporaries, who, according to Geminus, in admiration for his Conics gave him the title of The Great Geometer.[28]

In Book VII of his mathematical thesaurus, Pappus includes summaries of and lemmas to six other works of Apollonius besides the *Conics*. Pappus' account is sufficiently detailed to permit tentative reconstructions of these works, all but one of which are entirely lost. All belong to "higher geometry," and all consisted of exhaustive discussion of the particular cases of one or a few general problems. The contrast with Apollonius' approach in the *Conics*, where he strives for generality of treatment, is notable. A brief indication of the problem(s) discussed in these works follows.

(1) *Cutting off of a Ratio* (*λόγον ʼαπομη*), in two books, is the only surviving work of Apollonius apart from the *Conics*. However, it is preserved only in an Arabic version which, by comparison with Pappus' summary, appears to be an adaptation rather than a literal translation. Pappus describes the general problem as follows; "To draw through a given point a straight line to cut off from two given straight lines two sections measured from given points on the two given lines so that the two sections cut off have a given ratio,"[29] Apollonius discusses particular cases before proceeding to the more general (e.g., in every case discussed in Book I the two given lines are supposed to be parallel) and solves every case by the classical method of "analysis" (in the Greek sense). That is, the problem is presumed solved, and from the solution is deduced some other condition that is easily constructible. Then, by "Synthesis" from this latter construction, the original condition is constructed. We may presume that Apollonius followed the same method in all six of these works, especially since Book VII of Pappus was named *ʼαναλλόμενος*("Field of Analysis"). In the *Cutting off of a Ratio* the problem was reduced to one of "application of an area." Zeuthen[30] points out the relevance of this work to *Conics* III, 41: If one regards the theorem proved there as a method of drawing a tangent to a given point in a parabola by determining the intercepts it makes on two other tangents to the curve, that is exactly the problem discussed by Apollonius in this work. Although there is no mention in it of conic sections, the connection is surely not a fortuitous one. In fact, many of the problems discussed by Apollonius in the six works summarized in Book VII of Pappus can be reduced to problems connected with conics. (This helps to explain the great interest shown in this part of Pappus' work by mathematicians of the sixteenth and seventeenth centuries.)

(2) *Cutting off of an Area* (*κψϱόον ʼαποτομη*), in two books, has a general problem similar to that of the preceding work. But in this case the intercepts cut off from the two given lines must have a given product (in Greek terms, contain a given rectangle) instead of a given proportion.[31] Here again Zeuthen has shown that *Conics* III, 42 and 43, which concern tangents drawn to ellipse and hyperbola, are equivalent to particular cases of the problem discussed by Apollonius in this work.[32]

(3) Determinate Section (*διωϱισμένη ιομη*) deals with with following general problem: Given four points— *A, B, C, D*—on a straight line l, to determine a point *P* on that line such that the ratio *AP* • *CP* / *BP* • *DP* has a given value.[33] Since this comparatively simple problem was discussed at some length by Apollonius, Zeuthen conjectured—plausibly—that he was concerned to find the limits of possibility of a solution for the various possible arrangements of the points (e.g., when two coincide).[34] We know from Pappus' account that it dealt, among other things, with maxima and minima. Whether, as Zeuthen claims, the work amounted to "a complete theory of involution" cannot be decided on existing evidence. But it is a fact that the general problem is the same as determining the intersection of the line *l* and the conic that is the "locus for four lines," the four lines passing through *A, B, C*, and *D*; and Apollonius must have known this. Here again, then, is a connection with the theory of conics.

(4) Tangencies (*ʼεπαφαί*), in two books, deals with the general problem characterized by Pappus[35] as follows: "Given three elements, either points, lines or circles (or a mixture), to draw a circle tangent to each of the three elements (or through them if they are points)." There are ten possible different combinations of elements, and Apollonius dealt with all eight that had not already been treated by Euclid. The particular case of drawing a circle to touch three given circles attracted the interest of Vieta and Newton, among others. Although one of Newton's solutions[36] was obtained by the intersection of two hyperbolaones that can be reconstructed with some probability from Pappus' accounts, and solutions to other cases can also be represented as problems in conics, Apollonius seems to have used only straight-edge and compass constructions throughout. Zeuthen provides a plausible solution to the three-circle problem reconstructed from Pappus' lemmas to this work.[37]

(5) *Inclinations (νεύσεις)*, in two books, is described by Pappus on pages 670–672 of the Hultsch edition. In Greek geometry, a *νεύσις* problem is one that consists in placing a straight line of given length between two given lines (not necessarily straight) so that it is inclined *(νεύει*) toward a given point. Pappus tells us that in this work Apollonius restricted himself to certain "plane" problems, i.e., ones that can be solved with straight–edge and compass alone. The particular problems treated by Apollonius can be reconstructed with some probability from Pappus' account.

(6)*Plane Loci (ιόποι ᾽επίπεδοι)*, in two books, is described on pages 660–670 of Pappus,. "Plane loci" in Greek terminology are loci that are either straight lines or circles. In this work, Apollonius investigated certain conditions that give rise to such plane loci. From them one can easily derive the equation for straight line and circle in Cartesian coordinates.[38]

A number of other works by Apollonius in the field of pure mathematics are known to us from remarks by later writers, but detailed information about the contents is available for only one of these: a work described by Pappus in Books II of his *Collectio*.[39] Since the beginning of Pappus' description is lost, the title of the work is unknown. It expounds a method of expressing very large numbers by what is in effect a place-value system with base 10,000. This way of overcoming the limitations of the Greek alphabetic numeral system, although ingenious, is not surprising, since Archimedes had already done the same thing in his *Ψαμμὲτης* (or "Sand Reckoner").[40] Archimedes' base is $10{,}000^2$. It is clear that Apollonius' work was a refinement on the same idea, with detailed rules of the application of the system to practical calculation. Besides this we hear of works on the cylindrical helix (*κοκίας*);[41]on the ratio between dodecahedron and eicosahedron inscribed in the same sphere;[42]and a general treatise (*καθόλον πραγμτεία*).[43]It seems probable that the latter dealt with the foundations of geometry, and that to it are to be assigned the several rematks of Apollonius on that subject quoted by Proclus in his commentary on the first book of Euclid (see Friedlein's Index).

Thus Apollonius' activity covered all branches of geometry known in his time. He also extended the theory of irrationals developed in Book X of Euclid, for several sources mention a work of his on unordered irrationals (*περί τών ᾽ατάκτων ᾽αλόγων*).[44] The only information as to the nature of this work comes from Pappus' commentary on Euclid X, preserved in Arabic translation;[45] but the exact connotation of "unordered irrationals" remains obscure. Finally, Eutocius, in his commentary on Archimedes' Measurement of a Circle,[46] informs us that in a work called *᾽ωκντόκτον*, meaning "rapid hatching" or "quick delivery," Apollonius calculated limits for π that were closer than Archimedes' limits of 3–1/7 and 3–10/71. He does not tell us what Apollonius' limits were; it is possible to derive closer limits merely by extending Archimedes' method of inscribing and circumscribing regular 96-gons to polygons with an ever greater number of sides (as was frequently done in the sixteenth and seventeenth centuries).[47]Very probably this was Apollonius' procedure, but that cannot be proved.

In applied mathematics, Apollonius wrote at least one work on optics. The evidence comes from a late Greek mathematical work preserved only fragmentarily in a palimpsest (the "Bobbio Mathematical Fragment"). Unfortunately, the text is only partly legible at the crucial point,[48]but it is clear that Apollonius wrote a work entitled *On the Burning Mirror* (*περὶ τούς πνρ* ⟨ε⟩ *ίον*), in which he showed to what points parallel rays striking a spherical mirror would be reflected. The same passage also appears to say that in another work, entitled *To the Writers on Catoptrics*(*πρὸς τούς κατοπτρικού*), Apollonius proved that the supposition of older writers that such rays would be reflected to the center of sphericity is wrong. The relevance of his work on conics to the subject of burning mirrors is obvious. We may conjecture with confidence that Apollonius treated of parabolic as well as of spherical burning mirrors. But the whole history of this subject in antiquity is still wrapped in obscurity.

Several sources indicate that Apollonius was noted for his astronomical studies and publications. Ptolemaeus Chennus (see n. 2) made the statement that Apollonius was called Epsilon, because the shape of the Greek letter ϵ is similar to that of the moon, to which Apollonius devoted his most careful study. This fatuous remark incidentally discloses some valuable information. "Hippolytus," in a list of distances to various celestial bodies according to different authorities, says that Apollonius stated that the distance to the moon from the earth is 5,000,000 stades (roughly 600,000 miles).[49] But the only specific information about Apollonius' astronomical studies is given by Ptolemy (fl. a.d. 140) in the *Almagest*.[50] While discussing the determination of the "station" of a planet (the point where it begins or ends its apparent retrogradation), he states that Apollonius proved the following theorem. In Figure 12, *O* is the observer (earth), the center of a circle on the circumference of which moves an epicycle, center *C*, with (angular) velocity v_1; the planet moves on the circumference of the epicycle about *C* with velocity v_2, and in the same sense as *C* moves about *O*. Then Apollonius' theorem states that if a line *OBAD* is drawn from *O* to cut the circle at *B* and *D*, such that

B will be that point on the epicycle at which the planet is stationary. Ptolemy also indicates that Apol-lonius proved it both for the epicycle model and for an equivalent eccenter model (depicted in Figure 13; here the planet *P* moves on a circle, center *M*, eccentric to the earth *O*, such that $OM/MP = CD/OC$ in Figure 12; *M* moves about O with speed $[v_1 + v_2]$,

P about *M* with speed v_1). Even this much information is valuable, for it shows that Apollonius had already gone far in the application of geometrical models to explain planetary phenomena, and that he must have been acquainted with the equivalence of epicyclic and eccentric models (demonstrated by Ptolemy in *Almagest* III, 3); yet he was still operating with a simple epicycle/eccentric for the planets, although this would, for instance, entail that the length of the retrograde arc of a planet is constant, which is notoriously not the case. Neugebauer (see Bibliography) supposes, however, that the whole of the passage in which Ptolemy himself proves the above theorem is taken from Apollonius. That proof combines the two models of epicycle and eccenter in one by the ingenious device of using the same circle as both epicycle and eccenter; in other words, the epicycle model is transformed into the eccentric model by inversion on a circle. The procedure is worthy of Apollonius, and is indeed a particular case of the polepolar relationship treated in *Conics* III, 37. But Ptolemy (who of all ancient authors is most inclined to give credit where it is due) seems to introduce this device as his own,[51] and to return to Apollonius only later.[52] Fortunately, this uncertainty does not affect the main point: that Apollonius represents an important stage in the history of the adaptation of geometrical models to planetary theories. His real importance may have been much greater than we can ever know, since not only his astronomical works, but also those of his successor in the field, Hipparchus (*fl*. 130 b.c.), are lost.

It is not clear how far Appollonius applied his theoretical astronomical models to practical prediction (i.e., assigned sizes to the geometrical quantities and velocities). For the fact that he "calculated" the absolute distance of the moon need imply no more than imitation of the crude methods of Aristarchus of Samos (early third century b.c.); for "Hippolytus" also lists figures for distances in stades between the spheres of the heavenly bodies as given by Archimedes which cannot be reconciled with any rational astronomical

system.[53] We should not assume without evidence that Apollonius had any better basis for his lunar distance. There is, however, a passage in the astrologer Vettius Valens(*fl*. a.d. 160) that has been taken to show that Apollonius actually constructed solar and lunar tables.[54] The author says that he has used the tables of Hipparchus for the sun; of Sudines, Kidenas, and Apollonius for the moon; and also of Apollonius for both. But there is no certainty that "Apollonius" here refers to Apollonius of Perga. At least as likely is the suggestion of Kroll that it may be Apollonius of Mynda, who is known to us only from a passage of Seneca, from which it appears that he claimed to have studied with the "Chaldaeans" and that he was "very experienced in the examination of horoscopes."[55] The Apollonius of the Vettius Valens passage is also associated with Babylonian names and practices.

Although the mathematical stature of Apollonius was recognized in antiquity, he had no worthy successor in pure mathematics. The first four books of his *Conics* became the standard treatise on the subject, and were duly provided with elementary commentaries and annotations by succeeding generations. We hear of such commentaries by Serenus (fourth century a.d.?) and Hypatia (*d*. a.d. 415). The commentary of Eutocius (early sixth century a.d.) survives, but it is entirely superficial. Of surviving writers, the only one with the mathematical ability to comprehend Apollonius' results well enough to extend them significantly is Pappus (*fl*. a.d. 320), to whom we owe what knowledge we have of the range of Apollonius' activity in this branch of mathematics. The general decline of interest in the subject in Byzantium is reflected in the fact that of all Apollonius' works only *Conics* I-IV continued to be copied (because they were used as a textbook). A good deal more of his work passed into Islamic mathematics in Arabic translation, and resulted in several competent treatises on conics written in Arabic; but so far as is known, no major advances were made. (Ibn alHaytham discusses the focus of the parabola in his work on parabolic burning mirrors;[56] but this, too, may be ultimately dependent on Greek sources.) The first real impulse toward advances in mathematics given by study of the works of Apollonius occurred in Europe in the sixteenth and early seventeenth centuries. The *Conics* were important. but at least as fruitful were Pappus' reports on the lost works, available in the excellent Latin translation by Commandino, published in 1588. (We must remember in this context that Books V-VII of the *Conics* were not generally available in Europe until 1661, [57] too late to make a real impact on the subject.) The number of "restorations" of the lost works of Apollonius made in the late sixteenth and early seventeenth centuries, some by outstanding mathematicians(e.g., Vieta, whose *Apollonius Gallus*[1600] is a reconstruction of the *Tangencies*, and Fermat, who reconstructed the *Plane Loci*) attests to the lively interest that Pappus' account excited. It is hard to overestimate the effect of Apollonius on the brilliant French mathematicians of the seventeenth century, Descartes, Mersenne, Fermat, and even Desargues and Pascal, despite their very different approach. Newton's notorious predilection for the study of conics, using Apollonian methods, was not a chance personal taste. But after him the analytic methods invented by Descartes brought about a lack of interest in Apollonius which was general among creative mathematicians for most of the eighteenth century. It was not until Poncelet's work in the early nineteenth century, picking up that of Desargues, Pascal, and la Hire, revived the study of projective geometry that the relevance of much of Apollonius' work to some basic modern theory was realized. It is no accident that the most illuminating accounts of Apollonius' geometrical work have been written by mathematicians who were themselves leading exponents of the revived "synthetic" geometry, Chasles and Zeuthen.

The contribution of Apollonius to the development of astronomy, although far less obvious to us now, may have been equally important but, unlike his geometrical work, it had an immediate effect on the progress of the subject. Hipparchus and Ptolemy absorbed his work and improved on it. The result, the Ptolemaic system, is one of the most impressive monuments of ancient science (and certainly the longest-lived), and Apollonius' work contributed some of its essential parts.

NOTES

1. Eutocius, *Commentary*, Heiberg, II, 168, quoting one Heraclius.

2. Photius, *Bibliotheca*, p. 151b18 Bekker, quoting th dubious authority Ptolemaeus Chennus of the second century a.d.

3. Pappus, *Collectio* VII, Hultsch, p. 678

4. Heiberg, I, 192.

5.*Ibid*., II, 2.

6.*Ibid*., I, 192.

7.*Ibid*., Preface to Bk. IV, II, 2, 4.

8. Pappus, *Collectio* VII, Hultsch, p.990 ff.

9. Heiberg, 1, 2.

10.*De quadratura parabolae* III, Heiberg, II^2, 268; *cf. De conoidibus et sphaeroidihus III, Heiherg, 1*^{2}, 270.

11. E.g., *De conoidibus et sphcteroidibus* III, Heiherg, 1^2. 272.

12. for the ellipse, see. e.g. *De conoidibus et sphaeroidihus* VIII, Heiberg, I^2, 294, 22–26; for the hyperbola, *ibid*, XXV, Heiberg, I^2, 376, 19–23.

13. Heiberg, I, 2, 4.

14. In modern terms, the locus for four lines is the locus of a point whose distances *x, y, z, u* from four given straight lines, measured along a given axis, satisfy the equation xz/yu = constant. This locus is a conic. (The locus for three lines is just a particular case of the above: for the distances *x, y, z* from three lines, xz/y^2 = constant.) This is, in modern terms, an *anharmonic* ratio: it can be shown that the theorem that this locus is a conic is equivalent to some basic theorems of projective geometry. (See Michel Chalsles, *Aperçu historique*, pp. 58, 354 ff.)

15.*Collectio VII, Hultsch*, p. 672.

16. "Apollonius-Studien," p. 219.

17. Heiberg, II, 2.

18.*Ibid*., I, 6.

19.*Phaenomena*, ed. H. Menge (*Euclidis Opera Omnia VIII*) (Leipzig, 1916), p. 6.

20. Heiberg, II^2, 266–268.

21.*Ibid*., I^2, 270

22. Dijksterhuis, *Archimedes*, pp. 66, n.1, 106.

23.*De corporibus fluitantibus* II, 4, Heiberg, II^2, 357.

24. Chester Beatty MS. Ar.5255, f.4v.

25.*Collectio* VII, Hultsch, pp. 1006–1014.

26. See, e.g., Zeuthen, *Kegelschnitte*, p. 367 ff.

27. For a method of proving the focal property of the parabola exactly parallel to Apollonius' procedure for those of hyperbola and ellipse, see Neugebauer, "Apollonius-Studien," pp. 241–242.

28. Eutocius, *Commentary*, Heiberg, II, 170.

29. Hultsch, p.640.

30.*Kegelschnitte*, p. 345.

31. See Pappus, ed. Hultsch, pp. 640–642.

32.*Kegelschnitte*, p. 345 ff.

33. Pappus, ed. Hultsch, pp. 642–644.

34.*Kegelschnitte*, p. 196 ff.

35. Hultsch, p. 644.

36. Principia, Bk., I, Lemma XVI (Motte-Cajori trans., pp. 72–73).

37.*Kegelschnitte*, p. 381 ff.

38. See T.L. Heath, *A History of Greek Mathematics*, II, 187–189.

39. Hultsch, p. 2ff.

40. Heiberg, II^2, 216ff.

41. Proclus, *Commentary on Euclid*, ed. Friedlein, p. 105.

42. “Euclid,” Bk, XIV, ed, Heiberg, V, 2:the problem is solved by the author of this part of the *Elements*, a man named Hypsicles(*fl.ca*. 150 b.c.), but we cannot tell exactly how much he owes to Apollonius.

43. See the commentary on Euclid’s Data by Marinus (fifth century a.d.), in *Euclidis opera*, ed. Heiberv-Menge, Vi, 234.

44. Proclus, *op. cit*., p. 74

45. Ed. Junge-Thomson, p. 219.

46.*Archimedis opera*, ed. Heiberg, III^2, 258.

47. See E.W. Hobson, *Squaring the Circle* (CAmbridge, 1913), pp. 26–28.

48.*Mathematici Graec* Minores, ed. Heiberg, p. 88.

49.*Refutation of all Heresies* IV, 8, ed, Wendland, III, 41.

50. XII, l, ed, Heiberg, II, 450 ff.

51.*Ibid*., pp. 451,22.

52.*Ibid*., pp. 456,9.

53.*Refutation*, ed, Wendland, pp. 41–42.

54.*Anthologiae* XI II, ed. Kroll, 354.

55.*Questiones naturales* VII, 4, I, ed. Oltramarde, II, 304.

56. Ed. Heiberg-Wiedemann, in *Bibliotheca mathematica*, **10** (1910), 201–237.

57. 1661 is the date of the publication at Florence of Abraham Ecchellensis’ unsatisfactory version. Some knowledge of it had trickled out before, for Mersenne mentions some of the propositions in a book published in **1644** (see Introduction, xlvi, of ver Eecke’s translation of the *Conics*).

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