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(*b*. Groningen, Netherlands, 8 February 1700; *d*. Basel. Switzerland, 1 March 1782)

medicine, mathematics, physics.

Life. Daniel Bernoulli was the second son of Johann I Bernoulli and Dorothea Falkner, daughter of the patrician Daniel Falkner. At the time of Bernoulli'sbirth his father was professor in Groningen, but here turned to Basel in 1705 to occupy the chair of Greek. Instead, he took over the chair of mathematics, which had been made vacant by the death of his brother Jakob (Jacques) I. In 1713 Daniel began to study philosophy and logic, passed his b accalaureate in 1715, and obtained his master's degree in 1716. During this period he was taught mathematics by his father and, especially, by his older brother Nikolaus II. An attempt to place young Daniel as a commercial apprentice failed. And he was allowed to study medicine—first in Basel, then in Heidelberg (1718) and Strasbourg (1719). In 1720 he returned to Basel, where he obtained his doctorate in 1721 with a dissertation entitled *De respiratione* (1). That same year he applied for the then vacant professor ship in anatomy and botany (2), but the drawing of the lot went against him. Bad luck also cost him the chair of logic (3). In 1723 he journeyed to Venice, when he his brother Nikolaus had just departed and continued his studies in practical medicine under Pietro Antonio Michelotti. A severe illness prevented him from realizing his plan to work with G. B. Morgagni in Padua.

In 1724 Bernoulli published his *Exercitationes mathematicae* (4) in Venice, which attracted so much attention that he was called to the St. Petersburg Academy. He returned to Basel in 1725 and declared his readiness to go to the Russian capital with Nikolaus. That same year, he won the prize awarded by the Paris Academy, the first of the ten he was to gain. Bernfoulli's stay in St. Petersburg was marred by the sudden death of his be loved brother and by the rigorous climate, and he applied three times for a professorship in Basel, but in vain. Finally, in 1732, he was able to obtain the chair of anatomy and botany there.

His Petersburg years (1725–1733 [after 1727 he worked with Euler]) appear to have been Bernoulli's most creative period. During these years he outlined the *Hydrodynamica* and completed his first important work on oscillations (23) and an original treatise on the theory of probability (22). In 1733 he returned to Basel in the company of his younger brother Johann II, after a long detour via Danzig, Hamburg, and Holland, combined with a stay of several weeks in Paris. Everywhere he went, scholars received him most cordially.

*See p. 56 for genealogy chart.

Although largely occupied with his lectures in medicine, Bernoulli continued to publish in mathematics and mechanics, which interested him much more intensely. His principal work, the *Hydrodynamica* (31), had been completed as early as 1734 but was not published until 1738. About the same time his father published *Hydraulica*, predated to 1732.[1] This unjustifiable attempt to insure priority for himself was one among many instances that exhibited Johann I Bernoulli's antagonism toward his second son.

In 1743 Daniel Bernoulli was able to exchange his lectures in botany for those in physiology, which were more to his liking. Finally, in 1750, he obtained the chair of physics, which was his by rights. For almost thirty years (until 1776) he delivered his lectures in physics, which were enlivened by impressive experiments and attended by numerous listeners. He was buried in the Peterskirche, not far from his apartment in the Kleine Engelhof.

Works. Daniel Bernoulli's works include writings on medicine, mathematics, and the natural sciences, especially mechanics. His works in these different areas were usually conceived independently of each other, even when simultaneous. As a consequence itis legitimate to distinguish them by subject matter and to consider them in chronological order with in each subject.

Medicine. Bernoulli saw himself, against his inclination, limited to the field of medicine. Thus the future physicist promptly turned his interest to the mechanical aspects of physiology.[2] In his in augural dissertation of 1721(1), as a typical iatro physicist under the decisive influence of Borelli and Johann Bernoulli, he furnished a comprehensive review of the mechanics of breathing. During the same year he applied for the then vacant chair of anatomy and botany, presenting pertinent theses (2) in support of his candidacy. In St. Petersburg in 1728 he published a strictly mechanical theory of muscular contraction (10), which disregarded the hypothesis of fermentation in the blood corpuscles assumed by Borelli and Johann Bernoulli. That same year he furnished a beautifully clear contribution to the determination of the shape and the location of the entrance of the optic nerve into the bulbus, or blind spot (II). Also of great importance is a lecture on the computation of the mechanical work done by the heart (*vis cordis*). Bernoulli gave this address in 1737 at the graduation exercises of two candidates in medicine, and it was thus that he first developed a correct method for such calculations. Because of its lasting significance, this lecture was published with its German translation in 1941 (75). Contributions to the physiology of work, more particularly to the determination of the maximum work that a man can perform, are found in *Hydrodynamica* (sec. 9) and the prize winning treatise of 1753 (47). (In this context, Bernoulli meant by “maximum work” the quantity that a man could do over a sustained period of e.g., a working day.)

Mathematics. Medical research, however, did not divert Bernoulli from his primary interest, the mathematical sciences. This is evidenced by the publication in 1724 of his *Exercitationes mathematicae*, which he wrote during his medical studies in Italy. This treatise combined four separate works dealing, respectively, with the game of faro, the outflow of water from the openings of containers, Riccati’s differential equation, and the lunulae (figures bounded by two circular arcs). Ultimately, Bernoulli’stalent proved to lie primarily in physics, mechanics, and technology, but his mathematical treatises originated partly from external circumstancescati’s differential equation) and partly from applied mathematics (recurrent series, mathematics of probability).

The discussions on Jacopo Riccati’s differential equation were initiated in 1724 by the problem presented by Riccati in the *Supplement a to the Acta eruditorum*. Immediately thereafter Daniel Bernoulli offered a solution in the form of an anagram (5). In the two following papers, published in the *Actaeruditorum* (6, 7), as well as in the *Exercitation es mathematicae*, Bernoulli demonstrated that Riccati’s special differential equation $ax^n dx + u^2 dx = b$ du could be integrated through separation of the variables for the values $n = -4c/2c\pm1$), where c takes on all integral values—positive, negative, and zero.

In the first part of the *Exercitationes* (4), dealing with faro, Bernoulli furnished data on recurrent series that later proved to have no practical application. According to De Moivre, these series result from the generative fraction

Bernoulli made use of these series in (16) for the approximate calculation roots of algebraic equations. For this purpose, the fraction is broken up into partial fractions, which are then developed into power series yielding, in the case of simple roots$1/p$, $1/q$, and so on, the general term

$P = (Ap^n + Bq^n + \ldots)z^n$

and the following member

$Q = (Ap^{n+1} + Bq^{n+1} \ldots)z^{n+1}$.

If P is considerably larger than $q\ldots$, etc., then, for sufficiently large n, P is approximated by Ap^n, Q by Ap^{n+1}, and thus the smallest root, $1/p$, is approximated by P/Q. In treatise (20) this method is applied to infinite power series.

Divergent sine and cosine series are treated by Bernoulli in three papers (62, 64, 66). The starting point is the thesis formulated by Leibniz and Euler that the equation $1 - 1 + 1 - 1\ldots = 1/2$ is valid, which they base on the equation $1/(1 + x) = 1 - x + x^2\ldots$ for $x = 1$ and by observing that the arithmetic mean of the two possible partial sums of the series equals 1/2. In reality, however, this divergent series can be summed to many values, depending on the expression from which it is derived. On the other hand, it can be demonstrated that the mean value method for the equations found by Euler,

leads to a correct result. For if x is commensurable with π, but not a multiple of π, then the terms of these series for a definite p and for each n satisfy the conditions $a_{n+p} = a_n$ and $a_1 + a_2 + a_3 + \ldots + a_p = 0$. For this case, according to the Leibniz-Bernoulli rule, the sum of the series becomes equal to the arithmetic mean of the values $a_1, a_1 + a_2 a_1 + a_2 + a_3 \ldots a_1 + a_2 + a_3 + \ldots + a_p$

Interestingly, in (64) the integration of the above cosine series, with application of Leibniz’ series for $\pi/2$, yields the convergent series:

In (66) Bernoulli let the formulas derived by Bossut [3] for the sums of the finite sine and/or cosineseries n extend to the infinite. He assigned the value zero to the corresponding cos ∞x and sin ∞x and there by obtained the correct sums.

In his later years Bernoulli contributed two additional papers (70, 71) to the theory of the infinite continued fractions.

Rational mechanics. In order to appreciate Daniel Bernoulli’s contributions to mechanics, one must consider the state of this branch of science in the first half of the eighteenth century. Newton’s great work was already available but could be rendered fruit fulonly by means of Leibniz’ calculus. Collaterally there appeared Jakob Hermann’s *Phoronomia*(1716), a sort of textbook on the mechanics of solids and liquids that used only the formal geometrical method. Euler’sexcellent *Mechanica* (1736) dealt only with the mechanics of particles. The first theory on the movement of rigid bodies was published by Euler in 1765. The fields of oscillations of rigid bodies and the mechanics of flexible and elastic bodies were new areas that Daniel Bernoulli and Euler dominated for many years.

In his earliest publication in mechanics (9), Bernoulli attempted to prove the principle of the parallelogram of forces on the basis of certain case, assumed to be self-evident, by means of a series of purely logical extensions; this was in contrast with

Newton and Varignon, who attempted to derive this principle from the composition of velocities and acelerations. Like all attempts at logical derivation Bernoulli's was circular, and today the principle the parallelogram of forces is considered an axiom. This was one of the rate instances when Bernoulli discussed the basic principles of mechanics. General he took for granted the principles established by Newton; only in cosmology or astronomy (gravity)and magnetism was he unable to break away completely from a modified vortex theory of subtle matter propounded by Descartes and Huygens. The deductionof gravity from the rotation of the subtle matter can be found in (79) and (31, ch, 11) and the explanation of magnetism in (41).

Treatise (13), inspired by Johann I Bernoulli's reports, is a contribution to the theory of rotating bodies, which at that time, considering the state of the dynamics of rigid bodies, was no trivial subject. The starting point was the simple case of a system consisting of two rigidly connected bodies rotating around a fixed axis. By means of geometric-mechanical considerations based on Huygens, Bernoulli solved a number of pertinent problems. Let us mention here only a special case of König's theorem (1751), derived by formal geometrical means. Written analytically, it states that

where v_1 and v_2 represent velocities in a fixed system, V the velocity of the center of gravity, and v_1' and v_2' the velocities around the center of gravity.

The determination of a movement imparted to a body by an eccentric thrust and the calculation of the center of instantaneous rotation were accomplished by Bernoulli in 1737 (27). At his invitation Euler took up the problem simultaneously, with similar results. In this problem, Bernoulli limited himself to the simplest case, that involving rigid, infinitely thin rods. The motion caused by an impact on elastic rods was dealt with only much later (61).

The principle of areas and an extended version of the principle concerning the conservation of live force, both of which furnished integrals of Newton's basic equations, were published by Bernoulli, probably with Euler's assistance, in the *Berlin Mémoires* in 1745 and 1748 (40, 43). The principle of areas (40) was used and clearly formulated almost simultaneously by Bernoulli and Euler in their treatments of the problem involving the movement of a tube rotating around a fixed point and containing freely moving bodies.

The principle of conservation of live force (43) was developed by Bernoulli not only—as had been done before him—for the movements within a field of uniform gravity or within a field of one or several fixed centers of force, but also for a system of mobile, mutually attracting mass points. For example, given three centers with the masses m_1m_2, m_3, whose mutual distances change from initial *a, b, c*, to *x, y, z*, Bernoulli finds that if the gravity constant equals p^2/μ: for the difference of live forces,

Most probably this is the first time that the doublesum of m_hm_k/r_{hk} appears. However, the force function for conservative systems was first discovered by Lagrange.

Bernoulli also investigated problems of friction of solid bodies (36, 57.60). In his first such paper (36) he studied the movement of a uniformly heavy sphere rolling down an inclined plane and calculated the inclination at which the pure rotation changes into a motion composed of a rotatory and a sliding part.

The main problem of (60) consists in determining the progressive and rotatory motion of a uniformly heavy rod pressing upon a rough surface while a force oblique to the axis of the rod acts upon the rod.

A group of papers (14, 18, 21) dealing with the movement of solid bodies in a resisting medium is based on the presentation given by Newton in the *Principia*. The first two papers (14) deal with a rectilinear motion, the three subsequent ones (18, 21) with movement along a curve (pendulum swing). Here Bernoulli started with the usual premise that the resistance is largely proportional to the square of the velocity. At the same time he denied Newton's affirmation of a partial linear relation between resistance and velocity, but considered as probable the assumption that part of the resistance, at least for viscous fluids, is proportional to time (i. e., independent of speed). The value of these five papers rest sprimarily on their consistent analytical presentation and on the treatment of certain special problems.

Hydrodynamics.[4] Traditionally, Bernoulli's famerests on his, *Hydrodynamica*(31)—a term he himself introduced. The first attempt at solving the problem of out flow as presented in the *Exercitationes mathematicae* was conceived in accordance with the concepts of the time, and did little to advance them. Essentially it contained a controversy with Jacopo Riccati over Newton's two different views on the force of a liquid issuing from an opening. But as early as 1727 Bernoulli succeeded in breaking through to an accurate calculation of the problem (12). Further progress was represented by the published experiments on the pressure exerted on the walls of a tube by a fluid flowing through it (19). In 1733 Bernoullileft behind in St. Petersburg a draft of the *Hydrodynamica* that agrees extensively in substance although not in form with the final version. Only the thirteenth chapter of the definitive work is missing (82).

The treatise opens with an interesting history of hydraulics, followed by a brief presentation of hydrostatics. The following three chapters contain formulas for velocity, duration, and quantity of fluid flowing out of the opening of a container. The author treats both the case of a falling level of the residual fluid and that of a constant level in the reservoir, and takes into consideration the starting process (nonstationary flow)and radial contraction of the stream. Bernoulli based these deductions on the principle of the conservation of live force or, as he says, the quality of the *descensus actualis* (actual descent) and *ascensus potentialis* (potential ascent), whereby these physical magnitudes, which pertain to the center of gravity, are obtained from the former through division by the mass of water in the container. If weequate the changes in *ascensus potentialis* and *descensus actualis* resulting from the water outflow, we obtain, in the case of a dropping water level, a linear differential equation. The kinematic principle used was, the hypothesis of the parallel cross sections, which states that all particles of the liquid in a plan evertical to the flow have the same velocity, and that this velocity is inversely proportional to the cross section (principle of continuity).

Chapter 7 deals with the oscillations of the water in a tube immersed in a water tank and considers mainly the energy loss. Many years later Borda resumed these investigations, but arrived at another formula for the loss.

Chapter 9 contains a theory of machinery, lifting devices, pumps, and such, and their performance, as well as an extensive theory of the screw of Archimedes. A spiral pump related to the latter was discussed by Bernoulli much later (65). A theory of wind mill sails concludes the chapter.

Chapter 10 is devoted to the properties and motions of "elastic fluids" (i.e., gases), and its main importance lies in its sketch of a "kinetic gas theory." which enabled Bernoulli to explain the basic gas laws and to anticipate —in incomplete form—Van der Waals'equation of state, which was developed some hundred years later. Further on, Bernoulli examined the pressure conditions in the atmosphere, established a formula for relating pressure to altitude, provided a formula for the total refraction of light rays from various stellar heights, and was the first to derive a formula for the flow velocity of air streaming from a small opening.

Chapter 12 contains the some what questionable derivation of a rather unusual form of the so-called Bernoulli equation for stationary currents. For the wall pressure p in a horizontal tube, connected to an infinitely wide container filled with water to the level a and having the cross section n and an outlet with the cross section I, he determined the expression $p = [(n^2 - 1)/n^2]a$. Since $a/n^2 \sim u^2$ represents the height from which a body must fall to obtain the velocity u at the point observed, that expression becomes the equation $p + u^2 = a =$ const. More generally, for a current in a tube of any shape and inclination, u^2 must equal A/n^2, A or a being the distances between water surface and discharge opening or anycross section n. We then obtain the equation $p + A/n^2 = a$ and— with $A - z = a$ (z = distance between n and opening)—the term $p + z + u^2 = A$ = const. for the stationary current. Because of the system of measures used by Bernoulli, the constant factors have values other than those customarily used.

Chapter 13 is concerned with the calculation of the force of reaction of a laterally discharged fluid jet as well as with the determination of its pressure upon a facing plate. With the aid of the impulse theorem, Bernoulli proved that both pressures pare equal to the weight of the cylinder of water whose base equals the area n of the opening for the discharge and whose length is double the height a of the water. It is thus $p = 2\ gan = nu^2$. In contrast, Johann I Bernoulli advocated throughout his life the erroneous assumption of a cylinder length equal to the height of the water. A complicated calculation of the pressure of a water jet on an inclined plate is contained in (26). Toward the end of chapter 13 Daniel Bernoulli discusses the question or whether the traditional propelling forces of sail and oar could be replaced by such a force of reaction. This principle was converted to practice only many years later.

The weaknesses in the deduction of the so-called Bernoulli equation and Daniel's incomplete concepts of internal pressure can only be mentioned here. In this respect, Johann Bernoulli's *Hydraulica* represents a certain progress, which in turn inspired Euler in his work on hydrodynamics.

Vibrating systems. From 1728, Bernoulli and Euler dominated the mechanics of flexible and elastic bodies, [5] in that year deriving the equilibrium curves for these bodies. In the first part of (15) Bernoulli determined the shape that a perfectly flexible thread assumes when acted upon by forces of which one component is vertical to the curve and the other is parallel to a given direction. Thus, in one stroke he derived the entire series of such curves as the velaria, lintearia, catenaria etc.

More original was the determination of the curvature of a horizontal elastic band fixed at one end—aproblem simultaneously under taken by Euler. Bernoulli showed that the total moment of a uniform band around point s, by virtue of the weight P atits free end and of its own weight p acting on the center of gravity, relates to the curvature radius R by means of the equation

where by the are length s and the abscissa x are to be taken starting from the free end, with m being the modulus of bending and I the length of the string. A case involving a variable density and anoptionally directed final load is quite possible.

When he departed from St. Petersburg in 1733, Bernoulli left behind one of his finest works (23), ready for the printer. Here, for the first time, he defined the "simple modes" and the frequencies of oscillation of a system with more than one degree of freedom, the points of which pass their positions of equilibrium, at the same time. The inspiration for this work must have been the reports made by Johann I, toward the end of 1727, on treatment of a similar problem. In the first part of the treatise, Daniel Bernoulli discussedan arrangement consisting of a hanging rope loaded with several bodies, determined their amplitude rates and frequencies, and found that the number of simple oscillations equals the number of bodies (i.e., the degrees of freedom).

For a uniform, free-hanging rope of length l he found the displacement, y, of the oscillations at distance x from the lower end by means of the equation where a has to be determined from the equation and J_o is the first appearance of Bessel's function. It shows that α is the length of the simple pendulum of equal frequency. The above equation has an infinite number of real roots. Thus the rope can perform an infinite number of small oscillations with the frequencies . These theorems were demonstrated in (25) on the basis of a principle that is equivalent to that subsequently named after d' Alembert.

Immediately following Bernoulli's departure from St. Petersburg, there began between him and Euler one of the most interesting scientific correspondences of that time. In its course, Bernoulli communicated much important information from which Euler, through his analytical gifts and tremendous capacity for work, was able to profit within a short time.

The above results were corroborated by Bernoulli and Euler through additional examples. Thus Bernoulli, in extending paper (30), investigated small vibrations of a plate immersed in water (32)and those of a rod suspended from a flexible thread (34). Bothworks stress the difference between simple and composite vibrations. He investigated only the former, however, for composite vibrations ultimately change into the slower ones.

The following two papers (37, 38), dating from 1741–1743, deal with the transversal vibrations of elastic strings, with (37) discussing the motion of a horizontal rod of length *I*, fastened at one end to avertical wall. In order to derive the vibration equation, whose form he had known since 1735 (35), Bernoulli used the relation between curvature and moment, as detailed in (15): $m/R = M$. The resulting differential equation is $f^4\ d^4y = y\ dx^4$, where y becomes the amplitude at distance x from the band end, and $f^4 = m^4L/g$, if L is the length of the simple pendulum isochronal with the band vibrations and g is the load per unit of length. Bernoulli used the solution $y = y(x/f)$ through infinite series as well as in closed expression by means of exponential and trigonometric functions. The series of the roots l/f is an example of nonharmonic oscillations. In (38) Bernoulli discusses the differential equation in the Case of free ends.

Treatise (45), on vibrating strings, represented a reaction to the publications of d'Alembert and Euler, who calculated the form of the vibrating string from the partial differential equation

They thus moved the inference from the finite to the infinite up into the hypothesis, whereas Bernoulli always made this transition without thinking about it in the final, completed formula.

His deliberations in (45) started from the assumption that the single vibrations of a string of length *a* were furnished by $y = \alpha_n$, sin $n\pi x/a$ (n = any integral number). From this and from his previous deliberations he deduced that the most general motion could be represented by the super position of these single vibrations, i.e., by a series of the form

This equation appears nowhere explicitly, but it can be derived from a combination of various passages of this work and is valid only with the assumption of an initial velocity equaling zero.

In (46) Bernoulli determined the vibrations of a weightless cord loaded with n weights. He shows that in the case of $n = 2$, two simple vibrations, either commensurable or incommensurable, are possible, depending on the position and value of the two weights.

Treatise (53) is a beautiful treatment of the oscillations inside organ pipes, using only elementary mathematics. It is assumed that the movement of the particles parallel to the axis, the, velocities, and the pressure are equal at all points of the same cross section and that the compression at the open end of the pipes equals zero. Among other things, this work contains the first theory of conical pipes and an arrangement consisting of two coaxial pipes of different cross sections as well as a series of new experiments.

In paper (54), on the vibrations of strings of uneven thickness, Bernoulli inquires about cases where oscillations assume the form $y = Aq$ sin p Sin vt, where p and q are functions of x only and v is a constant. Here, for the first time, are solutions for the inverse problem, the determination of vibration curves from the distribution of density. In (63) he treats a specialcase in which the string consists of two parts of different thickness and length.

In treatise (67) Bernoulli compared the two possible oscillations of a body suspended from a flexible thread with the movement of a body bound with a rigid wire, and showed that one of the two oscillations of the first arrangement closely approximated the oscillation of the second arrangement. The method followed by Bernoulli is applicable only to infinitely small vibrations, and thus represents only a special case of the problem treated simultaneously by Eulerby means of the Newtonian fundamental equations of mechanics.

In (68, 69) Bernoulli once more furnished a comprehensive presentation of his views on the superposition principle, which he clarified by means of the example of the frequently studied double pendulum. These last papers show that he had nothing new to add to the problem of vibrations.

Probability and statistics. A number of valuable papers were published by Bernoulli on probability theory and on population statistics.[6] True, his youthful work on faro within the framework of the *Exercitationes mathematicae* contributed hardly anything new, but it was evidence of his early interest in the work on the theory of probability done by his predecessors Montmort and De Moivre, which had been nourished by discussions with his cousin Nikolaus 1. The most important treatise, and undoubtedly the most, influential, was the *De mensura sortis* (22), conceived while he was in St. Petersburg, which contains an unusual evaluation of capital gains, and thus also contains the mathematical formulation of a new kind of value theory in political economy.[7]

The basic idea is that the larger a person's fortuneis, the smaller is the moral value of a given increment in that fortune. If we assume, with Bernoulli, the special case, that a small increase of assets *dx* implies a moral value, *dy*, that is directly proportionalto *dx* and inversely proportional to the fortune *a—i.e., dy = b dx/a*—then it follows that the moral value *y* of the gains *x - a* complies with the formula *y =log x/a*.

If a person has the chances $p_1, p_2, p_3\ldots$, to make the gains $g_1, g_2, g_3\ldots$, where $p_1 + p_2 + \ldots = 1$, which reflects one and only one gain, then the mean value of the moral values of the gains is equal to

$$bp_1 \log a(a + g_1) + bq_2 \log a(a + g_2) + \ldots - b \log a$$

and the moral expectation (hope)

$$H = (a + g_1)^{p1} (a + g_2)^{p2} \ldots - a.$$

If gains are very small in comparison with the assets, then the moral hope converts to the mathematical expectation $H = p_1g_1 + p_2g_2 + \ldots$, There follow some applications of the preceding to risk insurance and a discussion of the Petersburg paradox.

Only in 1760 did Bernoulli again treat a problem of this sort: medical statistics concerning the rate of mortality resulting from smallpox in the various age groups (51). If ξ is the number of Survivors and *s* the number of those who at age *x* have not yet had smallpox, there results—given certain conditions—adilferential equation containing three variables that defines the ratio s/ξ as a function of *x*. A table calculated on that basis contains the values of $\xi\ s,\ \xi - s$, and so on valid for the first twenty-four years: ξ was taken from Halley's mortality table.[8] In (52) Bernoulli ardently advocated inoculation as a means of prolonging the average lifetime by three years.

In paper (55) Bernoulli treats, by means of urn models, problems of probability theory as applied to his treatise on population statistics (56). Their main purpose was to determine for every age the expected average duration of a marriage. Here and in his subsequent papers (58, 59) Bernoulli preferred to make use of infinitesimal calculus in probability theory by assuming continuously changing states. The problem treated in (58) is as follows: Given several urns, each of which contains *n* slips of the same color, but of a different color for each urn, one slip is taken from each urn and deposited in the next one, with the slip taken from the last urn deposited in the first. The question is, How many slips of each color do the various urns contain after a number *r* of such "permutations"? The problem treated in (59) belongs in the field of the theory of errors, and Concerns the determination of the probability with which (expressed in modern terms) a random variable subject to binomial distribution would assume values between two boundaries on either side of the mean value.

In paper (72) Bernoulli seeks to deal with the theory of errors in observation as a branch of probability theory. He challenges the assumption of Simpson and Lagrange that all observations are of equal importance. Rather, he maintains that small errors are moreprobable than large ones. Thus Bernoulli approximates the modern concept, except that he selects the semicircle instead of Gauss's probability curve.

Treatise (73) deals with errors to be considered inpendulum clocks, which are calculated partially by means of the method presented in (59).

Prizes of the Paris Academy. Bernoulli was highly esteemed for clarifying problems for a general public interested in the sciences. Of his essays entered in the competitions of the Paris Academy, ten were awarded prizes. Most of them concerned marine technology, navigation, and oceanology; but astronomy and magnetism were also represented.

His prize-winning paper of 1725 (8) dealt with the most appropriate shape for and the installation of hourglasses filled with sand or water. The subject of the 1728 contest was the cause and nature of gravity, on which Bernoulli prepared a manuscript, but the prize went to the Cartesian G. B. Bilfinger (79). In his entry for the 1729 competition Bernoulli indicated several methods for determining the height of the pole, particularly at sea, when only one unknown star is visible, or when one or more known stars are visible. The essay did not win a prize (80), but the manuscript is extant.

The prize of 1734 (24) was shared with his father, who begrudged Daniel his share of success. Here Daniel postulated an atmosphere resembling air and rotating around the solar axis, resulting in an increasing inclination of the planetary orbits toward the equator of the sun.

Bernoulli shared the 1737 prize for the best form of an anchor with Poleni (28). The 1740 prize on the tides was shared with Euler and several others. This important paper (33) on the relationship, recognized by Newton, between the tides and solar and lunar attraction, respectively, is still of interest, inasmuch as it furnishes a complete equilibrium theory of these phenomena.

The prize-winning papers of 1743 (39) and 1746 (41) deal with problems of magnetism. In the first paper Bernoulli considered all possibilities for reducing the sources of error in the inclination compass by improving construction. According to his instructions, the Basel mechanic Dietrich constructed such needles (49). The 1746 paper, written with his brother Johann **II** contains an attempt to establish a theory of magnetism. Both authors believed that there is a subtle matter which moves in the direction of the magnetic meridian and forms a vortex around the magnet.

The next prize, for the best method of determining the time at sea with the horizon not visible, was offered in 1745 for the first time. It was offered for a second time in 1747, And Bernoulli won (42). Included in the wealth of information contained in this paper are the proposals for improving pendulum and spring clocks and the description of a mechanism for holding a rod equipped with diopter in a vertical position, even in a turbulent sea. A detailed account of the determination of the time, with the position of a given star known, concludes this paper (see 17).

The 1748 prize, for the irregular movements of Saturn and Jupiter (81), went to Euler. Bernoulli'smanuscript has been preserved. The prize essay for 1749–1751 (44) discussed the question of the origin and nature of ocean currents, and added suggestions for measuring current velocities.

The problem treated by the prize essay of 1753 (47), the effect on ships of forces supplementary to that of the wind (e.g., rudder forces) was, answered by Bernoulli, mainly by means of detailed data on the maximum work that could be performed by a man in a given unit of time. Among other things, he calculated the number of oars men required for attaining a given ship velocity.

The subject of the prize essay for 1757 (48), proposals for reducing the roll and pitch of ships, gave Bernoulli the opportunity to air his views on the pertinent works of Bouguer and Euler, published several years earlier. Whereas Euler had limited himself to the free vibrations of a ship, Bernoulli extended his views to the behavior Of ships in turbulent seas, i.e., to forced vibrations. His findings prevailed for almost a century.

Evaluation and Appreciation. In order to appreciate both Bernoulli's importance in science, as indicated by the above summaries of his published works, and his private life, it is necessary to consider his extensive correspondence.[9] This includes his exchange of letters with Christian Goldbach (1723–1730), Euler (1726–1767, especially (1734–1750), and his nephew Johann III (1763–1774). Also important are his contemporaries' evaluations of Bernoulli and of his work. Unfortunately, his extremely popular lectures on experimental physics, in which he often introduced unproved hypotheses that have since been confirmed, apparently are not extant. Among them was his assertion of the validity of the relation later known as Coulomb's law in electrostatics. All of these achievements brought Bernoulli considerable fame in intellectual circles during his lifetime. He was a member of the leading learned societies and Académies, including Bologna (1724), St. Petersburg (1730), Berlin (1747), Paris (1748), London (1750), Bern (1762), Turin (1764), Zurich (1764), and Mannheim (1767).

We can now assert that Bernoulli was the first to link Newton's ideas with Leibniz' calculus, which he had learned from his father and his brother Nikolaus. He did not, however, attempt to solve the problems that confronted him by means of the fundamental Newtonian equations; rather, he preferred to use the first integrals of these equations, especially Leibniz'principle of the conservation of living force, which his father had emphasized. Like Newton, whose battles he fought on the Continent, Bernoulli was first and foremost a physicist, using mathematics primarily as a means of exploring reality as it was revealed through experimentation. Thus he was interested in physical apparatus as well as the practical application of the results of physics and other sciences.

Bernoulli's active and imaginative mind dealt with the most varied scientific areas. Such wide interests, however, often prevented him from carrying some of his projects to completion. It is especially unfortunate that he could not follow the rapid growth of mathematics that began with the introduction of partial differential equations into mathematical physics. Nevertheless, he assured himself a permanent place in the history of science through his work and discoveries in hydrodynamics, his anticipation of the kinetic theory of gases, a novel method for calculating the value of an increase in assets, and the demonstration that the most common movement of a string in a musical instrument is composed of the superposition of an infinite number of harmonic vibrations (proper oscillations).

Otto Spiess instituted the publication of editions of the works and correspondence of the Bernoullis, a project that has continued since Spiess's death.

NOTES

1.*Johannis Bernmilli Hyderaulica nunc primum detecta ac demonstratedirect ex fundamentis pure mechanicis*, in his *Operaimnia*, IV (Lausanne-Geneva, 1742), 387–488.

2.*Friedrich Huber, Daniel Bernoulli* (*1700–1782*) *als Physiologeund Statistiker*, Basler Veröffentlichungen zur Geschichte derMedizin und der Biologie, fasc, 8 (Basel, 1958).

3. Charles Bossut, “Maniéxe de sommer les suites…,” in *Mémoriesde mathématiques et de physique de l’Académie royale dessciences, Paris*, 1769 (1772), 453–466.

4. For Daniel Bernoulli’s hydrodynamic studies see Clifford Truesdell, *Rational Fluid Mechanics*, intro, to Euler’s *Opera Omnia*, 2nd ser., XII, Xiii (Zurich, 1954–1955).

5. Two excellent works are Clifford Truesdell. “The Rational Mechanics of Flexible or Elastic Bodies (1638–1788)“; hisIntro, to Euler’s *Opera omina*, 2nd ser., X, XI (Zurich, 1960);and H, Burkhardt, “Entwicklungen nach oscillierenden Functionenund Integration der Dfferentialglechungen der mathematichen Physik,” an *Jahresberichi der Deutschen Mathmematiker-Vereinigung*, **10** , no. 2 (1908), 1–24.

6. I. Todhunter, *A History of the Mathematical Theory of Probability From the Time of the Pascal to That of Laplace* (Cambridge-London, 1865; repr. New York 1949), pp. 213–238.

7. An English trans, by Louise Sommer appeared in *Econometrica*, **22** (Jan. 1954). There is also a German trans, with extensive commentary by Alfred Opringsheim (Leipzig, 1896).

8. Edmund Halley, “An Estimate of the Degrees of the Mortality of Mankind”,: in *Philosophical Transactions of the Royal Society of London*, no. 196 (1694), 596–610.

9. Correspondence with Euler and Christian Goldbch—as far asavailable—appeared in *Correspondence mathématique et physiquede quelques célébres géométres du XUVIIéme siécle*, II (St. Petershurg, 1843). Letters exchanged by Bernoulli and his nephew Johann III have not yet been published.

BIBLIOGRAPHY

The following abbreviations of journal titles are used in the listing of Bernoulli’s published works: *AE, Actaeruditorum; AP,. Acta Academiae Scientiarum Imperialis Petropolitanae; CP* (or *NCP*), *Commentarii* (or *Norvicommentarii*) *Academiae Scientiarum Imperialis Petropolitanae; Prix Piéces qui ont remporté les prix de l’Académieroyale des sciences* (Paris); *Hist, Berlin, Histoire de l’Académie royale des sciences et belles letters, Berlin, Mem. Berlin, Memoires de l’Académie royale des sciences et belles letters, Berlin; Mem, Paris, Mémories de mathématiques etde physique de l’Académie royale des sciences. Paris.*

The first year following a journal title is the serial year; the second is the year of publication.

1. *Dissertatio inauguralis physico-medica de respiratione* (Basel, 1721).

2. *Positiones miscellaneae medico-anatomico-botanicae* (Basel, 1721).

3. *Theses logicae Sistentes methodum examinandi syllogismorum validitatem* (Basel, 1722).

4. Execercitationes quaedam mathematicae (Venice, 1724).

5. “Notata in praecedens schediasma III. Co. Jacobi Riccati,” in *AE*, supp. 8 (1724).

6. “Danielis Betnoulli explanatio notationum suarum, quae exstant Supplem. Tomo VIII Sect II,” *ibid.*, 1725 (1725), also published in (4).

7. “Solutio problematis Riecatiani propositi in Act. Lips. Suppl. Tom. VIII p. 73,” *ibid*;, 1725 (1725), also publishedin (4).

8. “Discours sur la maniére la plus parfaite de conserversur merl’égalité du mouvement des clepsidres ou sabliers,“in. *Prix*, 1725 (1725).

9. “Examen principiorum mechanicae, et demonstrations geometricae de compositione et resolutione virium,“in *CP*, **1** , 1726 (1728).

10. “Tentamen novae de motu musculorum theoriae. “*ibid.*

11. “Experimentum circa nervum opticum,” *ibid.*

12. “Theoria nova de motu aquarum per canales quoscunque fluentium,” *ibid.*, **2** , 1727 (1729).

13. “De mutua relatione centri virium, centri oscillationiset centri gravitatis,” *ibid*.

14. “Dissertatio de actione fluidorum in corpora solidaet motu solidorum in fluidis,” *ibid*.; “Continuatio,” *ibid*., **3** , 1728 (1732).

15. “Methodus universalis determinandue curvaturaefili,” *ibid*.

16. “Observationes de seriebus quae formantur ex additione vel subtracione quacunque terminorum se mutuoconsequentium,” *ibid*.

17. “Problema astronomicum inveniendi altitudinem poli una cum declinalione stellae ejusdemque culminatione,“*ibid*., 4. 1729 (1735).

18. “Theorema de motu curvilineo corportum, quaeresistentiam patiuntur velocitatis suae quadrato proportionalem,“*ibid*.,; “Additamentum,” *ibid*., **5** , 1730/1731(1738),

19. “Experimenta coram societate instituta in confirmationem theoriae pressionum quas latera canalis ab aquatransfluente sustinent,” *ibid*., **4** , 1729 (1735).

20. “Notationes de aequationibus, quae progrediunturin infinitum, earumque resolutione per methodum serierum recurrentium,” *ibid*., **5** , 1730/1731 (1738).

21. “Dissertatio brevis de motibus corporum reciprocisseu oscillatoris, quae ubique resistentiam patiuntur quadratovelocitatis suae proportionlem,” *ibid*.

22. “Specimen theoriae novae de mensura sortis,“*ibid*.

23. “Theoremata de oscillationibus corporum filo flexili connexorum et catenae verticaliter suspensae,” *ibid*., **6** , 1732/1733 (1938).

24. “Quelle est la cause physique de l’inclinaison des plans des orbites des planétes par rapport au plan del’équttteur de la révolution du soleil author de son axe,“in *Prix* 1734 (1735).

25. “Demonstrationes theorematum Suorom de oscillation ibuscorpurum filo flexili connexorum et catenae verticaliter suspensae,” in *CP*, **7** , 1734/1735 (1740).

26. “De legibus quibusdam mechanicis, quas natura constanter affectat, nondum descriptis, earumque usu hydrodynamico, pro determinanda vi venae aqueae contra planum incurrentis,” *ibid*., **8** , 1736 (1741).

27. “De variatione motuum a percussione excentrica,“*ibid*., **9** , 1737 (1744).

28. “Reflexions sur la meilleure figure à donner auxancres,” in *Prix*, 1737(1731).

29. “Commentationes de immutatione et extensione principii conservationis virium vivarum, quae pro motu corporum coelestium requiritur,” in *CP*, **10** , 1738 (1747).

30. “Commentationes de statu aequilibrii corporumhumido insidentium,” *ibid*.

31. *Hydrodynamica, sive de viribus et motibus fluidorumcommentarii* (Strasbourg, 1738). The following truns, exist: German, with extensive commentary by Karl Flied, in theseries Veröffentlichungen des Forschungs instituts des Deutschen Museums für die Geschichte der Naturwissenschaftenund der Tedmik, Reihe C: QueUellentexte und Uebersetzungen, nos, la, Ib (Munich, 1965); English, *Hydrodynamics by Daniel Bernoulli, trans*, Thomas Carmodyand Helmut Kobus (New York, 1968), bound with Johann I Bernoulli’s *Hydraulics* (pp. 343–451); and Russian, *Daniel Bernoulli*. *Gidrodinamika ili zapiski o silakhi dvizheniakh zhidkostei*, trans. A. I. Nekrasov, K. K. Baumgart, and V. I. Smirnov (Moscow, 1959).

32. “De motibus oscillatoriis corporum humido iunsidentium,“in *CP*, **11** , 1739 (1750).

33. “Traité sur le flux et reflux de la mer.” In *Prix*, 17401741).

34. “De oscillationibus composites praesertim iss quae flunt in corporibus ex filo flexili suspensis, in *CP*, **12** , 1740(1750).

35. “Excerpta ex litteris ad Leonhardum Euler,” *ibid*.,**13** , 1741–1143 (1751).

36, “De motu mixto, quo corpora sphaeroidica super plano inclinato descendunt,” *ibid.*

37. “De vibrationibus et sono laminarum elasticarum,“*ibid.*

38. “De sonis multifariis quos laminae elasticae diversimodeedunt disquisitiones mechanico-geometricae experimentis acusticis illustratae et confirmatae,” *ibid.*

39, “Mémoire sur la maniére de construire les boussolesd’inclinaison,” in *Prix*, 1143 (1748).

40, “Nouveau probléme de mécanique,” in *Mem. Berlin*, 1745 (1746), trans, into German in Ostwald’s Klassiker der Exacten Wissenschaften, no. 191 (Leipzig, 1914), pp. 29–43.

41. “Nouveaux principes de mécaniqueet dephysique, tendans à expliquer la nature & les propriétés de l’aiman, “written with Johann II, in *Prix*, 1146 (1748).

42. “La meilleure maniérede trouver l’heure en mer,“*ibid.*, 1745 and 1747(1750).

43. “Remarques sur Ie principe de la conservation des forces vives pris dans un sens général,’ in *Mem. Berlin*, 1748 (1750), trans, into German in Ostwald’s Klassiker der Exacten Wissenschaften, no. 191 (Leipzig, 1914), pp. 67–75.

44. “Sur la nature et la cause des courans,” in *Prix*, 1749 and 1751 (1169).

45. “Réflexions et éclaircissemens sur les nouvelles vibrations des cordes,” in *Mem. Berlin*. 1753 (1755).

46. “Sur le mélange de plusieurs espéces de vibrations simples isochrones, qui peuvent coexister duns un mêmesystème de corps,” *ibid.*

47. “Recherches sur la manière la plus avantage use de supplüer ä l’action du vent sur les grands vaisseaux,” in *prix*, 1753 (1769).

48.“Quelle est fa meilleure maniere de diminuer leroulis & le tangage d’un navire,” *ibid.*, 1757 (1771).

49. “Sur les nouvelles aiguilles d’inclinasion,” in *Journaldes scavans*, 1757 (1757).

50.“Lettre de monsieut Daniel Bernoulli à M. Clairaut, au sujet des nouvelles découvertes faites sur les vibrations des cordes tendues,” *ibid.*, 1758 (1758).

51. “Essai d’ une nouvelle analyse de la mortalité causée par la petite vérole, & des avantages de l’inoculation pour la prevenir,” in *Mem. Paris*, 1760 (1766)

52.“Réfiexions sur les avantages de l’inoculation,” or *Mereure de France* (June 1760.).

53. “Sur le son & sur les tons des tuyax d’orguses, in *Mem. Paris*, 1762 (1764).

54. “Mémoire sur les vibrations des cordes d’une épaisseur inégale,” in *Mem. Berlin*, 1765 (1761).

55. “De usu algorithmi infini tesimlis in arte coniectandispecimen,” in *NCP*, **12** , 1766/1767 (1768).

56; “De duratione media matrimoniorum, pro quacunqueconiugum aelate,” *ibid.*, 5.

57. “Commentatio de utilissima accommodissima directione: potentiarum frictionibus mechanicis adhibendarum,“*ibid.*, **13** , 1768 (1769).

58. “Disqulsitiones analyticae de novo problemate coniecturali,“*ibid.*, **14** , 1769, pt. 1 (1170).

59; “Mensura sortis ad fortuitam successionem rerum naturaliter contingenutium applicata,” *ibid.*,: “Continuatio,” *ibid.*, **15** , 1770 (11).

60. “Commentationes physico-mechanicae de frictionibus,” *ibid.*, **14** , 1769, pt. I (1770.).

61. “Examen physico-mechanicum de motu mixto qui laminis elasticis a percussione simul imprimitur,” *ibid.*, **15** , 1770 (1771).

62. “De summationibus serierum quaruudam incongrue veris,” *ibid*.,**16** , 1771(1772).

63. “De vibrationibus chordarum,” *ibid*.

64. “De indole singulari serierum infinitarum quas sinus vel cosinus angulorum arithmetice progredientium formant, earumque summatione et usn,” *ibid*., **17** , 1172(1773).

65. “Expositio theoretica singularrs machinae hydraulieae,” *ibid*.

66. “Theoria elementaria serierum, ex sinibus atque cosinibus arcuum arithmetic progredientium diversimode compositarum, dilucidata,” *ibid*., **18** . 1773 (1774).

67. “Vera determinatio centri oscillationis in corporibusqualibuscunque filo fiexili suspensis eiusque ab regulacommuni discrepantia,” *ibid*.

68. “Commentatio physico-mechanica generalior principiide coexistentia vibrationum simplicium haud perturbatarumin systemate composito,“*ibid*., **19** , 1774 (1775).

69. “Commentatio physico-mechanica specialior demotibus reciprocis compositis,” *ibid*.

70. “Adversaria analytica miscellanea de fraction ibus continuis,” *ibid*., **20** , 1775(1776).

71. “Disqwsitiones ulteriores de indole fractionum continuum,“*ibid*.

72. “Diiudicatio maxime probabilis plurium observationum discrepantium atque verisimillima inductio inde formanda,” in *AP*, 1777, pt 1 (1778}.

73. “Specimen philosophicum de compensation ibushorologids, et veriori mcnsura temporis,” *ibid*., pt. 2(1780).

74. “Sur la cause des vents,” in Berlin Academy’s *Recueil des prix* without Bernoulli’s name. Also at University of Basel, Lla753E5.

75. “Oratio physiologica de vita.,” with German trans.and historical essays, ed. O. Spiess and F. Verzar, in *Verhandlungen der Naturforschenden Gesellschaft Basel*, **52** 1940/1941), 189–266. Also at University of Basel, Lia753E18.

The library of the University of Basel has most of Bernoulli’s original MSS, and photocopies of the rest.

76. “Method us isoperimetricorum ad novam problematum classem promota,” LIa751C3. From all curves of equal length lying between two fixed points, to find the ones for which $\int R^m ds$ is a minimum, where $R^m$is the *m*th power of the radius of curvature. *R* and *ds* is the arc element. This was first satisfactorily solved by Euler.

77. “Solutio problematis inveniendi curvam, quae cum aliis data sit tatuochrona” (1729), LIa751C4. Solution to the problem of finding the curves in which the oscillations of a center of mass moving in a vacuum are isochronous regardless of starting point. Also treated by Euder.

78. “De legibus motus mixti variati, qua corpus sphaericumsuper plano aspero progredietur,” LIaC 19.

79. “Discours sur la cause et la nature de la pesantur,“LIa752D2, submitted in the 1728 prize competition of the Paris Academy. The prize was awarded to Bilfinger.

80. “Quelle est fa meilleure méthode d’observer les hauteurs sur mer par de soleil et par les étoilcs,” Lta752D3, submitted in the 1729 prize competition of the ParisAcademy. The prize was awarded to Bouguer.

81. “Recherches mécanques et astronomiques sur la théorie de Satume et de Jupiter,” LIa33, submitted in the1748 prize competition of the Paris Academy. The prizewas awarded to Euler.

82. An outline of the *Hydrodynamica* (1733) is in Archivesof the Academy of Sciences, Leningrad; a photocopyis in Basel.

Biographical works on Bernoulli include Daniel II Bemoulli, “Vita banielis Bernoulli:’ in *Acta Helvelica*, **9** (1787), 1–32, a memorial address with an almost completebibliography of printed works; “Die Basler Mathematiker Bernoulli und Leonhard Euler. Vorträge von Fr. Burckhardtu.a.,” in *Verhandlungen der Naturforschenden Gesellseliaft Basel*, **7** (1884), appendix; Marquis de Condorcet. “Eloge de M. Bernoulli,” in *Hist*. Paris, 1782 (1785). 82–107, and in *Oeuvres de Condorcet*, II (Paris, 1847), 545–585; *Gedenkbuch der Familie Bernoulli zum 300. Jahresrage ihrer Aufnahme in das Basler. Bürgerrecht. 1622,* *1922* (Basel, 1922); Peter Merian, *Die Mathematiker Bernoulli. Jubelschrift Zur 4. Säcularjeier der Universität Basel. 6 September 1860* (Basel, 1860); Otto Spiess. Basel *anno 1760* (Bnsel, 1936); “Daniel Bernoulli,” in Eduard Fueter, ed., Gross Schweize Forscher (Zurich, 1939), pp. 110–112; “Johann Bernoulli und seine Soehne,” in *Atlantis* (1940). 663–669; “Die Mathematikerfamilie Bernoulli,” in Martin Huerlimann, ed., *Grosse Schweizer*, 2nd ed (Zurich, 1942), pp. 112–119; and “Bernoulli, Basler Gelehrten familie,” in *Neue deutsch Biographie* (1955), pp. 128–131); and RudolfWolf, “Daniel Bernoulli von Basel, 1700–1782,” in *Biographienzur Kuturgeschichte der Schweiz*, 3rd ser. (Zurich.1860), pp. 151–202.

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