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(*b*. Nemours, France, 31 March 1739; *d*. Basses-Loges, near Fontainebleau, France, 27 September 1783)

mathematics.

Étienne Bezout, the second son of Pierre Bezout and Hélène-Jeanne Filz, belonged to an old family in the town of Nemours. Both his father and grandfather had held the office of magistrate (*procureur aux baillage et juridiction)* there. Although his father hoped Étienne would succeed him, the young man was strongly drawn to mathematics, particularly through reading the works of Leonhard Euler. His accomplishments were quickly recognized by the Académie des Sciences, which elected him *adjoint* in 1758, and both *associé* and *pensionnaire* in 1768. He married early and happily: although he was reserved and somewhat somber in society, those who knew him well spoke of his great kindness and warm heart.

In 1763, the due de Choiseul offered Bezout a position as teacher and examiner in mathematical science for young would-be naval officers, the Gardes du Pavillon et de la Marine. By this time, Bezout had become a father and needed the money. In 1768 he added similar duties for the Corps d'Artillerie. Among his published works are the courses of lectures he gave to these students. The orientation of these books is practical, since they were intended to instruct people in the elementary mathematics and mechanics needed for navigation or ballistics. The experience of teaching nonmathematicians shaped the style of the works: Bezout treated geometry before algebra, observing that beginners were not yet familiar enough with mathematical reasoning to understand the force of algebraic demonstrations, although they did appreciate proofs in geometry. He eschewed the frightening terms "axiom," "theorem," "scholium," and tried to avoid arguments that were too close and detailed. Although criticized occasionally for their lack of rigor, his texts were widely used in France. In the early nineteenth century, they were translated into English for use in American schools; one translator, John Farrar, used them to, teach the calculus at Harvard University. The obvious practical orientation, as well as the clarity of exposition, made the books especially attractive in America. These translations considerably influenced the form and content of American mathematical education in the nineteenth century.

A conscientious teacher and examiner, Bezout had little time for research and had to limit himself to what was, for his time, a very narrow subject—the theory of equations. His first two papers (1758–1760) were investigations of integration, but by 1762 he was devoting all his research time to algebra. In his mathematical papers, Bezout often followed a "method of simplifying assumptions," concentrating on those specific cases of general problems which could be solved. This approach is central to the conception of Bezout's first paper on algebra, "Sur plusieurs classes d'équations" (1762).

This paper provides a method of solution for certain *n*th-degree equations. Bezout related the problem of solving *n*th-degree equations in one unknown to the problem of solving simultaneous equations by elimination: "It is known that a determinate equation can always be viewed as the result of two equations in two unknowns, when one of the unknowns is eliminated."[1] Since an equation can be so formed, Bezout investigated what information could be gained by assuming that it actually was so formed. Such a procedure resembles the eighteenth-century study of the root-coefficient relations in an *n*th-degree equation by treating it as formed by the multiplication of *n* linear factors. Now, if one of the two composing equations had some very simple form—for instance, had only the *n*th-degree term and a constant—Bezout saw that he could determine the form of its solution. Conversely, if the coefficients of a given *n*th-degree equation in one unknown had the form built up from such a special solution, that *n*th-degree equation could be solved. Bezoue's principal example considers

$x^n + mx^{n-1} + px^{n-2} + \cdots + M = 0$

as resulting from the equations

The importance of this paper lay in drawing Bezout's attention from the problem of explicitly solving the *n*th-degree equation—an important concern of eighteenth-century algebraists—to the theory of elimination, the area of his most significant contributions. The central problem of elimination theory for Bezout was this: given *n* equations in *n* unknowns, to find and study what Bezout called the resultant equation in one of the unknowns. This equation contains all values of that unknown that occur in solutions of the *n* given equations. Bezout wanted to find a resultant equation of as small degree as possible, that is, with as few extraneous roots as possible. He wanted also to find its degree, or at least an upper bound on its degree.

In his 1764 paper, "Sur le degré des équations résultantes de l'évanouissement des inconnues;" he discussed Euler's method for finding the equation resulting from two equations in two unknowns, and computed an upper bound on itsdegree.[2] He extended this method to N equations in N unknowns. But, although Euler's method yielded an upper bound on the degree of the resultant equation, Bezout observed that it was too clumsy to use for equations of high degree.

Another procedure, which gives a resultant equation of lower degree (now called the Bezoutiant) is given at the end of the 1764 paper. The equations to be solved are

where A, A′, B, B′,... are functions of y, and where $m \geqslant m'$. From these, he obtained m polynomials in x, of degree less than or equal to m-1, which have among their common solutions the solutions of (1) and (2). For the case $m = m'$, these polynominals are

He considered these polynomial equations as m linear equalions in the unknowns $x, x^2, \ldots, x^{m-1}$. And he observed that (1) and (2) have a common solution if these linear equations do. But when can the linear equations be solved?

At the beginning of this 1764 paper, Bezout had expressed what we would call a determinant by means of permutations of the coefficients, in what is sometimes called the Table of Bezout. He described the use of this table in solving simultaneous linear equations and, in particular, as a criterion for their solvability. This gave him a criterion for finding the resultant of (1) and (2). J. J. Sylvester, in 1853, explicitly gave the determinant of the coefficients of these m linear equations, and called it the Bezoutiant. The Bezoutiant, considered as a function of y, has as its zeros all the y's that are common solutions of equations (1) and (2).

It was not until 1779 that Bezout published his *Théorie des équations algébriques*, his major work on elimination theory. Its best-known achievement is the statement and proof of Bezout's theorem: "The degree of the final equation resulting from any number of complete equations in the same number of unknowns, and of any degrees, is equal to the product of the degrees of the equations."[3] Bezout, following Euler, defined a complete polynomial as one that contains each possible combination of the unknowns whose degree is no more than the degree of the polynomial. Bezout also computed that the degree of the resultant equation is less than the product of the degrees for various systems of incomplete equations. Here we shall consider only the complete case.

The proof makes one marvel at the ingenuity of Bezout, who, like Euler, not only could manipulate formulas but also had the ability to choose those manipulations that would be fruitful. He was compelled to justify his nth-order results by a naive "induction" from the observed truth of the statements for 1, 2, 3 ···. Also, numbered subscripts had not yet come into use, and the notations available were clumsy.

Here is Bezout's argument. Given n equations in n unknowns, of degrees t, t', t'' ···. Let us call the equations $P_1(u,x,y,\cdots)$ $P_2(u, x, y,\cdots)$, ··· (Bezout wrote them $(u\cdots n)^t$, $(u\cdots n)^{t'}$, ···). Suppose now that P_1 is multiplied by an indeterminate polynomial, which we shall designate as Q for definiteness, of degree T. If a Q can be found such that P_1Q involves only the unknown u, P_1Q will be the resultant; Bezout's problem then becomes to compute the smallest possible degree of such a P_1Q.

Bezout stated, and later[4] gave an argument to show, that he could solve the equations $P_2 u, x, y, \cdots) = 0$, ··· $P_n(u, x, y, \cdots) = 0$ to determine, respectively, $x^{t'}, y^{t''}, z^{t'''}\cdots$ in terms of lower powers of the unknowns. Substituting the values for $x^{t'}, y^{t''},\cdots$ in the product P_1Q would eliminate all the terms divisible by those powers of the unknowns.

The key to Bezout's proof was in counting the number of terms in the final polynomial, P_1Q. Bezout began his book with a derivation, by means of finite differences, of a complicated formula for the number of terms in a complete polynomial in several unknowns which are not divisible by the unknowns to particular powers; that is, for a complete polynomial in u, x, y, z, ··· of degree T, he gave an expression for the number of terms not divisible by $u^p, x^q, y^r, \cdots$, where $p + q + r + \cdots < T$. Bezout used this formula to compute the number of terms in the polynomials P_1Q and Q which remained after the elimination of $x^{t'}, y^{t''}, \cdots$.

Let us write N (instead of Bezout's complicated expression) for the number of terms remaining in P_1Q, M for those remaining in Q. If the degree of the resultant is to be D, then it will have D+1 terms, since it is an equation in the single unknown y. Then the coefficients of Q must be such that $N-(D+1)$terms in the product P_1Q will be annihilated by them. But, since Q or any multiple of Q would have the Same effect, one of the coefficients of Q may be taken arbitrarily. Thus, Bezout argued, there were $M - 1$Coefficients at his disposal to annihilate the number of terms beyond D+1 remaining in the product P_1Q In other words, Bezout had to solve $N-(D+1)$Linear equations in $M-1$ unknowns—these unknowns being the coefficients of Q. This can be done if the number of equations equals the number of unknowns, although Bezout did not explicitly state this. Equating $N-(D-1)$ with $M-1$, and using his formulas for N and M, Bezout was able to compute that $D=t, t', t'', t'''\cdots$[5] Bezout briefly noted that his theorem has a geometric interpretation: "The surfaces of three bodies whose nature is expressible by algebraic equations cannot meet each other in more points than there are units the product of the degrees of the equations,"[6]. We should note that Bezout did not show that the equations for the coefficients of Q form a consistent, independent set of linear equations, or that extraneous roots can never occur in the resultant equation. Further, the geometric statement must be modified to deal with special cases, since, for instance, three planes can have a straight line in common.

Later on in the work, [7] Bezout discussed another method of finding the resultant equation; this was by finding polynomials, which we may write $Q_1,\cdots, Q_n$, such that

$$P_1Q_1 + P_2Q_2 + \cdots + P_nQ_n = 0$$

is the resultant equation. Each Q_k has indeterminate coefficients, which Bezout explicitly determined for many systems of equations by comparing powers of the unknowns $x, y, z, \cdots$.

Bezout's work on resultants stimulated many investigations in the modern theory of elimination, including Cauchy's refinements of elimination procedure and Sylvester's work on resultants and inertia forms. Bezout's theorem is crucial to the study of the intersection of manifolds in algebraic geometry. In the preface to *Théorie des équations*, Bezaut had complained that algebra was becoming a neglected science. But his accomplishment showed that the fact that his contemporaries could not solve the general equation of *n*th degree did not mean that there were no fruitful areas of investigation remaining in algebra.

NOTES

1. "Sur plusieurli classes d'équations," 20.

2. For Euler's method, see *Introductio in analysin infinitorum* (Lausanne, 1748), **2** sees. 483 ff.

3. *Théorie des équations algébriques*, 32.

4. *Ibid*., 206.

5. *Ibid*., 32.

6. *Ibid*., 33.

7. *Ibid*., 187 ff.

BIBLIOGRAPHY

I. Original Works. Bezout's major works are the following: "Sur plusieurs classes d'equations de tous les degrés qui admettent une solution algébrique," in *Mémoirés de l'Académie royale des sciences*(1762), 17–52; *Cours de mathématiques a l'usage des Gardes du Pavillon et de la Marine*, 6 vols. (Paris, 1164–1769), reprinted many times with slight variations in the title, often translate or revised in parts; "Sur Ie degré des equations résultantesde l'évanouissement des inconnues," in *Mémmoriesde l, Académie royale des sciences*(1164;). 288–338; "Sur la resolution des équations de tous les degrés, *ibid* (1765), 533–552: and *Théorie géneralé des équations algébriques* (Paris, 1779)

II. Secondary Works. Secondary works are Georges Bouligand, "À une étape décisive de l'algébre. L'oeuvre scientifique d'Etienne Bezout," in *Revue générale des sciences*, **55** (1948), 121–123; Marquis de Condorcet. "Elogede M. Bezout," in *Eloges des académiciens, de l'AcadémieRoyale des Sciences*. 3 (l799), 322–337: E. Netto and R. Le Vavasseur," Les *fonctions rationnelles in Encyclopédiedes, sciences mathéematiques pures at appliqués*, I, pt. 2 (Paris-Leipzig, 1907), 1–232; and Henry S. White, "Bezout's Theory of Resultants and Its Influence on Geometry," in *Bulletin of the American Mathematical Society*, **15** (1909), 325–338.

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