George David Birkhoff | Encyclopedia.com

(*b*. Overisel, Michigan, 21 March 1884; *d*. Cambridge, Massachusetts, 12 November 1944).

mathematics.

The son of a physician, Birkhoff studied at the Lewis Institute (now Illinois Institute of Technology) in Chicago from 1896 to 1902. After a year at the University of Chicago he went to Harvard, where he received the A.B. in 1905. Returning to the University of Chicago, he was awarded the Ph.D., *summa cum laude*, in 1907. His graduate study at the University of Chicago was followed by two years as an instructor in the University of Wisconsin. In 1908 he was married to Margaret Elizabeth Grafius.

In 1909 Birkhoff went to Princeton as a preceptor and in 1911 was promoted to a full professorship in response to a call from Harvard. The following year he accepted an assistant professorship at Harvard, where he became professor in 1919, Perkins professor in 1932, and dean of the Faculty of Arts and Sciences from 1935 to 1939. As Perkins professor, the major part of his academic life was devoted to mathematical research and direction of graduate students.

Birkhoff was very generally regarded, both in the United States and abroad, as the leading American mathematician of his day. Honors came early in life and from all over the world. He was president of the American Mathematical Society in 1925 and or the American Association for the Advancement of Science in 1937.

Of Birkhoff's teachers, Maxime Bôcher of Harvard and E. H. Moore of the University of Chicago undoubtedly influenced him most. He was introduced by Bôcher to classical analysis and algebra. From Moore he learned of "general analysis." There are indications that Birkhoff preferred the approach of Bôcher to that of Moore. Through his reading, Birkhoff made Henri Poincaré his teacher and took over poincaré's problems in differentiate equations and celestial mechanics. Like Moore, Birkhoff was a pioneer among those who felt that American mathematics had come of age.

He had many close friends among his colleagues in Europe. With Hadamard he shared a deep interest and understanding of Poincaré. Neils Nörlund and Birkhoff had common ground in their study of difference equations. Between Levi-Cività and Birkhoff there were deep ties of friendship cemented by their common interest in the problem of three bodies. The correspondence of Sir Edmund Whittaker and Bitkhoff on the existence of periodic orbits in dynamics was intense, illuminating, and friendly.

Birkhoff's thesis was concerned with asymptotic expansions, boundary-value problems, and the Sturm-Liouville theory. Nonself-adjoint operators

were introduced with continuous coefficients and n boundary conditions, $U_i(u) = 0$, $i = 1,\ldots, n$, linear and homogeneous in u and its first n-1 derivatives at x = a and x = b. Birkhoff defined an operator M(z) adjoint to L(z), and boundary conditions $V_i(v) = 0$, $i = 1,\ldots, n$ adjoint to the conditions $U_i(u)= 0$, Forn >2 he introduced a parameter λ, as in the classical Sturm-Liouville equations, and, with suitable conditions on the matrix of coefficients in the boundary conditions, obtained an expansion of a prescribed real function $x \rightarrow f(x)$, "piecewise" of class C^1. This expansion was shown to converge essentially as does the classical Fourier expansion. This work of depth admitted extension both by Birkhoff and by such pupils as Rudolph Langer and Marshall Stone; he collaborated with Langer on "The Boundary Problems and Developments…" (1923).

Birkhoff next devoted his attention to linear differential equations, difference equations, and the generalized Riemann problem. With Gauss, Riemann, and Poincaré showing the way, second order differential equations of the Fuchsian type with regular singular points have become central in conformal mapping, in the theory of automorphic functions, and in mathematical physics, including quantum mechanics. Linear differential systems with irregular singular points appeared as a challenging new field, and Birkhoff turned to it.

Thomé had used formal solutions; Poincaré and Jakob Horn, asymptotic expansions; Hilbert and Josef Plemelj, unknown to Birkhoff, had solved one of the relevant matrix problems; and Ebenezer Cunningham had generalized Poincare's use of Laplace's transformation. It remained for Birkhoff to formulate a program of so vast a scope that it is still an object of study today.

Among analytic systems with a finite number of irregular singular points with prescribed ranks, Birkhoff defined a “canonical system” and a notion of the “equivalence” of singular points. Under the title “generalized Riemann problem,” Birkhoff sought to construct a system of linear differential equations of the first order with prescribed singular points and a given monodromy group. That he carried his program as far as he did is remarkable. The total resources of modern function space analysis are now involved.

Carmichael’s thesis, done under Birkhoff’s supervision at Princeton in 1911, was perhaps the first significant contribution on difference equations in America. Birkhoff extended his notioπ of a “generalized Riemann problem” to systems of difference equations. In “Analytic Theory of Singular Difference Equations,” he collaborated with Trjitzinsky in an extension and modification of earlier work.

Birkhoff’s major interest in analysis was in dynamical systems. He wished to extend the work of Poincaré, particularly in celestial mechanics. One can divide his dynamics into formal and nonformal dynamics. The nonformal portion includes the metrical and topological aspects.

Birkhoff was concerned with a real, analytic Hamiltonian or Pfaffian system. A periodic orbit gives rise, after a simple transformation, to a “generalized equilibrium point” at which the “equations of variation” are independent of *t*. First-order formal stability at such a point requires that the characteristic multipliers at the point be purely imaginary. Formal trigonometric stability is then defined. It is a major result of Birkhoff’s work that under the limitations on generality presupposed by Poincaré, first-order formal stability at a generalized equilibrium point implies formal trigonometric stability.

Possibly the most dramatic event in Birkhoff’s mathematical life came when he proved Poincaré’s “last geometric theorem.” In “Sur un théorème de géométrie” (1912), Poincaré had enunciated a theorem of great importance for the restricted problem of three bodies, acknowledging his inability to prove this theorem except in special cases. The young Birkhoff formulated this theorem in “Proof of Poincare’s Geometric Theorem” (1913, p. 14):

Let us suppose that a continuous one-to-one transformation T takes the ring R, formed by concentric circles C_a and C_b of radii a and b (a > b > 0), into itself in such a way as to advance the points of C_a in a positive sense, and the points of C_b in a negative sense, and at the same time preserve areas. *Then there are at least two invariant points.*

Birkhoff’s proof of this theorem was one of the most exciting mathematical events of the era.

In 1912, in “Quelques théorèmes sur le mouvement des systèmes dynamiques,” Birkhoff introduced his novel conceptions of minimal or recurrent sets of motions and established their existence under general conditions. This was the beginning of a new era in the theory of dynamical systems. Birkhoff continued by introducing the concepts of wandering, central, and transitive motions.

Metric transitivity, as defined by Birkhoff and Paul Smith in “Structure Analysis of Surface Transformations” (1928), requires that the only sets that are invariant under the “flow” in phase space be sets of measure zero or measure of the space. Metric transitivity implies topological transitivity (i.e., the existence of a transitive motion). Great problems abound and are today the object of research. On a compact regular analytic manifold it is not known, even today, whether topological transitivity implies metric transitivity.

From these concepts of Birkhoff’s the main body of modern dynamics has emerged, together with such branches as symbolic dynamics and topological dynamics. Other concepts of Birkhoff’s, his minimax principle and his theorem on the fixed points of surface transformations, have motivated some of the greatest advances in global analysis and topology.

One of Birkhoff’s theorems of major current interest in his “ergodic theorem.” Following an idea of Bernard Koopman, Von Neumann established his “mean ergodic theorem” in 1931. Stimulated by these ideas, Birkhoff presented his famous “pointwise ergodic theorem.” As formulated by Khintchine, Birkhoff’s theorem takes the form “The space *M* is assumed to have a finite measure m invariant under the flow. Let *f* be integrable over *M* and let *P* be a point of *M*. Then

exists for almost all *P* on *M*.”

Birkhoff thought critically for many years about the foundations of relativity and quantum mechanics. His philosophical and scientific ideas found vivid expression in “Electricity as a Fluid” (1938), where he described a “perfect fluid” that he proposed as a model from which to deduce the observed spectrum of hydrogen without postulating “energy levels.” In “El concepto matematico…” (1944) he formulated a theory of gravitation in flat space-time, and deduced from it the three “crucial effects.” Both of these models were consistent with special relativity; both avoided the general curvilinear coordinates basic to Einstein’s general relativity but always considered by Birkhoff to be unnecessary and difficult to interpret experimentally.

Although Birkhoff’s physical models may be controversial, his original critiques and interpretations are stimulating and illuminating.

Birkhoff wrote on many subjects besides those of his major works; for example, he devised a significant formula for the ways of coloring a map. At sixteen, he began a correspondence with H. S. Vandiver, who was eighteen, on number theory. A significant paper resulted in 1904.

Another paper, written in collaboration with his colleague Oliver Kellogg (1922), was one of the openers of the age of function spaces. Schauder and Leray acknowledged this paper as an inspiration for their later, more powerful theorem.

In 1929 Birkhoff and Ralph Beatley joined in writing a textbook on elementary geometry, which they called “basic geometry.” After a period of revision and development, the pedagogical conceptions of this book have been widely adopted in current teaching of high school geometry.

Birkhof’s lifelong interest in music and the arts culminated in his book *Aesthetic Measure* (1933), in preparation for which he had spent a year traveling around the world, observing objects of art, ornaments, tiles, and vases, and recording impressions of music and poetry.

BIBLIOGRAPHY

I. Original Works. Among Birkhoff’s works are “On the Integral Divisors of a” — b^n in *Annals of Mathematics*, **5** (1904), 173–180, written with H. S. Vandiver: “On the Asymptotic Character of the Solutions of Certain Differential Equations Containing a Parameter,” in *Transactions of the American Mathematical Society*, **9** (1908), 219–231; “Boundary Values and Expansion Problems of Ordinary Linear Differential Equations,” *ibid*., 373–395; “Quelques théorémes sur Ie mouvement des systemes dynamiques.” in Bulletin de la Société mathématique de France, **40** (1912), 305–323; “Proof of Poincaré’s Geometric Theorem.” in *Transactions of the American Mathematical Society*, **14** (1913). 14–22: “Invariant Points in Function Space,” **ibid.** , 23 (1922), 96–115, written with O. D. Kellogg; “The Boundary Problems and Developments Associated With a System of Ordinary Linear Differential Equations of First Order,” in *Proceedings of the American Academy of Arts and Sciences*, **58** (1923), 49–128, written with R. E. Langer; “Structure Analysis of Surface *Transformations.” in Journal de mathéematiques pures et appliquées*, 9th ser., 7 (1928), 345–379, written with P. A. Smith; “On the Number of Ways of Coloring a Map,” in *Proceedings of the Edinburgh Mathematical Society, 2nd ser., 2 (1930), 83–91: Aesthetic Measure* (Cambridge, Mass., 1933); “Analytic Theory of Singular Difference Equations,” in *Acta mathematica*, **60** (1933), 1–89, written with W. J. Trjitzinsky: “Electricity as a Fluid,” in *Journal of the Franklin Institute*, **226** (1938). 315–325; *Basic Geometry* (1940), written with Ralph Beatley; “El concepto matemático de tiempo y la gravitaciün,” in *Bolelin de la Sociedad matemática nexicana*, 1(1944), 1–24: and *Dynamical Systems*, rev. ed. (Providence, R.L. 1966), with introduction, bibliography, and footnotes by Jürgen Moser.

His works have been brought together by the American Mathematical Society as Collected Mathematical Works of George David Birkhoff, 3 vols. (Providence, R. I., 1950).

II. Secondary Literature. Works on Birkhoff are American Mathematical Society Semicentennial Publications, I, *History* (Providence, R. I., 1938), p. 212, a list of his honors up to 1938; P. Masani, “On a Result of G. D. Birkhoff on Linear Differential Systems,” in *Proceedings of the American Mathematical Society*, **10** (1959), 696–698; and H. L. Turrittin, “Reduction of Ordinary Differential Equations to the Birkhoff Canonical Form,” in *Transaclions of the American Mathematical Society*, **107** (1963), 485–507.

Marston Morse