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(*b*. Paris, France, 21 August 1789; *d* Sceaux [near Paris J, France, 22 May 1857),

*mathematics, mathematical physics, celestial mechanics*.

**Life** . Cauchy's father, Louis-François Cauchy, was born in Rouen in 1760. A brilliant student of classics at Paris University, after graduating he established himself as a barrister at the *parlement* of Normandy. At the age of twenty-three he became secretary general to Thiroux de Crosnes, the *intendant* of Haute Normandie. Two years later he followed Thiroux to Paris, where the latter had been appointed to the high office of *lieutenant de police*.

Louis-François gradually advanced to high administrative positions, such as that of first secretary to the Senate. He died in 1848. In 1787 he married Marie-Madeleine Desestre, who bore him four sons and two daughters. She died in 1839, Of their daughters Thérèse died young and Adèdle married her cousin G. de Neuburg. She died in 1863. The youngest son Amédée, died in 1831, at the age of twenty-five; Alexandre (1792–1857) held high judicial posts; and Eugène (1802–1877) held administrative posts and became known as a scholar in the history of law. Augustin was the eldest child.

Cauchy enjoyed an excellent education; his father was his first teacher. During the Terror the family escaped to the village of Arcueil, where they were neighbors of Laplace and Berthollet, the founders of the celebrated Société d'Arcueil. Thus, as a young boy Augustin became acquainted with famous scientists. Lagrange is said to have forecast his scientific genius while warning his father against showing him a mathematical text before the age of seventeen.

After having completed his elementary education at home, Augustin attended the École Centrale du Panthéon. At the age of fifteen he completed his classical studies with distinction. After eight to ten months of preparation he was admitted in 1805 to the École Polytechnique (at the age of sixteen). In 1807 he entered the École des Ponts et Chaussées, which he left (1809?) to become an engineer, first at the works of the Oureq Canal, then the Saint-Cloud bridge, and finally, in 1810, at the harbor of Cherbourg, where Napoleon had started building a naval base for his intended operations against England. When he departed for Cherbourg, his biographer says, Cauchy carried in his baggage Laplace's *Mécanique céleste*, Lagrange's *Traité des fonctions analytiques*, Vergil, and Thomas à Kempis' *Imitatio*. Cauchy returned to Paris in 1813, allegedly for reasons of health, although nothing is known about any illness hesuffered during his life.

Cauchy had started his mathematical career in 1811 by solving a problem set to him by Lagrange: whether the angles of a convex polyhedron are determined by its faces. His solution, which surprised his contemporaries, is still considered a clever and beautiful piece of work and a classic of mathematics. In 1812 he solved Fermat's classic problem on polygonal numbers: whether any number is a sum of *n n* gonal numbers. He also proved a theorem in what later was called Galois theory, generalizing a theorem of Ruffini's. In 1814 he submitted to the French Academy the treatise on definite integrals that was to become the basis of the theory of complex functions. In 1816 he won a prize contest of the French Academy on the propagation of waves at the surface of a liquid; his results are now classics in hydrodynamics. He invented the method of characteristics, which is crucial for the theory of partial differential equations, in 1819; and in 1822 he accomplished what to the heterodox opinion of the author is his greatest achievement and would suffice to assure him a place among the greatest scientists: the founding of elasticity theory.

When in 1816 the republican and Bonapartist Gaspard Monge and the "regicide" Lazare Carnot were expelled from the Académie des Sciences, Cauchy was appointed (not elected) a member. (Even his main biographer feels uneasy about his hero's agreeing to succeed the highly esteemed and harshly treated Monge.) Meanwhile Cauchy had been appointed *répétiteur*, adjoint professor (1815), and full professor (1816) at the École Polytechnique [11];[1] at some time before 1830 he must also have been appointed to chairs at the Faculté des Sciences and at the Collège de France.[2] His famous textbooks, which date from this period, display an exactness unheard of until then and contain his fundamental work in analysis, which has become a classic. These works have been translated several times.

In 1818 Cauchy married Aloïse de Bure, daughter (or granddaughter) of a publisher who was to publish most of Cauchy's work. She bore him two daughters, one of whom married the viscount de l'Escalopier and the other the count of Saint-Pol. The Cauchys lived on the rue Serpente in Paris and in the nearby town of Sceaux.

Cauchy's quiet life was suddenly changed by the July Revolution of 1830, which replaced the Bourbon king, Charles X, with the Orléans king of the bourgeoisie, Louis-Philippe. Cauchy refused to take the oath of allegiance, which meant that he would lose his chairs. But this was not enough: Cauchy exiled himself. It is not clear why he did so: whether he feared a-new Terror and new religious persecutions, whether he meant it as a demonstration of his feelings against the new authority, or whether he simply thought he could not live honestly under a usurper.

Leaving his family, Cauchy went first to Fribourg, where he lived with the Jesuits. They recommended him to the king of Sardinia, who offered him a chair at the University of Turin.[3] Cauchy accepted. In 1833, however, he was called to Prague, where Charles X had settled, to assist in the education of the crown prince (later the duke of Chambord). Cauchy accepted the offer with the aim of emulating Bossuet and Fénelon as princely educators. In due time it pleased the ex-king to make him a baron. In 1834 Mme. Cauchy joined her husband in Prague—the biographer does not tell us for how long.

The life at court and journeys with the court took much of Cauchy's time, and the steady flow of his publications slowed a bit. In 1838 his work in Prague was finished, and he went back to Paris. He resumed his activity at the Academy, which meant attending the Monday meeting and presenting one or more communications to be printed in the weekly *Comptes rendus;* it is said that soon the Academy had to put a restriction on the size of such publications. In the course of less than twenty years the *Comptes rendus* published 589 notes by Cauchy—and many more were submitted but not printed. As an academician Cauchy was exempted from the oath of allegiance An effort to procure him a chair at the Collège de France foundered on his intransigence, however. In 1839 a vacancy opened at the Bureau des Longitudes which legally completed itself by cooption. Cauchy was unanimously elected a member, but the government tied the confirmation to conditions that Cauchy again refused to accept. Biot [7] tells us that two subsequent ministers of education vainly tried to build golden bridges for Cauchy. Bertrand [11] more specifically says that the only thing they asked of him was to keep silent about the fact that he had not been administered the oath. But, according to Biot [7, p. 152], "even such an appearance terrified Cauchy, and he tried to make it impossible by all diplomatic finesses he could imagine, finesses which were those of a child."

When the February Revolution of 1848 established the Second Republic, one of the first measures of the government was the repeal of the act requiring the oath of allegiance. Cauchy resumed his chair at the Sorbonne (the only one that was vacant). He retained this chair even when Napoleon III reestablished the oath in 1852, for Napoleon generously exempted the republican Arago and the royalist Cauchy.

A steady stream of mathematical papers traces Cauchy's life. His last communication to the Academy closes with the words "C'est ce que j'expliquerai plus au long dans un prochain mémoire." Eighteen days later he was dead. He also produced French and Latin poetry, which, however, is better forgotten. More than a third of his biography deals with Cauchy as a devout Catholic who took a leading part in such charities as that of Françcois Régis for unwed mothers, aid for starving Ireland, rescue work for criminals, aid to the Petit Savoyards, and important activity in the Society of Saint Vincent de Paul, Cauchy was one of the founders of the Institut Catholique, an institution of higher education; he served on a committee to promote the observance of the sabbath; and he supported works to benefit schools in the Levant. Biot [7] tells us that he served as a social worker in the town of Sceaux and that he spent his entire salary for the poor of that town, about which behavior he reassured the mayor: "Do not worry, it is only my salary; it is not my money, it is the emperor's."

Cauchy's life has been reported mainly according to Valson's work [8], which, to tell the truth, is more hagiography than biography. It is too often too vague about facts which at that time could easily have been ascertained; it is a huge collection of commonplaces; and it tends to present its hero as a saint with all virtues and no vices. The facts reported are probably true, but the many gaps in the story will arouse the suspicion of the attentive reader. The style reminds one of certain saccharine pictures of saints. Contrary to his intention Valson describes a bigoted, selfish, narrow-minded fanatic. This impression seems to be confirmed by a few contemporary anecdotal accounts. N. H. Abel [10] called Cauchy mad, infinitely Catholic, and bigoted. Posthumous accounts may be less trustworthy, perhaps owing their origin or form to a reaction to Valson's book.

A story that is hardly believable is told by Bertrand [9] in a review of Valson, Bertrand, who deeply admired Cauchy's scientific genius, recalled that in 1849, when Cauchy resumed his chair of celestial mechanics at the Sorbonne,

… his first lessons completely deceived the expectation of a selected audience, which was surprised rather than charmed by the somewhat confused variety of subjects dealt with. The third lesson, I remember, was almost wholly dedicated to the extraction of the square root, where the number 17 was taken as an example and the computations were carried out up to the tenth decimal by methods familiar to all auditors but belies ed new by Cauchy, perhaps because he had hit upon them the night before. I did not return; but this was a mistake, since the next lectures would have introduced me ten years earlier to some of the most brilliant discoveries of the famous master.

This story was a vehement reaction to Valson's statement that Cauchy was an excellent teacher who "… never left a subject until he had completely exhausted and elucidated it so he could satisfy the demands of the most exacting spirits" [8, I, 64]. Clearly, this is no more than another of Valson's many cliché epithets dutifully conferred upon his subject. In fact Cauchy'smanner of working was just the contrary of what is here described, as will be seen.

According to Valson, when Mme. Cauchy joined her husband in Prague [8. I, 90), he complained that he was still separated from his father and mother and did not mention his daughters. Yet according to Biot [7] his wife brought their daughters with her and the family stayed in Prague and left there together. It is characteristic of Valson that after the report of their birth he never mentions Cauchy's daughters, except for the report that one of them was at his bedside when he died. His wife is not mentioned much more. The result of such neglect is that despite the many works of charity one feels a disturbing lack of human relationships in Cauchy's life. Possibly this was less Cauchy's fault than that of his biographer. But even Bertrand [11], who was more competent than Valson, more broad-minded, and a master of the *éloge*, felt uneasy when he had to speak about Cauchy's human qualities. Biot [7) was more successful. It is reassuring to learn from him that Cauchy befriended democrats, nonbelievcrs, and odd fellows such as Laurent. And it is refreshing to hear Biot call Cauchy's odd behavior "childish."

A prime example of his odd behavior is his exile. One can understand his refusal to take the oath, but sharing exile with the depraved king is another matter. It could have been a heroic feat; unfortunately it was not. The lone faithful paladin who followed his king into exile while all France was gratified at the smooth solution of a dangerous crisis looks rather like the Knight of the Rueful Countenance. Yet his quixotic behavior is so unbelievable that one is readily inclined to judge him as being badly melodramatic. Stendhal did so as early as 1826, when he said in *New Monthly Magazine* (see [13], p. 192) about a meeting of the Académic des Sciences:

After the lecture of a naturalist, Cauchy rose and protested the applause, "Even if these things would be as true as I think thev are wrong"—he said—"it would not be convenient to disclose them to the public… this can only prejudice our holy religion." People burst out laughing at this talk of Cauchy, who… seems to seek the role of a martyr to contempt.

Probably Stendhal was wrong. Biot knew better: Cauchy was a child who was as naïve as he looked. Among his writings one finds two pieces in defense of the Jesuits [8, pp. 108–121) that center on the thesis that Jesuits are hated and persecuted because of their virtue. It would not be plausible that the man who wrote this was really so naïve if the author were not Cauchy.

Another story about Cauchy that is well confirmed comes from the diary of the king of Sardinia, 16 January 1831 [13, p. 160]. In an audience that the king granted Cauchy, five times Cauchy answered a question by saying, "I expected Your Majesty would ask me this, so I have prepared to answer it." And then he took a memoir out of his pocket and read it.

Cauchy's habit of reading memoirs is confirmed by General d'Hautpoul [12; 13, p. 172], whose memoirs of Prague shed an unfavorable light on Cauchy as an educator and a courtier.

Sometimes, in the steady flow of his Academy publications, Cauchy suddenly turned or returned to a different subject; after a few weeks or months it would become clear why he did so. He would then submit to the Academy a report on a paper of a *savant étranger* (i.e., a nonmember of the Academy), which had been sent to him for examination. Meanwhile he had proved anew the author's results, broadened, deepened, and generalized them. And in the report he never failed to recall all his previous investigations related to the subject of the paper under consideration. This looks like extremely unfair behavior, and in any other case it would be—but not with Cauchy. Cauchy did not master mathematics; he was mastered by it. If he hit on an idea—and this happened often—he could not wait a moment to publish it. Before the weekly *Comptes rendus* came into being this was not easy, so in 1826 he founded a private journal, *Exercices de mathématiques*. Its twelve issues a year were filled by Cauchy himself with the most improbable choice of subjects in the most improbable order. Five volumes of the *Exercices* appeared before he left Paris. In Turin he renewed this undertaking—and even published in the local newspaper— and continued it in Prague and again in Paris, finally reaching a total of ten volumes. He published in other journals, too; and there are at least eighteen memoirs by him published separately, in no periodical or collection, as well as many textbooks. Sometimes his activity seems explosive even by his own standards. At the meetings of the Academy of 14, 21, and 28 August 1848 he submitted five notes and five memoirs—probably to cover the holiday he would take until 9 October. Then in nine meetings, until 18 December, he submitted nineteen notes and ten memoirs. He always presented many more memoirs than the Academy could publish.

On 1 March 1847 Lamé presented to the Academy a proof of Fermat's last theorem. Liouville pointed out that the proof rested on unproved assumptions in the arithmetic of circle division fields. Cauchy immediately returned to this problem, which he hadconsidered earlier. For many weeks he informed the Academy of all his abortive attempts to solve the problem (which is still unsolved) by proving Lamé's assumption. On 24 May, Liouville read a letter from Ernst Kummer, who had disproved Lame's assumption. Even such an incident would not silence Cauchy, however, and a fortnight later he presented investigations generalizing those of Kummer.

The story that Abel's Paris memoir went astray [10] through Cauchy's neglect rests on gossip. It has been refuted by D. E. Smith, who discovered Legendre and Cauchy's 1829 report on Abel's work [10a; 10b]. In general it would be wrong to think that Cauchy did not recognize the merits of others. When he had examined a paper, he honestly reported its merits, even if it overlapped his own work. Of all the mathematicians of his period he is the most careful in quoting others. His reports on his own discoveries have a remarkably nave freshness because he never forgot to sum up what he owed to others. If Cauchy were found in error, he candidly admitted his mistake.

Most of his work is hastily, but not sloppily, written. He was unlike Gauss, who published *pauca sed matura* only—that is, much less than he was able to, and many things never. His works still charm by their freshness, whereas Gauss's works were and still are turgid. Cauchy's work stimulated new investigations much earlier than Gauss's did and in range of subject matter competes with Gauss's. His publishing methods earned for Gauss an image of an almost demonic intelligence who knew all the secrets better and more deeply than ordinary men. There is no such mystery around Cauchy, who published lavishly—although nothing that in maturity could be compared with Gauss's publications. He sometimes published the same thing twice, and sometimes it is evident that he was unfamiliar withsomething he had brought out earlier. He published at least seven books and more than 800 papers.

More concepts and theorems have been named for Cauchy than for any other mathematician (in elasticity alone [35] there are sixteen concepts and the-orems named for Cauchy). All of them are absolutely simple and fundamental. This, however, is an objective assessment and does not consider the subjective value they had for Cauchy. In the form that Cauchy discovered and understood them, they often were not so simple; and from the way that he used or did not use them, it often appears that he did not know that they were fundamental. In nearly all cases he left the final form of his discoveries to the next generation. In all that Cauchy achieved there is an unusual lack of profundity. He was one of the greatest mathematicians—and surely the most universal—and also contributed greatly to mathematical physics. Yet he was the most superficial of the great maithematicians, the one who had a sure feeling for what was simple and fundamental without realizing it.

**Writings** . Cauchy's writings appeared in the publications of the Academy, in a few scientific journals, separately as books, or in such collections as the *Exercices*. Some of his courses were published by others [3; 4]. There were, according to Valson, eighteen *memoires detaches*. A Father Jullien (a Jesuit), under the guidance of Cauchy, once catalogued his work. The catalog has not been published. Valson's list was based on it, but he was not sure whether it was complete;[4] and it is not clear whether Valson ever personally saw all of the *mémoires détachés* or whether all of them really existed. Some of them, which have been lithographed, are rare.[5] Cauchy must have left an enormous quantity of anecdota, but nothing is known of what happened to them. What the Academy possesses seems to be insignificant.

In 1882 the Academy began a complete edition of Cauchy's work [1]. Volume XV of the second series is still lacking. In the second of the second series, which appeared as late as 1957, the commissioners of the Academy declared their decision to cease publication after still another volume. They did not say whether the edition would be stopped because it is finished or whether it will remain unfinished. The missing volume seems to have been reserved for the *mémoires détachés*, among which are some of Cauchys most important papers. Fortunately, one of them has been reprinted separately [2; 2a; 2b].

The Academy edition of Cauchy's works contains no anecdota. Cauchy's papers have been arranged according to the place of original publication. The first series contains the Academy publications; the second, the remainder. This makes use of the edition highly inconvenient. Everything has been published without comment: there is no account of how the text was established and no statement whether printing errors and evident mistakes were corrected (sometimes it seems that they were not). Sometimes works have been printed twice (such as (1, 1st ser., V, 180–198], which is a textual extract of [1,2nd ser., XI, 331–353]). In other cases such duplications have been avoided, but such avoidances of duplication have not regularly been accounted for. Since Cauchy and his contemporaries quote the *Exercices* according to numbers of issues it is troublesome that this subdivision has not been indicated. This criticism, however, is not to belittle the tremendous value of the Academy edition.

Important bibliographic work on Cauchy was done by B. Boncompagni [5].

Since Valson's biography [8] and the two biographic sketches by Biot and Bertrand [7; 11] no independent biographical research on Cauchy has been undertaken except for that by Terracini [13]. (It would be trouble-some but certainly worthwhile to establish a faithful picture of Cauchy from contemporary sources. He was one of the best-known people of his time and must have been often mentioned in newspapers, letters memoirs.[6]) Valson's analysis of Cauchy's work is unsatisfactory because sometimes he did not understand Cauchy's mathematics; for instance, he mistook his definition of residue. Lamentably no total appreciation of Cauchy's work has been undertaken since. There are, however, a few historical investigation of mathematical fields that devote some space to Cauchy. See Casorati [19] on complex functions (not accessible to the present reporter); Verdet [20] on optics; Studnicka [21a] on determinants (not accessible); Todhunter [21 a] on elasticity; Brill and Noether [22] on complex functions (excellent); Stäckel andJourdain [23, 24, 25] on complex functions; the *Encyclopädie der mathematischen Wissenschaften* [26] on mathematical physics and astronomy; Burkhardt [27] on several topics (chaotic, with unconnected textual quotations, but useful as a source); Miller [28] on group theory; Jourdain [29] on calculus (not accessible); Love [30] on elasticity (fair); Lamb [31] on hydrodynamics (excellent); Whittaker [32] on optics (excellent); Carruccio [33] on complex functions; Courant and Hilbert [34] on differential equations (fair); Truesdell and Toupin [35] on elasticity (excellent).

Because of the great variety of fields in which Cauchy worked it is extremely difficult to analyze his work and properly evaluate it unless one is equally experienced in all the fields. One may overlook important work of Cauchy and commit serious errors of evaluation. The present author is not equally experienced in all the fields: in number theory less than in analysis, in mathematical physics less than in mathematics, and entirely inexperienced in celestial mechanics.

**Calculus.** The classic French *Cours d'analyse* (1821), descended from Cauchy's books on calculus [1, 2nd ser., III; IV; IX, 9–184], forcefully impressed his contemporaries. N. H. Abel [17] called the work [ 1, 2nd ser., III] "an excellent work which should be read by every analyst who loves mathematical rigor." In the introduction Cauchy himself said, "As to the methods, I tried to fill them with all the rigor one requires in geometry, and never to revert to arguments taken from the generality of algebra." Cauchy needed no metaphysics of calculus. The "generality of algebra," which he rejected, assumed that what is true forreal numbers is true for complex numbers; that what is true for finite magnitudes is true for infinitesimals; that what is true for convergent series is true for divergent ones. Such a remark that looks trivial today was a new, if not revolutionary, idea at the time.

Cauchy refused to speak about the sum of an infinite series unless it was convergent, and he first defined convergence and absolute convergence of series, and limits of sequences and functions [1, 2nd ser., III, 17–19]. He discovered and formulated *convergence criteria:* the Cauchy principle of $s_{m+n} - s_n$ becoming small[7] [1, 2nd ser., VII, 269], the root criterion using the lowest upper limit of [1, 2nd ser., III, 121], the quotient criterion using that of $|a_{n+1}|/|a_n|$ [1,2nd ser., III, 123], their relation, the integral criterion [1, 2nd ser., VII, 267–269]. He defined upper and lower limits [1, 2nd ser., III, 121], was first to prove the convergence of $(1 + 1/n)^n$, and was the first to use the limit sign [1, 2nd ser., IV, 13 f.]. Cauchy studied convergence of series under such operations as addition and multiplication [1, 2nd ser., III, 127–130] and under rearrangement [1, 1st ser., X, 69; 1st ser., IX, 5–32]. To avoid pitfalls he defined convergence of double series too cautiously [1, 2nd ser., III, 441; X, 66]. Explicit estimations of convergence radii of power series are not rare in his work. By his famous example $\exp(-x^{-2})$ he warned against rashness in the use of Taylor's series [1, 2nd ser., II, 276–282]. He proved Lagrange's and his own remainder theorem, first by integral calculus [1, 2nd ser., IV, 214] and later by means of his own generalized mean-value theorem [1, 2nd ser., IV, 243, 364; VI, 38–42], which made it possible to sidestep integral calculus. In the first proof he used the integral form of the remainder that is closely connected to his famous formula [1, 2nd ser., IV, 208–213],

An important method in power series arising from multiplication, inversion, substitution, and solving differential equations was Cauchy's celebrated *calculdes limites* (1831–1832), which in a standard way reduces the convergence questions to those of geometrical series [1, 2nd ser., II, 158–172; XI, 331– 353; XII, 48–112].

Cauchy invented our notion of continuity[8] and proved that a continuous function has a zero between arguments where its signs are different [1, 2nd ser., III, 43, 378], a theorem also proved by Bolzano. He also did away with multivalued functions. Against Lagrange he again and again stressed the limit origin of the differential quotient. He gave the first adequate definition of the definite integral as a limit of sums [1, 2nd ser., IV, 122–127] and defined improper integrals [1, 2nd ser., IV, 140–144], the well-known Cauchy principal value of an integral with a singular integrand [1, 1st ser., I, 288–303, 402–406], and closely connected, singular integrals (i.e., integrals of infinitely large functions over infinitely small paths [δ functions]) [1, 1st ser., I, 135, 288–303, 402–406; 2nd ser., I, 335–339; IV, 145–150; XII, 409–469]. Cauchy made much use of discontinuous factors 1, 1st ser., XII, 79–94] and of the Fourier transform (see under Differential Equations). Cauchy also invented what is now called the Jacobian, although his definition was restricted to two and three dimensions [1, 1st ser., I, 12].

In addition, Cauchy gave the proof of the fundamental theorem of algebra that uses the device of lowering the absolute value of an analytic function as long as it does not vanish [1, 2nd ser., I, 258–263; III, 274–301; IV, 264; IX, 121–126]. His investigations (1813, published in 1815) on the number of real roots [1, 2nd ser., I, 170–257] were surpassed by Sturm's (1829). In 1831 he expressed the number of complex roots of $f(z)$ in a domain by the logarithmic residue formula, noticing that the same expression gives the number of times $\operatorname{Re} f(z)/\operatorname{Im} f(z)$ changes from $-\infty$ to $\infty$ along a closed curve—in other words how often the $f$-image of the curve turns around 0—which led to a new proof of the fundamental theorem that was akin to Gauss's first, third, and fourth (reconstructed from [8, II, 85–88]—see also [1, 1st ser., IV, 81–83], since the 1831 *mémoire détaché* has not yet been republished). In [1, 2nd ser., I, 416–466] the proof has been fashioned in such a way that it applies to mappings of the plane into itself by pairs of functions.

With unsurpassed skill and staggering productivity he calculated and transformed integrals and series developments.

In mathematics Cauchy was no dogmatist. Despite his insistence on the limit origin of the differential quotient, he never rejected the formal approach, which he called symbolic [1, 2nd ser., VII, 198–254; VIII, 28–38] and often justified by Fourier transformation. On a large scale he used the formal approach in differential and difference equations. Cauchy admitted semiconvergent series, called "limited" [1, 1st ser., VIII, 18–25; XI, 387–406], and was the first to state their meaning and use clearly. By means of semiconvergent series in 1842 he computedall the classic integrals such as $\int^{\infty} v \cos 1/2\pi v^2 dv$ [1, 1st ser., VII, 149–157] and, in 1829 [1, 1st ser., II, 29–58], asymptotics of integrals of the form $\int u^n v\, dx$, particularly those such as $\int (1-x)^m x^n f(x)\, dx$, where beta functions are involved if $f(x)$ is duly developed in a series [1, 1st ser., IX, 75–121; II, 29–58]; he used rearrangements of conditionally convergent series [1, 1st ser., IX, 5–14] in the same way.

In a more profound sense Cauchy was rather more flexible than dogmatic, for more often than not he sinned against his own precepts. He operated on series, Fourier transforms, and improper and multiple integrals as if the problems of rigor that he had raised did not exist, although certainly he knew about them and would have been able to solve them. Although he had been first to define continuity, it seems that Cauchy never proved the continuity of any particular function. For instance, it is well known that he asserted the continuity of the sum of a convergent series of continuous functions [1, 2nd ser. III, 120]; Abel gave a counterexample, and it is clear that Cauchy himself knew scores of them. It is less known that later Cauchy correctly formulated and applied the uniform convergence that is needed here [1, 1st ser., XII, 33]. He proved

by a popular but unjustified interchange of limit processes [1, 2nd ser., III, 147], although he was well acquainted with such pitfalls; it is less well-known that he also gave a correct proof [1, 2nd ser., XIV, 269–273]. Terms like "infinitesimally small" prevail in Cauchy's limit arguments and epsilontics still looks far away, but there is one exception. His proof [1, 2nd ser., III, 54–55] of the well known theorem.

If $\lim_{x\to\infty} (f(x)) = \alpha$,

then $\lim_{x\to\infty} x^{-1} f(x) = \alpha$, is a paragon, and the first example, of epsilontics— the character ε even occurs there. It is quite probable that this was the beginning of a method that, after Cauchy, found general acceptance. It is the weakest point in Cauchy's reform of calculus that he never grasped the importance of uniform continuity.

**Complex Functions.** The discoveries with which Cauchy's name is most firmly associated in the minds of both pure and applied mathematicians are without doubt his fundamental theorems on complex functions.

Particular complex functions had been studied by Euler, if not earlier. In hydrodynamics d'Alembert had developed what are now called the Cauchy-Riemann differential equations and had solved them by complex functions. Yet even at the beginning of the nineteenth century complex numbers were not yet unanimously accepted; functions like the multivalued logarithm aroused long-winded discussions. The geometrical interpretation of complex numbers, although familiar to quite a few people, was made explicit by Gauss as late as 1830 and became popular under his name. It is, however, quite silly to doubt whether, earlier, people who interpreted complex functions as pairs of real functions knew the geometric interpretation of complex numbers. Gauss's proofs of the fundamental theorem of algebra, although reinterpreted in the real domain, implicitly presupposed some facts from complex function theory. The most courageous ventures in complex functions up to that time were the rash ideas of Euler and Laplace of shifting real integration paths in the complex domain (for instance, that of $e^{-x^2}$ from $-\infty$ to $\infty$) to get new formulas for definite integrals [24], then an entirely unjustified procedure. People sometimes ask why Newton or Leibniz or the Bernoullis did not discover Cauchy's integral theorem and integral formula. Historically, however, such a discovery should depend first on some geometrical idea on complex numbers and second on some more sophisticated ideas on definite integrals. As long as these conditions were not fulfilled, it was hardly possible to imagine integration along complex paths and theorems about such kinds of integrals. Even Cauchy moved slowly from his initial hostility toward complex integration to the apprehension of the theorems that now bear his name. It should be mentioned that Gauss knew most of the fundamental facts on complex functions, although he never published anything on them [ 22, pp. 155–160].

The first comprehensive theory of complex numbers is found in Cauchy's *Cours d'analyse* of 1821 [1, 2nd ser., III, 153–256]. There he justified the algebraic and limit operations on complex numbers, considered absolute values, and defined continuity for complex functions. He did not teach complex integration, although in a sense it had been the subject of his *mémoire* submitted to the French Academy in 1814 and published in 1825 [1, 1st ser., I, 329–506]. It is clear from its introduction that this *mémoire* was written in order to justify such rash but fruitful procedures as those of Euler and Laplace mentioned above. But Cauchy still felt uneasy in the complex domain. He interpreted complex functions as pairs of real functions of two variables to which the Cauchy-Riemann differential equations apply. This meant bypassing rather than justifying the complex method. Thanks to Legendre's criticism Cauchy restored the complex view in footnotes added to the 1825 publication, although he did not go so far as to admit complex integration paths. The problem Cauchy actually dealt with in this *mémoire* seems strange today. He considered a differentiable function $f= u + iv$ of the complex variable $z = x + iy$ and, using one of the Cauchy–Riemann differential equations, formed the double integral

(2) $\int\int u_x \, dx \, dy = \int\int v_y \, dx \, dy$

over a rectangle $x_0 \leqslant x \leqslant x_1, y_0 \leqslant y \leqslant y_1$. Performing the integrations, he obtained the fundamental equality

Using the other Cauchy-Riemann differential equation, he obtained a second equality; and together they yielded

the Cauchy integral theorem for a rectangular circuit, as soon as one puts the *i* between the *d* and the *y*.

Of course regularity is supposed in this proof. Cauchy had noticed, however, that (3) and (4) may cease to hold as soon as there is a singularity within the rectangle; this observation had even been his point of departure. He argued that when drawing conclusions from (2), one had interchanged integrations; and he decided that this was not generally allowed. He tried to compute the difference between the two members of (4), but his exposition is quite confused and what he means is elucidated elsewhere [1, 2nd ser., VI, 113–123].

Let $a + ib$ be the (simple polar) singularity. Then the integrals in (3) and (4) have to be understood as their principal values, e.g.,

This means that the first member of (3) is the limit of the sum of the double integrals over the rectangles

$x_0 \leqslant x \leqslant x_1, y_0, \leqslant y \leqslant b - \varepsilon;$

$x_0 \leqslant x \leqslant x_1, \mathrm{b} + \varepsilon \leqslant y \leqslant y_1;$

and the difference between both members of (3) and instrument; a host of old and new definite integrals

$x_0 \leqslant x \leqslant x_1, b - \varepsilon \leqslant y \leqslant b + \varepsilon.$

In other words,

where $x_0, x_1$ may still be replaced by arbitrary abscissae around $a$. This expression is just what Cauchy calls a singular integral. In his 1814 paper he allows the singularity to lie on the boundary of the rectangle, and even in a corner. (To make the last step conclusive, one should define principal values in a more sophisticated way.)

Of course if there is one singularity $a + ib$ within the rectangle, then according to the residue theorem the difference of both members of (4) should be $2\pi i$ times the coefficient of $z - (a + ib))^{-1}$ in the Laurent series of $f(z)$. This knowledge is still lacking in Cauchy's 1814 paper. He deals with simple polar singularities only, taking $f(z)$ as a fraction $g(z)/h(z)$

In the 1825 footnotes he adds the expression

(6) $2\pi i \lim_{\varepsilon=0} \varepsilon f((a + ib) + \varepsilon)$,

which had already appeared in 1823 [1, 2nd ser., I, 337].

Cauchy's most important general result here is the computation of

(over the real axis) as a sum of expressions (5) from the upper half-plane; singularities on the real axis are half accounted for in such sums. The conditions under which he believes one is entitled to pass from the rectangle as an integration path to the real axis are not clearly formulated. It seems that he requires vanishing of $f(z)$ at infinity, which of course is too much; in any case, he applies the result to functions with an infinity of poles, where this requirement does not hold. In 1826 he stated more sophisticated but still too rigid conditions [1, 2nd ser., VI, 124–145]; strangely enough, at the end of this paper he returned to the useless older ones. In 1827 [1, 2nd ser., VII, 291–323] he discovered the "good conditions": $zf(z)$ staying bounded on an appropriate sequence of circles with fixed centers and with radii tending to infinity.

Even in the crude form of the 1814 *mémoire*, Cauchy's integral theorem proved to be a powerful instrument; a host of old and new definite integrals could be verified by this method. The approach by double integrals looks strange, but at that time it must have been quite natural; in fact, in his third proof of the fundamental theorem of algebra (1816), Gauss used the same kind of double integrals to deal with singularities [22, pp. 155–160].

Genuine complex integration is still lacking in the 1814 *mémoire*, and even in 1823 Poisson's reflections on complex integration [23] were bluntly rejected by Cauchy [1, 2nd ser., I, 354]. But they were a thorn in his side; and while Poisson did not work out this idea, Cauchy soon did. In a *mémoire détaché* of 1825 [2], he took a long step toward what is now called Cauchy's integral theorem. He defined integrals over arbitrary paths in the complex domain; and through the Cauchy-Riemann differential equations he derived, by variation calculus, the fact that in a domain of regularity of $f(z)$ such an integral depended on the end points of the path only. Curiously enough he did not introduce closed paths. Further, he allowed the changing path to cross a simple polar singularity $\gamma$, in which case the integral had to be interpreted by its principal value. Of course, the variation then would differ from zero; its value, equal toward both sides, would be

$\lim_{\varepsilon=0} \varepsilon f(\gamma + \varepsilon)\, \pi i$.

In the case of an $m$ tuple polar singularity $\gamma$ the integrals over paths on both sides of $\gamma$ would differ by

a formula that goes back at least as far as 1823 [1, 2nd ser., I, 337 n.]. (Notice that at this stage Cauchy did not know about power series development for analytic functions.) The foregoing yields the residue theorem with respect to poles; it was extended to general isolated singularities by P. A. Laurent in 1843 [22].

(It is a bewildering historical fact that by allowing for simple singularities upon the integration path, Cauchy handled his residue theorem as a much more powerful tool than the one provided by modern textbooks, with their overly narrow formulation.)

The important 1825 *mémoire* was neither used nor quoted until 1851 [1, 1st ser., XI, 328], a circumstance utterly strange and hard to explain. Did Cauchy not trust the variational method of proof? Was he bothered by the (unnecessary) condition he had imposed on the paths, staying within a fixed rectangle? Did he not notice that the statement could be transformed into the one about closed paths that he most needed? Or had he simply forgotten about that *mémoire détaché?* In any case, for more than twenty-five years he restricted himself to rectangular paths or circular-annular ones (derived from the rectangular kind by mapping), thus relying on the outdated 1814 *mémoire* rather than on that of 1825.

The circle as an integration path and Cauchy's integral formula for this special case had in a sense already been used in 1822 and 1823 (1, 2nd ser., II, 293–294; I, 338, 343, 348], perhaps even as early as 1819 [1, 2nd ser., II, 293 n.]. The well-known integral expression for the $n$th derivative also appeared, although of course in the form

since complex integration paths were still avoided. In 1840 (perhaps as early as 1831) such an expression of regularity of f(z) would be called an average (over the unit circle) and, indeed, constructed as the limit of averages over regular polygons [1, 2nd ser., XI, 337].

Indirect applications of Cauchy's definition of integrals were manifold in the next few years. In the *Exercices* of 1826–1827 [1, 2nd ser., V-VI] many papers were devoted to a rather strange formal calculus of residues. The residue of *f(z)* at γ is defined as the coefficient of $\pi i$ in (6); the residue in a certain domain, as the sum of those at the different points of the domain. A great many theorems on residues are proved without recurring to the integral expressions, and it often seems that Cauchy had forgotten about that formula.

By means of residues Cauchy arrived at the partial fractions development of a function *f(z)* with simple poles.

The trouble with this series is the same as that with the residue theorem. Originally the asymptotic assumption under which this would hold, reads: vanishing at infinity. This is much too strong and surely is not what Cauchy meant when he applied the partial fractions development under much broader conditions The condition in [1, 2nd ser., VII, 324–362] is still too strong. It is strange that in this case Cauchy did not arrive at the "good condition"; and it is stranger still that in 1843 he again required continuity at infinity, which is much too strong [1, 1st ser., VIII, 55–64].

From the partial fractions development of mero-morphic functions it was a small step to the product representation of integral functions; it was taken by Cauchy in 1829–1830 [1, 2nd ser., IX, 210–253]. In special cases Cauchy also noticed the exponential factor, needed in addition to the product of linear factors; the general problem, however, was not solved until Weierstrass. Poles and roots in such investigations used to be simple; Cauchy tried multiple @ones as well [1, 2nd ser., IX, 223], but this work does not testify for a clear view.

Cauchy skillfully used residues for many purposes. He expressed the number of roots of a function in a domain by logarithmic residues [1, 2nd ser., VII, 345–362] and, more generally, established a formula for sums over the roots $z_i$ of *F(z)*,

which had many applications. He was well aware of the part played by arrangement in such infinite sums. In 1827 he derived the Fourier inversion formula in this context [1, 2nd ser., VI, 144; VII, 146–159, 177–209].

In 1827 Cauchy devised a method to check the convergence of a special power series for implicit functions, the so-called Lagrange series of celestial mechanics [1, 1st ser., II, 29–66]. It is the method that in the general case leads to the power series development: a function in the complex domain with a continuous derivative can be developed into a power series converging in a circle that on its boundary contains the next singularity. It seems to have been proved in the Turin *mémoires détachés* of 1831–1832; a summary of these papers was published in 1837 as *Comptes rendus* notes [1, 2nd ser., IV, 48–80] and the papers themselves, or a substantial part of them, were republished in 1840–1841 [1, 2nd ser., XI, 331– 353; XII, 48–112; see also XI, 43–50]. Here Cauchy first derives in a remarkable way his integral formula from his integral theorem by means of

which is formulated for circular paths only, although it also applies to arbitrary circuits. The development of the integrand of

according to powers of $z$ yields the power series development of *f(z)*. Cauchy also finds an integral expression of the remainder if the development is terminated and the power series coefficients theorem

$|a_n| \leqslant \max_{|z|=r} |f(z)| \cdot r^{-n}$

(see also [1, Ist ser., VIII, 287–292]), which was to become the cornerstone of the powerful *calcul des limites*.

The results were applied to implicitly given functions. Using (8) a simple zero $w$ of $F(z, w) = 0$ or a sum of simple roots or a sum $\sum \Phi(w_i)$ over simple roots, $w_i$ is developed into a power series according to $z$. Cauchy also noticed that the power series for a simple root will converge up to the first branching point, which is obtained by $\partial F(z, w)/\partial w = 0$—of course it should be one of the same sheet, but this was not clear at the time. In one of his 1837 notes [1, 1st ser., IV, 55–56] Cauchy had gone so far as to state that in all points developments according to fractional powers were available; in 1840–1841, however, he did not come back to this point.

The foregoing summarizes some of Cauchy's tremendous production in this one area of his work. It is awe-inspiring and yet, in a sense, disappointing. One feels that Cauchy had no clear overall view on his own work. Proofs are usually unnecessarily involved and older papers, superseded by newer results, are repeatedly used and quoted. Often he seems to be blindfolded; for example, he did not notice such a consequence of his work as that a bounded regular function must be constant [1, 1st ser., VIII, 366–385] until Liouville discovered this theorem in the special case of doubly periodic functions—this is why it is now (incorrectly) called Liouville's theorem. One can imagine that Cauchy felt ashamed and confused, so confused, indeed, that he missed the point to which he should have connected Liouville's theorem. Instead of using the power series coefficients theorem he handled it with partial fractions development, which does not work properly because of the asymptotic conditions.

Cauchy also failed to discover Laurent's theorem and the simple theorem about a function with an accumulating set of roots in a regularity domain, which he knew only in crude forms [1, 1st ser., VIII,5–10]. He would have missed much more if others had cared about matters so general and so simple as those which occupied Cauchy. Most disappointing of all is, of course, the fact that he still did not grasp the fundamental importance of his 1825 *mémoire*. He confined himself to rectangular and circular integration paths and to a special case of his integral formula.

A sequence of *Comptes rendus* notes of 1846 [1, 1st ser., X, 70–74, 133–196] marks long-overdue progress. Cauchy finally introduced arbitrary closed integration paths, although not as an immediate consequence of his 1825 *mémoire*, which he did not remember until 1851 [1, 1st ser., XI, 328]; instead, he proved his integral theorem anew by means of what is now called Green's formula—a formula dating from 1828 but possibly rediscovered by Cauchy. A still more important step was his underst and ing of multivalued ana-lytic functions. The history of this notion is paradigmatic of what often happens in mathematics: an intuitive notion that is fruitful but does not match the requirements of mathematical rigor is first used in a naïve uncritical fashion; in the next phase it is ignored, and the results to which it led are, if needed, derived by cumbersome circumvention; finally, it is reinterpreted to save both the intuitive appeal and the mathematical rigor. In multivalued functions Cauchy embodied the critical phase. From 1821 he treated multivalued functions with a kill-or-cure remedy: if branched at the origin, they would be admitted in the upper half-plane only [1, 2nd ser., III, 267]. Fortunately, he more often than not forgot this gross prescription, which if followed would lead him into great trouble, as happened in 1844 [1, 1st ser., VIII, 264] — strangely enough, he wrote this confused paper just after he had taken the first step away from this dogmatism. Indeed [1, 1st ser., VIII, 156–160], he had already allowed for a plane slit by the positive axis as the definition domain of functions branched at the origin; and he had even undertaken integrations over paths pieced together from $|z| = r$ in the positive sense, $r \leqslant z \leqslant R$ in the positive sense, $|z| = R$ in the negative sense, $r \leqslant z \leqslant R$ in the negative sense, where the two rectilinear pieces are combined into one over the jump function. Such paths had long since been obtained in a natural way by a mapping of rectangular paths.

The progress Cauchy achieved in 1846 consisted in restoring the intuitive concept of a multivalued function. Such a function may now freely be followed along rather arbitrary integration paths, which are considered closed only if both the argument and the function return to the values with which they started (of course this was not yet fully correct). Integration over such closed paths produces the *indices de périodicité* that are no longer due to residues.

This is a revival of the old idea of the multivalued function, with all its difficulties. In 1851—the year of Riemann's celebrated thesis—after Puiseux's investigations on branchings, which again depended on Cauchy's work, Cauchy came back with some refinements [1, 1st ser., XI, 292–314]. He slit the plane by rectilinear *lignes d'arrêt* joining singularities and, as in the 1844 paper, proposed to compute the *indices de périodicité* by means of the integrals of the jump functions along such slits. This is too crude, and it gave Cauchy wrong ideas about the number of linearly independent periods. The correct reinterpretation of multivalued functions is by means of Riemann surfaces, with their *Querschnitte*; Cauchy's *lignes d'arrêt* are drawn in the plane, which means that they may be too numerous.

Nevertheless, the progress made in Cauchy's 1846 notes was momentous. The periodicity of elliptic and hyperelliptic functions had previously been understood as an algebraic miracle rather than by topological reasons. Cauchy's crude approach was just fine enough for elliptic and hyperelliptic integrals, and his notes shed a clear light of understanding upon those functions. Notwithstand ing Riemann's work, this seemed sufficient for the near future. Thus, Briot and Boucquet [18], when preparing the second edition of their classic work, saw no advantage in using Riemann surfaces and still presented Cauchy's theory in its oldform.

Cauchy's work on complex functions has to be pieced together from numerous papers; he could have written a synthetic book on this subject but never did. The first to undertake such a project were Briot and Boucquet [18], Nevertheless, complex function theory up to Riemann surfaces, with the sole exceptions of Laurent's theorem and the theorem on accumulating zeros, had been Cauchy's work. Of course he also did less fundamental work in complex function theory, such as generalizing Abel's theorem [1, 1st ser., VI, 149–175, 187–201], investigating "geometrical factorials" [1, 1st ser., VIII, 42–115] and so on.

**Error Theory.** Cauchy also made three studies of error theory, which he presented as logically connected; this, however, is misleading since to understand them one has to consider them as not connected at all.

The first seems to date from 1814, although it was not published until 1824 and 1831 [1, 2nd ser., II, 312–324; I, 358– 402]. Laplace [14, II, 147–180] had tried to fit a set of $n$ observational data $\ulcorner x_i, y_i \urcorner$ to a linear relation $y = ax + b$. Before Laplace, calculators proceeded by first shifting the average

to the origin to make the problem homogeneous, and then estimating $a$ by

$\sum \delta_i y_i / \sum \delta_i x_i$ where $\delta_i = x_i/|x_i|$.

Laplace proposed a choice of $a$ and $b$ that would make the maximal error $|y_i - ax_i - b|$ (or, alternatively, the sum of the absolute errors minimal). To do so Laplace developed a beautiful method, the first specimen of linear programming. Cauchy, following a suggestion of Laplace, extended his method to fitting triples of observational data $\ulcorner x_i, y_i, z_i \urcorner$ to a relation $z = ax + by + c$; where Laplace had reasoned by pure analysis, Cauchy presented his results in a geometrical frame, which shows him to be, as often, motivated by considerations of geometry.

At the time when Cauchy took up Laplace's problem, fitting by least squares had superseded such methods. Nevertheless, in 1837 [1, 1st ser., II, 5–17] Cauchy attempted to advocate the pre-Laplacian method. He postulated the maximal error (among the $|y_i - ax_i|$) to be "minimal under the worst conditions." It does not become clear what this means, although it is a principle vague enough to justify the older methods. Actually Cauchy now dealt with a somewhat different problem: fitting systems of observational data to polynomials (algebraic, Fourier, or some other kind)

$u = ax + by + cz\ldots$

where the number of terms should depend on the goodness of fit, reached during the course of the computation. What Cauchy prescribes is no more than a systematic elimination of $a, b, c, \cdots$. In 1853, when Cauchy again drew attention to this method [1, 1st ser., XII, 36–46], he was attacked by Bienaymé [16], a supporter of "least squares." Cauchy [1, 1st ser., XII, 63–124] stressed the advantage of the indeterminate number of terms in his own method, obviously not noticing that "least squares" could easily be adapted to yield the same advantage; it is, however, possible that it was the first time he had heard of "least squares."

In this discussion with Bienaymé, Cauchy took a strange turn. What looks like an argument in favor of his second method is actually a third attempt in no way related to the first and second. Cauchy assumes the errors

$\varepsilon_i = u_i - ax_i - by_i - cz_i - \cdots$

to have a probability frequency $f$. The coefficients $k_i$ by which $a$ has to be eliminated from $\sum k_i\varepsilon_i = 0$ should be chosen to maximize the probability of $\sum k_i\varepsilon_i$ falling within a given interval $(-\eta,\eta)$. This is an unhealthy postulate, since generally the resulting $k_i$ will depend on the choice of $\eta$. Cauchy's remedy is to postulate that $f$ should be so well adapted that $k_i$ would not depend on $\eta$. This is quite a strange assumption, since $f$ is not an instrument of the observer but of nature; but it does produce a nice result: the only $f$ that obey these requirements are those with a Fourier transform $\phi$ such that

$\phi(\xi) = \exp(-\alpha\xi^N)$,

where $\alpha$ and $N$ are constants. For $N = 1$ these are the celebrated Cauchy stochastics with the probability frequency

Their paradoxical behavior of not being improved by averaging was noticed by Bienayme and forged into an argument against Cauchy. In the course of these investigations Cauchy proved the central limit theorem by means of Fourier transforms in a much more general setting than Laplace had done. The present author adheres to the heterodox view that Cauchy's proof was rigorous, even by modern standards.

This was a muddy chapter of Cauchy's work, which shows him coining gold out of the mud.

**Algebra.** Cauchy published (1812) the first comprehensive treatise on determinants [1, 2nd ser., I, 64–169] it contains the product theorem, simultaneously discovered by J. Binet; the inverse of a matrix; and theorems on determinants formed by subdeterminants. He knew "Jacobians" of dimension 3 [1, 1st ser., I, 12]; generally defined "Vandermonde determinants"; and in 1829, simultaneously with Jacobi, published the orthogonal transformation of a quadratic form onto principal axes [1, 2nd ser., IX, 172–195], although he must have discovered it much earlier in his work on elasticity. Through his treatise the term "determinant" became popular, and it is strange that he himself later switched to "resultant." A more abstract approach to determinants, like that of Grassmann's algebra, is found in [1, 2nd ser., XIV, 417–466].

Cauchy gave the first systematic theory of complex numbers [1, 2nd ser., III, 153–301]. Later he confronted the "geometric" approach with the abstract algebraic one of polynomials in $x$ mod $x^2 + 1$ [1, 2nd ser., XIV, 93–120, 175–202].

One of Cauchy's first papers [1, 2nd ser., I, 64–169] generalized a theorem of Ruffinis; he proved that if under permutations of its $n$ variables a polynomial assumes more than two values, it assumes at least $p$ values, where $p$ is the largest prime in $n$—in other words, that there are no subgroups of the symmetric group of $n$ permutands with an index $i$ such that $2 < i < p$. Bertrand here replaced the $p$ with $n$ itself for $n > 4$, although to prove it he had to rely on a hypothetical theorem of number theory (Bertrand's postulate) that was later verified by P. L. Chebyshev. Cauchy afterward proved Bertrand's result without this assumption [I, 1st ser., IX, 408–417]. His method in [1, 2nd ser., I, 64–169], further developed in [1, 2nd ser., XIII, 171–182; 1st ser., IX, 277–505; X, 1–68], was the *calcul des substitutions*, the method of permutation groups. Fundamentals of group theory, such as the order of an element, the notion of subgroup, and conjugateness are found in these papers. They also contain "Cauchy's theorem" for finite groups: For any prime $p$, dividing the order there is an element of order $p$. This theorem has been notably reenforced by L. Sylow.

In 1812 Cauchy attacked the Fermat theorem on polygonal numbers, stating that every positive integer should be a sum of $n$ $n$ gonal numbers. At that time proofs for $n = 3{,}4$ were known. Cauchy proved it generally, with the addendum that all but four of the summands may be taken as 0 or 1 [1, 2nd ser., VI, 320–353]. Cauchy's proof is based on an investigation into the simultaneous solutions of

Cauchy contributed many details to number theory and attempted to prove Fermat's last theorem. A large treatise on number theory is found in [1, 1st ser., III].

**Geometry.** Cauchy's most important contribution to geometry is his proof of the statement that up to congruency a convex polyhedron is determined by its faces [1, 1st ser., II, 7–38]. His elementary differential geometry of 1826–1827 [1, 2nd ser., V] strongly influenced higher instruction in mathematics. Of course his elasticity theory contains much differential geometry of mappings and of vector and tensor fields, and the notions of grad, div, rot, and their orthogonal invariance.

**Differential Equations.** What is fundamentally new in Cauchy's approach to differential equations can be expressed in two ideas: (1) that the existence of solutions is not self-evident but has to be proved even if they cannot be made available in an algorithmic form and (2) that uniqueness has to be enforced by specifying initial (or boundary) data rather than by unimportant integration constants. The latter has become famous as the Cauchy problem in partial differential equations. It may have occurred to Cauchy in his first great investigation (1815), on waves in liquids [1, 1st ser., I, 5–318]. Indeed, the difficulty of this problem—and the reason why it had not been solved earlier—was that to be meaningful it had to be framed into a differential equation with initial and boundary data.

To solve ordinary differential equations Cauchy very early knew the so-called Cauchy-Lipschitz method of approximation by difference equations, although its proof was not published until 1840 [1, 2nd ser., XI, 399–404]. Several instances show that he was also acquainted with the principle of iteration [1, 1st ser., V, 236–260; 2nd ser., XI, 300–415 f.; 3, II, 702]. With analytic data the celebrated *calcul des limites* led to analytic solutions of ordinary differential equations [1, 1st ser., VI, 461–470; VII, 5–17; 3, II, 747].

Cauchy discovered (1819), simultaneously with J. F. Pfaff, the characteristics method for first-order partial differential equations [1, 2nd ser., II, 238– 252; see also XII, 272–309; 1st ser., VI, 423–461]. His method was superior to Pfaff's and simpler, but it still appears artificial. The geometrical language in which it is taught today stems from Lie. Of course Cauchy also applied the *calcul des limites* to partial differential equations [1, 1st ser., VII, 17–68]. It is not quite clear which class of equations Cauchy had in mind, in 1875 Sonja (Sophia) Kowalewska precisely formulated and solved the problem by an existence theorem that usually bears the names of Cauchy and Kowalewska.

Another way to solve a system

was by means of exp $tZ$, with

and by an analogous expression if $X_i$ depended on $t$ as well [1, 2nd ser., XI, 399–465; 1st ser., V, 236–250, 391–409]; the convergence of such series was again obtained by *calcul des limites*.

The greater part of Cauchy's work in differential equations was concerned with linear partial equations with constant coefficients, which he encountered in hydrodynamics, elasticity, and optics. The outstanding device of this research was the Fourier transform. It occurs in Cauchy's work as early as 1815, in his work on waves in liquids [1, 1st ser., I, 5–318], as well as in 1817 [1, 2nd ser., I, 223–232] and 1818 [1, 2nd ser., II, 278–279]. Fourier's discovery, while dating from 1807 and 1811, was published as late as 1824–1826 [27], so Cauchy's claim that he found the inversion formula independently is quite acceptable. It is remarkable that he nevertheless recognized Fourier's priority by calling the inversion formula Fourier's formula. Cauchy put the Fourier transform to greater use and used it with greater skill than anybody at that time and for long after—Fourier and Poisson included; and he was the first to formulate the inversion theorem correctly. He also stressed the importance of principal values, of convergence-producing factors with limit 1, and of singular factors (δ functions) under the integral sign [1, 2nd ser., I, 275–355]. His use of the Fourier transform was essentially sound—bold but not rash—but to imitate it in the pre-epsilontic age one had to be another Cauchy. After Weierstrass, Fourier transforms moved into limbo, perhaps because other methods conquered differential equations. Fourier transforms did not become popular until recently, when the fundamentals of Fourier integrals were proved with all desirable rigor; but so much time had elapsed that Cauchy's pioneering work had been forgotten.

From 1821 on, Cauchy considered linear partial differential equations in the operational form

with $F$ as a polynomial function in $u_1,\ldots, u_n, s$. Such a differential equation has the exponential solutions

$\exp (\sum u_i x_i + st)$,

which are functions of $x_i$, $t$ for every system of $u_i$, $s$ fulfilling

(10) $F(u_1,\ldots, u_n, s)$,

The Fourier transform method aims at obtaining the general solution by continuously superposing such exponential solutions, with imaginary $u_1,\ldots, u_n, s$. For wave equations this means wave solutions by superposition of plane harmonic waves. In the 1821 and 1823 papers [1, 2nd ser., II, 253–275; I, 275–333] a kind of interpolation procedure served to satisfy the initial conditions for $t = 0$. Another approach would be solutions arising from local disturbances (spherical waves under special conditions); they may be obtained from plane waves by superposition and in turn may give rise to general solutions by superposition. This idea is present in the 1815 papers on waves; it is neglected but not absent in the 1821 and 1823 papers.

In 1826 the residue calculus is introduced as a new device, first for solving linear ordinary differential equations with constant coefficients [1, 2nd ser., VI, 252–255; VII, 40–54, 255–266]. The general solution is obtained as the integral

performed around the roots of $F$, with an arbitrary polynomial $\psi(\zeta)$. Several times Cauchy stressed that this formula avoids casuistic distinctions with respect to multiple roots of $F$ [1, 1st ser., IV, 370].

In 1830, when Cauchy went into optics, this formula was applied to partial differential equations, which meant (10) and (9) explicitly and elegantly solved with respect to $s$ and $t$, whereas with respect to $x_1 \ldots, x_n$ the Fourier transform method prevailed. Again due care was bestowed on the initial conditions at $t = 0$. The formula, obtained in polar coordinates, is involved and not quite clear; its proof is not available because the *mémoire* of which the 1830 paper is a brief extract seems never to have been published and possibly is lost. The construction of wave fronts rested upon intuitive arguments, in fact upon Huygens' principle, although he did not say so and never proved it. According to the same principle Cauchy constructed ray solutions as a superposition of planar disturbances in planes that should be slightly inclined toward each other, as Cauchy says.

From June 1839 to March 1842 Cauchy, again drawing on optics, tried new approaches to linear partial differential equations with constant coefficients. This work was instigated by P. H. Blanchet's intervention (see [1: 1st ser. IV, 369–426; V, 5–20; VI, 202–277, 288–341, 375–401, 404–420; 2nd ser., XI, 75–133, 277–264; XII, 113–124]). It now starts with a system of first-order linear ordinary differential equations, in modern notation

where $x$ is an $n$-vector and $A$ a linear mapping. One considers

$S(s) = \det(A - s)$

and defines the *fonction principale* $\Theta$ as the solution of

which is obtained as an integral

around the roots of $S$. From $\Theta$ the solution of (11) with the initial vector $\ulcorner \alpha_1, \ldots, \alpha_n \urcorner$ is elegantly obtained by using

$Q(s) = \det(A - s) \cdot (A - s)^{-1}$

and applying $Q(d/dt)$ to the vector

This method is extended to partial differential equations, where it again suffices to solve (9) under the initial conditions

The formula obtained is much simpler than that of 1830, particularly if $F$ is homogeneous. (See the quite readable presentation in [1, 1st ser., VI, 244–420].) If, moreover, the initial value of $(d/dt)^{n-1}\omega$ prescribed is a function $\Sigma$ of

$\sigma = \sum u_i x_i$,

one obtains, by first assuming $\Pi$ to be linear and then using the homogeneity of $F$,

where $(d/dt)^{-1}$ means the integration over $t$ from 0. An analogous formula is obtained if the initial datum is assumed as a function of a quadratic function of $x$, say of $r = (\sum x_i^2)^{1/2}$.

The direct method of 1830 is again applied to the study of local disturbances and wave fronts, with Huygen's principle on the background. Cauchy apparently believed that disturbances of infinitesimal width stay infinitesimal, although this belief is disproved by $(d/dt)^{1-n}$ in (12) and by the physical argument of two or even three forward wave fronts in elasticity. It was only after Blanchet's intervention that Cauchy admitted his error.

It is doubtful whether Cauchy's investigations on this subject exerted strong immediate influence; perhaps the generations of Kirchhoff, Volterra, and Orazio Tedone were terrified by his use of Fourier transforms. Solutions arising from local disturbances, substituted into Green's formula, seemed more trustworthy. But even after Hadamard's masterwork in this field Cauchy's attempts should not be forgotten. The only modern book in which they are mentioned and used, although in an unsatisfactory and somewhat misleading fashion, is Courant and Hilbert [34].

It should be mentioned that Cauchy also grasped the notions of adjointedness of differential operators in special cases and that he attacked simple boundary problems by means of Green's function [1, 1st ser., VII, 283–325; 2nd. ser., XII, 378–408].

**Mechanics.** Cauchy can be credited with some minor contributions to mechanics of rigid bodies, such as the momental ellipsoid and its principal axes [1, 2nd ser., VII, 124–136]; the surfaces of the momentaneous axes of rigid motion [1, 2nd ser., VII, 119], discovered simultaneously with Poinsot; and the first rigorous proof that an infinitesimal motion is a screw motion [1, 2nd ser., VII, 116]. His proper domain, however, was elasticity. He created the fundamental mathematical apparatus of elasticity theory.

The present investigations have been suggested by a paper of M. Navier, 14 August 1820. To establish the equation of equilibrium of the elastic plane, the author had considered two kinds of forces, the ones produced by dilatation or contraction, the others by flection of that plane. Further, in his computations he had supposed both perpendicular to the lines or faces upon which they act. It came into my mind that these two kinds could be reduced to one, still called tension or pressure, of the same nature as the hydrodynamic pressure exerted by a fluid against the surface of a solid. Yet the new pressure should not be perpendicular upon the faces which undergo it, nor be the same in all directions at a given point…. Further, the pressure or tension exerted against an arbitrary plane is easily derived as to magnitude and direction from the pressures or tensions exerted against three rectangular planes. I had reached that point when M. Fresnel happened to speak to me about his work on light, which he had only partially presented to the Academy, and told me that concerning the laws according to which elasticity varies in the different directions through one point, he had obtained a theorem like mine. However, this theorem was far from sufficient for the purpose I had in mind at that time, that is, to form the general equation of equilibrium and internal motion of a body, and it is only recently that I have succeeded in establishing new principles, suited to lead to this goal and the object of my communication….

These lines, written by Cauchy in the fall of 1822 [1, 2nd ser., II, 300–304], announced the birth of modern elasticity theory. Rarely has a broad mathematical theory been as fully explained in as few words with as striking a lack of mathematical symbols. Never had Cauchy given the world a work as mature from the outset as this.

From Hooke's law in 1660 up to 1821, elasticity theory was essentially one-dimensional. Euler's theory of the vibrating membrane was one of the few exceptions. Another was the physical idea of internal shear stress, which welled up and died twice (Parent, 1713; Coulomb, 1773) with no impact upon the mathematical theory. Even in this one-dimensional setting, elasticity was a marvelous proving ground for Euler's analysis of partial derivatives and partial differential equations (see [36]). In 1821 Navier's paper on equilibrium and vibration of elastic solids was read to the Academy (published in 1827) [30, 32]. Navier's approach constituted analytic mechanics as applied to an isotropic molecular medium that should obey Hooke's law in molecular dimensions: any change in distance between two molecules causes a proportional force between them, the proportionality factor rapidly decreasing with increasing distance. Cauchy was one of the examiners of Navier's paper. It was not, however, this paper of Navier's to which Cauchy alluded in the above quotation. Cauchy's first approach was independent of Navier's it was nonmolecular but rather geometrically axiomatic.

Still another advance had taken place in 1821. Thomas Young's investigations on interference in 1801 had made it clear that light should be an undulation of a hypothetical gaseous fluid, the ether. Consequently light waves were thought of as longitudinal like those of sound in air, although the phenomena of polarization pointed to transverse vibrations. In 1821 Fresnel took the bold step of imagining an ether with resistance to distortion, like a solid rather than a fluid; and marvelously enough he found transmission by transverse waves (although longitudinal ones would subsist as well [32]). Fresnel's results encouraged Cauchy to pursue his investigations.

The short communication from which the above extract was taken was followed by detailed treatises in 1827–1829 (1, 2nd ser., VII, 60–93, 141–145; VIII, 158–179, 195–226, 228–287; IX, 342–369], but nearly all fundamental notions of the mechanics of continuous media were already clear in the 1822 note: the [stress tensor (and the concept of tensor at all), the strain tensor, the symmetry of both tensors, their principal axes, the principle of obtaining equilibrium and motion equations by cutting out and freezing an infinitesimal piece of the medium, and the striking idea of requiring Hooke's law for the principal stresses and strains. For homogeneous media this led to [Navier's equations, with one elastic constant, but [independently of Navier's molecular substructure. Soon Cauchy introduced the second elastic constant, which arose from an independent relation between volume stress and volume strain. It led to the now generally accepted elasticity theory of isotropic media. For anisotropic media Cauchy was induced by Poisson's intervention to admit a general linear dependence between stress and strain, involving thirty-six parameters. The only fundamental notion then lacking was the elastic potential, which allows one to reduce the number of parameters to twenty-one; it is due to G. Green (1837) [30; 32].

Meanwhile, in 1828–1829 Cauchy had pursued Navier's molecular ideas and had arrived at a fifteenparameter theory for anisotropic media [1, 2nd ser., VIII, 227–277; IX, 162–73].[9] The nineteenth-century discussions are long-since closed in favor of the axiomatic "multi-constant" theory and against the molecular "rari-constant" theory, if not against any molecular theory of elasticity at all.

Cauchy applied the general theory to several special problems: to lamellae [1, 2nd sen, VIII, 288–380], to the rectangular beam [1, 2nd ser., IX, 61–86] (definitively dealt with by Saint Venant), and to plane plates [1, 2nd ser., VIII, 381–426; IX, 9–60], in which Kirchhoff finally succeeded. The application on which Cauchy bestowed more pains than on any other subject was elastic light theory. The mathematical context of this theory was partial differential equations with constant coefficients. In the history of physics it was one of the great pre-Maxwellian efforts necessary before physicists became convinced of the impossibility of any elastic light theory.

Cauchy developed three different theories of reflection and refraction (1830, 1837, 1839) [1, 1st ser., II, 91–110; 2nd ser., II, 118–133; 1st ser., IV, 11–38; V, 20–39]. The problems were to explain double refraction (in which he succeeded fairly well), to adjust the elastic constants to the observational data on the velocity of light under different conditions in order to obtain Fresnel's sine and tangent laws of polarization by suitable boundary conditions, and to eliminate the spurious longitudinal vibrations. Whether he assumed the transverse vibrations to be parallel or orthogonal to the polarization plane (as he did in his first and second theories, respectively), he obtained strange relations between the elastic constants and was forced to admit unmotivated and improbable boundary conditions. His third theory, apparently influenced by Green's work, was based upon the curious assumption of an ether with negative compressibility—later called labile by Lord Kelvin—which does away with longitudinal waves. In 1835 Cauchy also attempted dispersion 1, 2nd ser., X, 195–464]; the problem was to explain the dependence of the velocity of light upon the wavelength by a more refined evaluation of the molecular substructure.

**Celestial Mechanics.** One not acquainted with the computational methods of astronomers before the advent of electronic apparatus can hardly evaluate Cauchy's numerous and lengthy contributions to celestial mechanics. In handbooks of astronomy he is most often quoted because of his general contributions to mathematics. Indeed, it must have been a relief for astronomers to know that the infinite series they used in computations could be proved by Cauchy to converge. But he also did much detailed work on series, particularly for the solution of the Kepler equation [1, 1st ser., VI, 16–48] and developments of the perturbative function [1, 1st ser., V, 288–321; VII, 86–126]; textbooks still mention the Cauchy coefficients. Cauchy's best-known contribution to astronomy (1845) is his checking of Leverrier's cumbersome computation of the large inequality in the mean motion of Pallas by a much simpler method [1, 1st ser., IX, 74–220; see also XI, 385–403]. His tools consisted of formulas for the transition from the eccentric to the mean anomaly [1, 1st ser., VI, 21]; the Cauchy "mixed method" [I, 1st ser., VIII, 168–188, 348–359], combining numerical and rational integrations when computing negative powers of the perturbative function; and asymptotic estimations of distant terms in the development of the perturbative function according to multiples of the mean anomaly—such asymptotics had interested Cauchy as early as 1827 [1, 1st ser., II, 32–58; see also IX, 5–19, 54–74; XI, 134–143].

# NOTES

1. All references in brackets are lo numbered works in the bibliography.

2. The data on his career in (7; 8; 9; 11] are incomplete and contradictory, even self-contradictory; but it still would not be too difficult to check them. According to Cauchy's own account [1, 2nd ser., II. 283] he taught in 1817 as Biot's *suppléant* at the Collège de France. However, on the title pages of his books published until 1830 he never mentions a chair at the Collège de France.

3. Turin was the capital of Piedmont; the dukes of Piedmont and Savoy had become kings of Sardinia.

4. In the library of the University of Utrecht I came across an unknown print of a work by Cauchy that must have appeared in a periodical. I noticed several quotations from papers missing from Valson's list and the Academy edition (e.g. [1, 2nd ser., II, 293, note]).

5. I know only two of them.

6. [1, 1st ser., IX, 186–190] shows him quarreling with the press.

7. Of course he did not prove it; he simply applied it. It had been discovered by Bolzano in 1817.

8. His own account of this invention (1, 1st ser., VIII, 145], although often repeated, is incorrect. With Euler and Lagrange, he said, "continuous" meant "defined by one single law." Actually, there was no serious definition of continuity before Cauchy.

9. I do not understand on what grounds Müller and Timpe [26, IV, 23] and Love [30] claim that Cauchy mistook his fifteenparameter theory for the twenty-one-parameter theory. The common source of this criticism seems to be Clausius (1849). Although Cauchy was sometimes less outspoken on this point, the charge is at least refuted by [1, 2nd ser., IX, 348].

# BIBLIOGRAPHY

1. Cauchy, *Oeuvres complètes*, 1st ser., 12 vols., 2nd ser., 14 vols. (Paris, 1882-). The final volume, 2nd ser., XV, is due to appear in 1970.

2. Cauchy, *Mémoire sur les intègrales définies prises entre des limites imaginaires* (Paris, 1825). To be republished in [1, 2nd ser., XV].

2a. Reprint of [2] in *Bulletin des sciences mathèmatiques*, **7** (1874), 265–304; **8** (1875), 43–55, 148–159.

2b. Cauchy, *Abhandlung über bestimmte Integrate zwischen imaginären Grenzen*, trans, of [2] by P. Stäekel, no. 112 in Ostwald's Kiassiker der exacten Wissenschaften (Leipzig, 1900).

3. F. N. M. Moigno and M. Lindelöf, eds., *Leçons du calcul différentiel et de calcul intégral, rédigées d'après les méthodes et les oeuvres publiés ou inédits d' A.-L. Cauchy*, 4 vols. (Paris, 1840–1861).

4. F. N. M. Moigno, *Leçons de mécanique analytique, rédigées principalement d'après les méthodes d'Augustin Cauchy…. Statique* (Paris, 1868).

5. B. Boncompagni, "La vie et les travaux du baron Cauchy," in *Bollettino di bibliografia e di storia delle scienze matematiche e fisiche*, **2** (1869), 1–102. Review of [8].

6. K. Rychlik, "Un manuscrit de Cauchy aux Archives de l'Académie tchécoslovaque des sciences," in *Czechoslovak Mathematical Journal*, **7** **[82** ] (1957), 479–481.

7. J. B. Biot, *Mélanges scientifiques et littéraires*, III (Paris, 1858), 143–160.

8. C. A. Valson, *La vie et les travaux du baron Cauchy*, 2 vols. (Paris, 1868).

9. J. Bertrand, "La vie et les travaux du baron Cauchy par C. A. Valson," in *Journal des savants* (1869), 205–215, and *Bulletin des sciences mathématiques*, **1** (1870), 105–116. Review of [8].

10. C. A. Bjerknes, *Niels-Henrik Abel, tableau de sa vie et de son action scientifique* (Paris, 1885), pp. 268– 322, 342–347.

10a. D. E. Smith, "Among My Autographs, 29. Legendreand Cauchy Sponsor Abel," in *American Mathematical Monthly*, **29** (1922), 394–395.

10b. F. Lange-Nielsen, "Zur Geschichte des Abelschen Theorems. Das Schicksal der Pariser Abhandlung," in *Norsk matematisk tidsskrift*, **9** (1927), 55–73.

11. J. Bertrand, *Éloges académiques*, new ed. (Paris, 1902), pp. 101–120.

12. A. d'Hautpoul, *Quatre mois à la cour de Prague* (Paris, 1912). Not accessible to the present reporter.

13. A. Terracini, "Cauchy a Torino," in *Rendiconti del Seminario matematico* (Turin), **16** (1956–1957), 159–205.

14. P. S. Laplace, *Traité de la mécanique céleste*, II (Paris1804).

15. C. Sturm, "Analyse d'un mémoire sur la résolution des équations numériques," in *Bulletin universel des sciences et de l'industrie*, **11** (1829), 419–422.

16. J. Bienaymé, "Considerations…, Remarques…," in *Comptes rendus de l'Académie des sciences*, **37** (1853), 5–13, 68–69, 309–324.

17. N. H. Abel, "Untersuchungen über die Reihe…," in *Journal für reine and angewandte Mathematik*, **1** (1826), 311–359, also published as no. 71 in Ostwald's Klassiker der exacten Wissenschaften (1895).

18. S.-H. Briot and J. C. Boucquet, *Théorie des fonctions doublement périodiques et, en particulier, des fonctions elliptiques* (Paris 1859); 2nd ed., *Théorie des fonctions elliptiques* (Paris 1873–1875).

19. F. Casorati, *Teorica delle funzioni di variabili complesse* (Pavia, 1868). Not accessible to the present reporter.

20. E. Verdet, *Leçons d'optique physique*, vols. V–VI of Verdet's *Oeuvres* (Paris, 1869–1872).

21. F. Studnicka, *A. L. Cauchy als formaler Begründer der Determinantentheorie* (Prague, 1876). Not accessible to the present reporter.

21a. I. Todhunter, *A History of the Theory of Elasticity and of the Strength of Materials from Galilei to the Present, Time*, vol. I (Cambridge, 1886).

22. A. Brill and M. Noether, "Die Entwicklung der Theorie der algebraischen Funktionen in älterer and neuerer Zeit," in *Jahresbericht der Deutschen Mathematiker-Vereinigung*, **3** (1894), esp. 155–197.

23. P. Stäckel, "Integration durch das imaginäre Gebiet," in *Bibliotheca mathematica*, 3rd ser., **1** (1900), 109–128.

24. P. Stäckel, "Beitrag zur Geschichte der Funktionentheorie im 18. Jahrhundert." in *Bibliotheca mathematica*, 3rd ser., **2** (1901), 111–121.

25. P. Jourdain, “The Theory of Functions With Cauchy and Gauss,” in *Bibliotheca mathematica*, 3rd ser., **6** (1905),190–207.

26. The following articles in the *Encyklopädie der mathematischen Wissenschaften:* C. H. Müller and A. Timpe, “Grundgleichungen der mathematischen Elastizitätstheorie,” IV (1907), I; A. Wangerin, “Optik, ältere Theorie,” V (1909), 21; K. F. Sundman. “Theorie der Planeten.” VI, pt. 2 (1912), 15; H. von Zeipel, “Entwieklung der Störungsfunktion,” VI, pt. 2 (1912). 557–665.

27. H. Burkhardt, “Entwicklungen nach oscillierenden Funktionen and Integration der Differentialgleichungen der mathematischen Physik,” in *Jahresbericht der Deutschen Mathematiker- Vereinigung*, **10** (1904–1908).

28. E. A. Miller, “Historical Sketch of the Development of the Theory of Groups of Finite Order,” in *Bibliotheca mathematica*, 3rd ser., **10** (1909). 317–329,

29. P. Jourdain, “The Origin of Cauchy’s Conception of a Definite Integral and of the Continuity of a Function,” in *Isis*, **1** (1913–1914), 661–703. Not accessible to the author.

30. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity* (Oxford, 1927), esp, the introduction.

31. H. Lamb, *Hydrodynamics* (Cambridge, 1932), pp. 384 f.

32. E. T. Whittaker, *A History of the Theories of Aether and Electricity* (Edinburgh, 1951), esp, pp. 128–169.

33. E. Carruccio, “I fondamenti delle analisi matematica nel pensiero di Agostino Cauchy.” in *Rendiconti del Seminario matematico* (Turin), **16** (1956–1957). 205–216.

34. R. Courant and D. Hilbert, *Methods of Mathematical Physics*, 11 (1962), esp. pp. 210–221.

35. C. Truesdell-R. Toupin, “The Classical Field Theories,” in *Handbuch der Physik*, III (Berlin-Güttingen-Heidelberg, 1960), pp. 226–793, esp. 259–261. 270–271, 306–308, 353–356, 536–556.

36. C. Truesdell, “The Rational Mechanics of Flexible and Elastic Bodies, 1638–1788,” in *Leonhardi Euleri Opera Omnia*, 2nd ser., XI, sec. 2 (Zurich, 1960), 7–435.

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