Delsarte, Jean Frédéric Auguste | Encyclopedia.com

(*b*. Fourmies, France, 19 October 1903; *d*. Nancy, France, 28 November 1968)

mathematics.

Delsarte was the oldest of three children of the head of a textile factory. After the German invasion in 1914, his father remained in Fourmies to preserve what he could of the factory, while the rest of the family fled to unoccupied France.

Delsarte entered the École Normale Supérieure in 1922. After graduating in 1925 and completing his military service, he was granted a research fellowship, and in a little more than one year he had written his dissertation. Although university teaching jobs were very scarce at that time, Delsarte obtained one at the University of Nancy immediately after receiving the doctorate in 1928; he remained there for the rest of his life, and was dean of the Faculty of Sciences from 1945 to 1949. In 1929 he married Thérèse Sutter; they had two daughters.

In mathematics Delsarte had a predilection for what has been called "hard" analysis, in the tradition of Leonhard Euler, Carl Jacobi, and G. H. Hardy. Like them, he was a superb calculator with a remarkable talent for seeing his way through a maze of computations. His ideas were strikingly original, and his work was completely uninfluenced by the mathematical fashions of his time. Since Delsarte had very few students, he did not receive the recognition he deserved; several of his pioneering ideas were rediscovered much later without his being given credit. Such was the case with his extension of the Möbius function to abelian groups and its use in enumeration problems, and with the generalization to Fuchsian groups of the formula giving the value of the number of lattice points within a ball.

Delsarte's main results in mathematical analysis stem from a common theme: the expansion of a function *f* by a series

where the ϕ_i are functions independent of f and the L_i linear functionals. The Taylor series with $\phi_k (x) =x^k/K!$ and $L_k (f)=D^kf(o)$ is the "classical" example, but many other types were known and were studied in George Watson and Edmund Whittaker's *A Course of Modern Analysis* and Watson's *Treatise on the Theory of Bessel Functions*, which were Delsarte's most cherished books. These works had convinced him that a good understanding of the formal properties of such expansions was necessary to a fruitful study of their domain of definition and their mode of convergence. This was the course he followed with remarkable success, opening up new fields of research that are still far from having been thoroughly explored.

The starting point is a vector space *E* of complex valued functions, which for simplicity one may assume to be defined in a neighborhood I of 0 in **R** , and to be $C^{\propto}$.

(A) There is given an endomorphism D of E that has a continuous spectrum. This means that for every $\lambda \in C$, there is a function $j_k \in E$ for which

(B) Each j_k has a formal expansion

where the ϕ_n are polynomials belong to E and $\phi_0 = 1$, so that

(C) Delsarte introduces the formal series depending on a parameter $y \in I$

these operators satisfy the relation

and if $\phi_f (x, y) = (T^y f)(x)$ for $y, z \in I$,

This is in general a partial differential equation, and its integration with the initial condition $\Phi_f (x, 0) = f(x)$ yields $T^y f$. With the initial condition $\Psi(x, 0) = 0$, the equation

similarly yields the "remainder," the difference between $T^y f$ and the first n terms in (5). In the classical case, $\phi_f (x, y) = f(x+y)$. A more interesting example (among many others studied by Delsarte) corresponds to for which Delsarte shows that

(D) Delsarte's main interest in these expansions was in the study of linear endomorphisms U of E that commute with all the T^y for $y \in \mathrm{I}$. In the classical case with $E = D(R)$, this condition and minimal continuity properties characterize the convolution operators $f \mapsto f * \mu$ where μ is a measure or distribution with compact support. In general,

where α is a linear form on E; this implies

(E) The introduction of these generalized convolution operators U leads to the subclasses J_U of functions $f \in E$ such that $U \cdot f = 0$.

In the classical case, in which μ is the difference of two Dirac measures $\mu = \delta_0 – \delta_0$, $J_U$consists of the functions of period a. For an arbitrary μ, Delsarte called J_U a space of mean-periodic functions; in the general case, he spoke of "J-mean periodic" functions. In many cases, $X(\lambda)$ is an entire function of λ with a sequence of zeroes $\lambda_1, \lambda_2, \ldots \lambda_n, \ldots$; with each J-mean periodic function is associated a formal expansion

for which Delsarte gave a general method of computing the c_n; it is similar to the Fourier series, to which it reduces for $U \cdot f = f * (\delta_a - \delta_0$. If $X(0) \neq 0$, the Taylor series $j_\lambda (x)/X(\lambda)$ introduces polynomials that generalize the Bernoulli polynomials. Delsarte also gave a generalized Euler-Maclaurin formula with explicit remainder using these polynomials.

Earlier, Delsarte had made a thorough study of the case $U \cdot f = f * \mu$; then X is the Fourier transform of μ, and therefore an entire function, and $j_{\lambda n}$ have the form $P_n (x) \exp (2\pi i \lambda_n x)$, with P_n a polynomial. Delsarte investigated the convergence of such expansions, and his results were extended in the later work of Laurent Schwartz and Jean-Pierre Kahane. For the operator D defined in (9), he also showed how, for different choices of U, the expansions (13) became various known expansions in Bessel functions, where usually the coefficients c_n were computed by ad hoc devices but Delsarte's general process applied in every case.

After 1950 Delsarte greatly enlarged the scope of these ideas. The classical translation operators generalize naturally to the operators S^y and T^y defined by $S^y f{:}x\, f \mapsto (yx)$ and $T^y f{:}x\, f \mapsto (xy)$ in any Lie group. Delsarte considered the more general situation of a Lie group G with a compact group, A of automorphisms of G. The operators S^y and T^y are defined by

and reduce to translations when A reduces to the identity element 1_G. In general those operators still satisfy $S^y T^z = T^z S^y$ and depend only on the orbit $\check{y}$ of y under A, so that they operate on functions on the space of orbits G/A. When G is the additive group R^2 and A the group of rotations, G/A can be identified with the positive reals, and the function on *(G/A) x (G/A)*.

is given by (10). When $A = \{1_G\}$, the relation (7) for Φ_f on $G \times G$ still holds when D is an invariant vector field (element of the Lie algebra of G). Delsarte showed that such equations still exist in general, but for invariant differential operators, which in general will be of order > I, as (9) shows. He considered this result as the beginning of an analog of Lie theory for G/A. This point of view was developed by B. Lewitan, and probably more remains to be done.

The fact that the operators (5) may be obtained by integrating the partial differential equation (7) led Delsarte to consider more generally, for two different differential operators D and D′ the equation

which yields what he called "transmutation" operators, changing D into D'. This was developed by Delsarte in collaboration with J.-L. Lions, and later was used by Lewitan and others for the Sturm-Liouville problem.

The same ideas enabled Delsarte to prove one of his most elegant and unexpected results. It has been known since Gauss that if a C^∞ function f in R^n has at each point x a value equal to its mean value on every sphere of center x, it is a harmonic function. Delsarte showed that the conclusion still holds if $f(x)$ is equal to its mean value on only two spheres of center x and radii $a > b > 0$, provided the ratio a/b is not a number in a finite set depending only on n.

From 1950 on, Delsarte often lectured at institutes and universities in India and North and South America. After 1960 his eyesight which had always been poor, began to deteriorate. At the end of his life he could read and write only with great difficulty. From 1962 to 1965 he was director of the Franco-Japanese Institute in Tokyo.

BIBLIOGRAPHY

Delsarte's works are collected in *Oeuvres de Jean Delsarte*, 2 vols. (Paris, 1971), which includes brief essays on his life and work. His writings also include *Lectures on Topics in Mean Periodic Functions and the Two-Radius Theorem* (Bombay, 1961).

Jean DieudonnÉ