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(*fl*. Alexandria [and Athens?], *ca*. 295 b.c.)

mathematics. The following article is in two parts: Life and Works; Transmission of the Elements.

Life and Works

Although Euclid (Latinized as Euclides) is the most celebrated mathematician of all time, whose name became a synonym for geometry until the twentieth century,[1] only two facts of his life are known, and even these are not beyond dispute. One is that he was intermediate in date between the pupils of Plato (*d*. 347 b.c.) and Archimedes (*b.ca*. 287 b.c.); the other is that he taught in Alexandria.

Until recently most scholars would have been content to say that Euclid was older than Archimedes on the ground that Euclid, *Elements* I.2, is cited in Archimedes, *On the Sphere and the Cylinder* I.2; but in 1950 Johannes Hjelmslev asserted that this reference was a naïve interpolation. The reasons that he gave are not wholly convincing, but the reference is certainly contrary to ancient practice and is not unfairly characterized as naïve; and although it was already in the text in the time of Proclus, it looks like a marginal gloss which has crept in.[2] Although it is no longer possible to rely on this reference,[3] a general consideration of Euclid's works such as that presented here still shows that he must have written after such pupils of Plato as Eudoxus and before Archimedes.

Euclid's residence in Alexandria is known from Pappus, who records that Apollonius spent a long time with the disciples of Euclid in that city.[4] This passage is also attributed to an interpolator by Pappus' editor, Friedrich Hultsch, but only for stylistic reasons (and these not very convincing); and even if the Alexandrian residence rested only on the authority of an interpolator, it would still be credible in the light of general probabilities. Since Alexander ordered the foundation of the town in 332 b.c. and another ten years elapsed before it began to take shape, we get as a first approximation that Euclid's Alexandrian activities lay somewhere between 320 and 260 b.c. Apollonius was active at Alexandria under Ptolemy III Euergetes (acceded 246) and Ptolemy IV Philopator (acceded 221) and must have received his education about the middle of the century. It is likely, therefore, that Euclid's life overlapped that of Archimedes.

This agrees with what Proclus says about Euclid in his commentary on the first book of the Elements. The passage, which is contained in Proclus' summary of the history of geometry,[5] opens:

Not much younger than these [Hermotimus of Colophon and Philippus of Medma, two disciples of Plato] is Euclid, who put together the elements, arranging in order many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefutable demonstration the things which had been only loosely proved by his predecessors. This man lived[6] in the time of the first Ptolemy;[7] for Archimedes, who followed closely upon the first [Ptolemy], makes mention of Euclid,[8] and further they say that Ptolemy once asked him if there were a shorter way to the study of geometry than the *Elements*, to which he replied that there was no royal road to geometry. He is therefore younger than Plato's circle, but older than Eratosthenes and Archimedes; for these were contemporaries, as Eratosthenes somewhere says.[9] In his aim he was a Platonist, being in sympathy with this philosophy, whence he made the end of the whole *Elements* the construction of the so-called Platonic figures.

Since Plato died in 347, Ptolemy I ruled from 323 and reigned from 304 to 285, and Archimedes was born in 287, the chronology of this passage is selfconsistent but allows a wide margin according to whether Euclid flourished in Ptolemy's rule or reign. It is clear, however, that Proclus, writing over six centuries later, had no independent knowledge, obviously relying upon Archimedes for his lower date. The story about the royal road is similar to a tale that Stobaeus tells about Menaechmus and Alexander.[10] Euclid may very well have been a Platonist, for mathematics received an immense impetus from Plato's encouragement; what Proclus says about his relationship to Plato's associates, Eudoxus and Theaetetus, is borne out by his own works; and if he were a Platonist, he would have derived pleasure from making the *Elements* end with the construction of the five regular solids. The testimony of so zealous a Neoplatonist as Proclus is not, however, necessarily conclusive on this point.

Confirmation of Proclus' upper dale comes from the relationship of Euclid to Aristotle, who died in 322 b. c. Euclid's postulates and axioms or "commonnotins" undoubtedly show the influence of Aristotle's elaborate discussion of these topics.[11] Aristotle, on the other hand, shows no awareness of Euclid, and he gives a proof of the proposition that the angles at the base of an isosceles triangle are equal which is pre-Euclidean and would hardly have been cited if *Elements* I.5 had been at hand.[12]

If exact dates could be assigned to Autolycus of Pitane, greater precision would be possible, for in his *Phanomena* Euclid quotes (but without naming his source) propositions from Autolycus' *On the Moving Sphere*. Autolycus was the teacher of Arcesilaus, who was born about 315 b.c. It would be reasonable to suppose that Autolycus was at the height of his activities about 300 b.c.,; and the date that would best fit the middle point of Euclid's active career is about 295 b.c.; but the uncertainties are so great that no quarrel can be taken with the conventional round date of 300 b.c.[13]

He is therefore a totally different person from Euclid of Megara, the disciple of Plato, who lived about a hundred years earlier.[14] His birthplace is unknown,[15] and the date of his birth can only be guessed. It is highly probable, however, quite apart from what Proclus says about his Platonism, that he attended the Academy, for Athens was the great center of mathematical studies at the time; and there he would have become acquainted with the highly original work of Eudoxus and Theaetetus. He was probably invited to Alexandria when Demetrius of Phalerum, at the direction of Ptolemy Soter, was setting up the great library and museum. This was shortly after 300 b.c., and Demetrius, then an exile from Athens, where he had been the governor, would have known Euclid's reputation. It is possible this had already been established by one or more books, but the only piece of internal or external evidence about the order in which Euclid wrote his works is that the *Optics* preceded the *Phenomena* because it is cited in the preface of the latter. Euclid must be regarded as the founder of the great school of mathematics at Alexandria, which was unrivaled in antiquity. Pappus or an interpolator[16] pays tribute to him as "most fair and well disposed toward all who were able in any measure to advance mathematics, careful in no way to give offense, and although an exact scholar not vaunting himself," as Apollonius was alleged to do; and although the object of the passage is to denigrate Apollonius, there is no reason to reject the assessment of Euclid's character. It was presumably at Alexandria, according to a story by Stobaeus,[17] that someone who had begun to learn geometry with Euclid asked him, after the first theorem, what he got out of such things. Summoning a slave, Euclid said, "Give him three obols, since he must needs make gain out of what he learns." The place of his death is not recorded—although the natural assumption is that it was Alexandria—and the date of his death can only be conjectured. A date about 270 b.c. would accord with the fact that about the middle of the century Apollonius studied with his pupils.

Arabic authors profess to know a great deal more about Euclid's parentage and life, but what they write is either free invention or based on the assumption that the so-called book XIV of the *Elements*, written by Hypsicles, is a genuine work of Euclid.

Geometry: Elements (Στοιχεῖα). . Euclid's fame rests preeminently upon the *Elements*, which he wrote in thirteen books[18] and which has exercised an influence upon the human mind greater than that of any other work except the Bible. For this reason he became known in antiquity as 'Ο Στοιχειωτής, '"the Writer of the Elements," and sometimes simply as 'Ο Γεωμέτρης, "the Geometer." Proclus explains that the "elements" are leading theorems having to those which follow the character of an all-pervading principle; he likens them to the letters of the alphabet in relation to language, and in Greek they have the same name.[19] There had been *Elements* written before Euclid—notably by Hippocrates, Leo, and Theudius of Magnesia—but Euclid's work superseded them so completely that they are now known only from Eudemus' references as preserved by Proclus. Euclid's *Elements* was the subject of commentaries in antiquity by Hero, Pappus, Porphyry, Proclus, and Simplicius; and Geminus had many observations about it in a work now lost. In the fourth century Theon of Alexandria reedited it, altering the language in some places with a view to greater clarity, interpolating intermediate steps, and supplying alternative proofs, separate cases, and corollaries. All the manuscripts of the *Elements* known until the nineteenth century were derived from Theon's recension. Then Peyrard discovered in the Vatican a manuscript, known as *P*, which obviously gives an earlier text and is the basis of Heiberg's definitive edition.

Each book of the *Elements* is divided into propositions, which may be theorems, in which it is sought to prove something, or problems, in which it is sought to do something. A proposition which is complete in all its parts has a general enunciation (πρότασıs); a setting-out or particular enunciation ('έκθεσıs), in which the general enunciation is related to a figure designated by the letters of the alphabet; a definition (διορισμός),[20] which is either a closer statement of the object sought, with the purpose of riveting attention, or a statement of the conditions of possibility; a construction (κατασκενή), including any necessary additions to the original figure; a proof or demonstration (ἀπόδεiξıs); and a conclusion (σνμέρασμα), which reverts to the language of the general enunciation and states that it has been accomplished. In many cases some of these divisions may be missing (particularly the definition or the construction) because they are not needed, but the general enunciation, proof, and conclusion are always found. The conclusion is rounded off by the formulas ὅπερ 'έδει δεῖξαι ("which was to be proved") for a theorem and ὅπερ 'έδει ποιῆσαι ("which was to be done") for a problem, which every schoolboy knows in their abbreviated Latin forms as Q.E.D. and Q.E.F. These formal divisions of a proposition in such detail are special to Euclid, for Autolycus before him—the only pre-Euclidean author to have any work survive entire—had normally given only a general enunciation and proof, although occasionally a conclusion is found; and Archimedes after him frequently omitted the general or particular enunciation.

The Greek mathematicians carefully distinguished between the analytic and the synthetic methods of proving a proposition.[21] Euclid was not unskilled in analysis, and according to Pappus he was one of the three writers—the others being Apollonius and Aristaeus the Elder—who created the special body of doctrine enshrined in the *Treasury of Analysis*. This collection of treatises included three by Euclid: his Data, *Porisms, and Surface Loci*. But in the Elements the demonstrations proceed entirely by synthesis, that is, from the known to the unknown, and nowhere is appeal made to analysis, that is, the assumption of the thing to be proved (or done) and the deduction of the consequences until we reach something already accepted or proved true. (Euclid does, however, make frequent use of *reductio ad absurdum or demonstratio per impossibile*, showing that if the conclusion is not accepted, absurd or impossible results follow; and this may be regarded as a form of analysis. There are also many pairs of converse propositions, and either one in a pair could be regarded as a piece of analysis for the solution of the other.) No hint is given by Euclid about the way in which he first realized the truth of the propositions that he proves. Majestically he proceeds by rigorous logical steps from one proved proposition to another, using them like steppingstones, until the final goal is reached.

Each book (or, in the case of XI-XIII, group of books) of the *Elements* is preceded by definitions of the subjects treated, and to book I there are also prefixed five postulates (αιτήατα) and five common notions (κοιναι 'έννοιαι) or axioms which are the foundation of the entire work. Aristotle had taught that to define an object is not to assert its existence; this must be either proved or assumed.[22] In conformity with this doctrine Euclid defines a point, a straight line, and a circle, then postulates that it is possible

1. To draw a straight line from any point to any point
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and radius.

In other words, he assumes the existence of points, straight lines, and circles as the basic elements of his geometry, and with these assumptions he is able to prove the existence of every other figure that he defines. For example, the existence of a square, defined in **I** , definition 22, is proved in 1.46.

These three postulates do rather more, however, than assume the existence of the things defined. The first postulate implies that between any two points only one straight line can be drawn; and this is equivalent to saying that if two straight lines have the same extremities, they coincide throughout their length, or that two straight lines cannot enclose a space. (The latter statement is interpolated in some of the manuscripts.) The second postulate implies that a straight line can be produced in only one direction at either end, that is, the produced part in either direction is unique, and two straight lines cannot have a common segment. It follows also, since the straight line can be produced indefinitely, or an indefinite number of times, that the space of Euclid's geometry is infinite in all directions. The third postulate also implies the infinitude of space because no limit is placed upon the radius; it further implies that space is continuous, not discrete, because the radius may be indefinitely small.

The fourth and fifth postulates are of a different order because they do not state that something can be done. In the fourth the following is postulated:

4. All right angles are equal to one another.

This implies that a right angle is a determinate magnitude, so that it serves as a norm by which other angles can be measured, but it is also equivalent to an assumption of the homogeneity of space. For if the assertion could be proved, it could be proved only by moving one right angle to another so as to make them coincide, which is an assumption of the invariability of figures or the homogeneity of space. Euclid prefers to assume that all right angles are equal.

The fifth postulate concerns parallel straight lines. These are defined in I, definition 23, as "straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction". The essential characteristic of parallel lines for Euclid is, therefore, that they do not meet. Other Greek writers toyed with the idea, as many moderns have done, that parallel straight lines are equidistant from each other throughout their lengths or have the same direction, [23] and Euclid shows his genius in opting for nonsecancy as the test of parallelism. The fifth postulate runs:

5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, will meet on that side on which are the angles less than two right angles.

In Figure 1 the postulate asserts that if a straight line (*PQ*) cuts two other straight lines (*AB, CD*) in *P, Q* so that the sum of the angles *BPQ, DQP* is less than two right angles, *AB, CD* will meet on the same side of *PQ* as those two angles, that is, they will meet if produced beyond *B* and *D*.

There was a strong feeling in antiquity that this postulate should be capable of proof, and attempts to prove it were made by Ptolemy and Proclus, among others. [24] Many more attempts have been made in modern times. All depend for their apparent success on making, consciously or unconsciously, an assumption which is equivalent to Euclid's postulate. It was Saccheri in his book *Euclides ab omni naevo vindicatus* (1733) who first asked himself what would be the consequences of hypotheses other than that of Euclid, and in so doing he stumbled upon the possibility of non-Euclidean geometries. Being convinced, as all mathematicians and philosophers were until the

nineteenth century, that there could be no geometry besides that delineated by Euclid, he did not realize what he had done; and although Gauss had the first understanding of modem ideas, it was left to Lobachevski (1826, 1829) and Bolyai (1832), on the one hand, and Riemann (1854), on the other, to develop non-Euclidean geometries. Euclid's fifth postulate has thus been revealed for what it really is—an unprovable assumption defining the character of one type of space.

The five common notions are axioms, which, unlike the postulates, are not confined to geometry but are common to all the demonstrative sciences. The first is "Things which are equal to the same thing are also equal to one another," and the others are similar.

The subject matter of the first six books of the *Elements* is plane geometry. Book I deals with the geometry of points, lines, triangles, squares, and parallelograms. Proposition 5, that in isosceles triangles the angles at the base are equal to one another and that, if the equal straight lines are produced, the angles under the base will be equal to one another, is interesting historically as having been known (except in France) as the *pons asinorum*; this is usually taken to mean that those who are not going to be good at geometry fail to get past it, although others have seen in the figure of the proposition a resemblance to a trestle bridge with a ramp at each end which a donkey can cross but a horse cannot.

Proposition 44 requires the student "to a given straight to apply in a given rectilineal angle a parallelogram equal to a given triangle," that is, on a given straight line to construct a parallelogram equal to a given area and having one of its angles equal to a

given angle. In Figure 3, AB is the given straight line and the parallelogram *BEFG* is constructed equal to the triangle *C* so that $\angle GBE = \angle D$. The figure is completed, and it is proved that the parallelogram *ABML* satisfies the requirements. This is Euclid's first example of the application of areas [25] one of the most powerful tools of the Greek mathematicians. It is a geometrical equivalent of certain algebraic operations. In this simple case, if $AL = x$, then $x \cdot AB \cos D = C$, and the theorem is equivalent to the solution of a first-degree equation. The method is developed later, as it will be shown, so as to be equivalent to the solution of second-degree equations.

Book I leads up to the celebrated proposition 47,

FIGURE 4. Book I, Proposition 47, "Phthagoras' Theorem," $BC^2 = CA^2+AB^2$

"Pythagoras' theorem," which asserts: "In right angled triangles the square on the side subtending the right angle is equal to the [sum of the] squares on the sides containing the right angle." In Figure 4 it is shown solely by the use of preceding propositions that the parallelogram *BL* is equal to the square *BG* and the parallelogram CL is equal to the square *AK*, so that the whole square *BE* is equal to the sum of the squares *BG, AK*. It is important to notice, for a reason to be given later, that no appeal is made to similarity of figures. This fundamental proposition gives Euclidean space a metric, which would be expressed in modern notation as $ds^2=dx^2 + dy^2$. It is impossible not to admire the ingenuity with which the result is obtained, and not surprising that when Thomas Hobbes first read it he exclaimed, "By God, this is impossible."

Book II develops the transformation of areas adumbrated in I.44, 45 and is a further exercise in geometrical algebra. Propositions 5, 6, 11, and 14 are the equivalents of solving the quadratic equations $ax - x^2 =b^2$, $ax + x^2 = b^2$, $x^2 + ax = a^2$, $x^2 = ab$. Propositions 9 and 10 are equivalent to finding successive pairs of integers satisfying the equations $2x^2 - y^2 = \pm 1$. Such pairs were called by the Greeks side numbers and diameter numbers. Propositions 12 and 13 are equivalent to a proof that in any triangle with sides a,b,c, and angle A opposite a,

$$a^2 =b^2 + c^2 - 2\, bc \cos A.$$

It is probably not without significance that this penultimate proposition of book II is a generalization of "Pythagoras' theorem," which was the penultimate proposition of book I.

Book III treats circles, including their intersections and touchings. Book IV consists entirely of problems about circles, particularly the inscribing or circumscribing of rectilineal figures. It ends with proposition 16: "In a given circle to inscribe a fifteen-angled figure which shall be both equilateral and equiangular." Proclus asserts that this is one of the propositions that Euclid solved with a view to their use in astronomy, "for when we have inscribed the fifteen-angled figure in the circle through the poles, we have the distance from the poles both of the equator and the zodiac, since they are distant from one another by the side of the fifteen-angled figure"—that is to say, the obliquity of the ecliptic was taken to be 24°, as is known independently to have been the case up to Eratosthenes.[26]

Book V develops the general theory of proportion. The theory of proportion as discovered by the Pythagoreans applied only to commensurable magnitudes because it depends upon the taking of aliquot parts, and this is all that was needed by Euclid for the earlier books of the *Elements*. There are instances, notably 1.47, where he clearly avoids a proof that would depend on similitude or the finding of a proportional, because at that stage of his work it would not have applied to incommensurable magnitudes. In book V he addresses himself at length to the general theory. There is no book in the *Elements* that has so won the admiration of mathematicians. Barrow observes: "There is nothing in the whole body of the *Elements* of a more subtile invention, nothing more solidly established and more accurately handled, than the doctrine of proportionals." In like spirit Cayley says, "There is hardly anything in mathematics more beautiful than this wondrous fifth book."[27]

The heart of the book is contained in the definitions with which it opens. The definition of a ratio as "a sort of relation in respect of size between two magnitudes of the same kind" shows that a ratio, like the elephant, is easy to recognize but hard to define. Definition 4 is more to the point: "Magnitudes are said to have a ratio one to the other if capable, when multiplied, of exceeding one another." The definition excludes the infinitely great and the infinitely small and is virtually equivalent to what is now known as the axiom of Archimedes. (See below, section on book X.) But it is definition 5 which has chiefly excited the admiration of subsequent mathematicians: "Magnitudes are said to be in the same ratio, the first to the second and the second to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order." It will be noted that the definition avoids mention of parts of a magnitude and is therefore applicable to the incommensurable as well as to the commensurable. De Morgan put its meaning very clearly: "Four magnitudes, *A* and *B* of one kind, and *C* and *D* of the same or another kind, are proportional when all the multiples of *A* can be distributed among the multiples of **B** in the same intervals as the corresponding multiples of *C* among those of *D*"; or, in notation, if m, n are two integers, and if mA lies between nB and $(n + 1)B$, $m C$ lies between nD and $(n + 1)D$.[28] It can be shown that the test proposed by Euclid is both a necessary and a sufficient test of proportionality, and in the whole history of mathematics no equally satisfactory test has ever been proposed. The best testimony to its adequacy is that Weierstrass used it in his definition of equal numbers;[29] and Heath has shown how Euclid's definition divides all rational numbers into two coextensive classes and thus defines equal ratios in a manner exactly corresponding to a Dedekind section.[30]

The remaining definitions state the various kinds of transformations of ratios—generally known by their Latin names: *alternando, invertendo, componendo, separando, convertendo, ex aequali*, and *ex aequali in proportione perturbata*—and with remorseless logic the twenty-five propositions apply these various operation to the objects of Euclid's definitions.

It is a sign of the abiding fascination of book V for mathematicians that in 1967 Friedhelm Beckmann applied his own system of axioms, set up in close accordance with Euclid, in such a way as to deduce all definitions and propositions of Euclid's theory of magnitudes, especially those of books V and VI. In his view magnitudes, rather than their relation of "having a ratio," form the base of the theory of proportions. These magnitudes represent a well-defined structure, a so-called Eudoxic semigroup, with the numbers as operators. Proportion is interpreted as a mapping of totally ordered semigroups. This mapping proves to be an isomorphism, thus suggesting the application of the modern theory of homomorphism.

Book VI uses the general theory of proportion established in the previous book to treat similar figures. The first and last propositions of the book illustrate the importance of V, definition 5, for by the method of equimultiples it is proved in proposition 1 that triangles and parallelograms having the same height are to one another as their bases, and in proposition 33 it is proved that in equal circles the angles at the center or circumference are as the arcs on which they stand. There are many like propositions of equal importance. Proposition 25 sets the problem "To construct a rectilineal figure similar to one and equal to another, given rectilineal figure." [31] In propositions 27–29 Euclid takes up again the application of areas. It has been explained above that to apply (παραβάλλειν) a parallelogram to a given straight line means to construct on that line a parallelogram equal to a given area and having a given angle. If the straight line is applied to only part of the given line, the resulting figure is said to be deficient (ἐλλείπειν); if to the straight line produced, it is said to exceed (ὑπερβάλλειν). Proposition 28 is the following problem: "To a given straight line to apply a parallelogram equal to a given rectilineal figure and deficient by a parallelogrammatic figure similar to a given one." (It has already been shown in proposition 27 that the given rectilineal figure must not be greater than the parallelogram described on half the straight line and similar to the defect.) In Figure 5 let the parallelogram *TR* be applied to the straight line *AB* so as to be equal to a given rectilineal figure having the area *S* and deficient by the parallelogram *PB*, which is similar to the given parallelogram *D*. Let $AB = a$, $RP = x$; let angle of *D* be α and the ratio of its sides $b{:}c$. Let *E* be the midpoint of *AB*

and let *EH* be drawn parallel to the sides. Then (the parallelogram *TR*) = (the parallelogram *TB*)

–(the parallelogram *PB*)

If the area of the given rectilineal figure is *S*, this may be written

Constructing the parallelogram *TR* is therefore equivalent to solving geometrically the equation

It can easily be shown that Euclid's solution is equivalent to completing the square on the left-hand side. For a real solution it is necessary that

i.e., $S \geq HE \sin \alpha \cdot EB$

i.e., $S \geq$ parallelogram *HB*,

which is exactly what was proved in VI.27. Proposition 29 sets the corresponding problem for the excess: "To a given straight line to apply a parallelogram equal to a given rectilineal figure and exceeding by a parallelogrammatic figure similar to a given one." This can be shown in the same way to be equivalent to solving geometrically the equation

In this case there is always a real solution. No διορισμός or examination of the conditions of possibility is needed, and Euclid's solution corresponds to the root with the positive sign.

This group of propositions is needed by Euclid for his treatment of irrationals in book X, but their chief importance in the history of mathematics is that they are the basis of the theory of conic sections as developed by Apollonius. Indeed, the very words "parabola," "ellipse," and "hyperbola" come from the Greek words for "to apply," "to be deicide," and "to exceed," It is significant that the antepenultimate proposition of the book, proposition 31, is a generalization of "Pythagoras' theorem" : "In right-angled triangles any [literally "the"] figure [described] on the side subtending the right angle is equal to the [sum of the] similar and similarly described figures on the sides containing the right angle."

Books VII, VIII, and IX are arithmetical; and although the transition from book VI appears sharp, there is a logical structure in that the theory of proportion, developed in all its generality in book V, is applied in book VI to geometrical figures and in book VII to numbers. The theory of numbers is continued in the next two books. The theory of proportion in book VII is not, however, the general theory of book V but the old Pythagorean theory applicable only to commensurable magnitudes.[32] This return to an outmoded theory led both De Morgan and W. W. Rouse Ball to suppose that Euclid died before putting the finishing touches to the *Elements*,[33] but, although the three arithmetical books seem trite in comparison with those that precede and follow, there is nothing unfinished about them. It is more likely that Euclid, displaying the deference toward others that Pappus observed, thought that he ought to include the traditional teaching. This respect for traditional doctrines can be seen in some of the definitions which Euclid repeats even though he improves upon them or never uses them.[34] Although books VII-IX appear at first sight to be a reversion to Pythagoreanism, it is Pythagoreanism with a difference. In particular, the rational straight line takes the place of the Pythagorean monad;[35] but the products of numbers are also treated as straight lines, not as squares or rectangles.

After the numerical theory of proportion is established in VII. 4–19, there is an interesting group of propositions on prime numbers (22–32) and a final group (33–39) on least common multiples. Book VIII deals in the main with series of numbers "in continued proportion," that is, in geometrical progression, and with geometric means. Book IX is a miscellany and includes the fundamental theorem in the theory of numbers, proposition 14: "If a number be the least that is measured by prime numbers, it will not be measured by any other prime number except those originally measuring it," that is to say, a number can be resolved into prime factors in only one way.

After the muted notes of the arithmetical books Euclid again takes up his lofty theme in book X, which treats irrational magnitudes. It opens with the following proposition (X.1): "If two unequal magnitudes be set out, and if there be subtracted from the greater a magnitude greater than its half, and from that which is left a magnitude greater than its half, and so on continually, there will be left some magnitude less than the lesser magnitude set out." This is the basis of the "method of exhaustion," as later used by Euclid in book XII. Because of the use made of it by Archimedes, either directly or in an equivalent form, for the purpose of calculating areas and volumes, it has become known, perhaps a little unreasonably, as the axiom of Archimedes. Euclid needs the axiom at this point as a test of incommensurability, and his next proposition (X.2) asserts: "If the lesser of two unequal magnitudes is continually subtracted from the greater, and the remainder never measures that which precedes it, the magnitudes will be incommensurable."

The main achievement of the book is a classification of irrational straight lines, no doubt for the purpose of easy reference. Starting from any assigned straight line which it is agreed to regard as rational—a kind of datum line—Euclid asserts that any straight line which is commensurable with it in length is rational, but he also regards as rational a straight line commensurable with it only in square. That is to say, if *m, n* are two integers in their lowest terms with respect to each other, and *I* is a rational straight line, he regards .l as rational because $(m/n)\ l^2$ is commensurable with l^2. All straight lines not commensurable either in length or in square with the assigned straight line he calls irrational. His fundamental proposition (X.9) is that the sides of squares are commensurable or incommensurable in length according to whether the squares have or do not have the ratio of a square number to each other, that is to say, if *a, b* are straight lines and *m, n* are two numbers, and if $a:b = m:n$, then $a^2:b^2 = m^2: n^2$ and conversely. This is easily seen in modern notation, but was far from an easy step for Euclid. The first irrational line which he isolates is the side of a square equal in area to a rectangle whose sides are commensurable in square only. He calls it a medial. If the sides of the rectangle are *l*, the medial is $k^{1/4}l$, Euclid next proceeds to define six pairs of compound irrationals (the members of each pair differing in sign only) which can be represented in modern notation as the positive roots of six biquadratic equations (reducible to quadratics) of the form

$$x^4 \pm 2alx^2 \pm bl^4 = 0.$$

The first pair are given the names "binomial"(or "biterminal") and "apotome," and Euclid proceeds to define six pairs of their derivatives which are equivalent to the roots of six quadratic equations of the form

$$x^2 + 2alx + bl^2 = 0.$$

In all, Euclid investigates in the 115 propositions of the book (of which the last four may be interpolations) every possible form of the lines which can be represented by the expression , some twenty-five in all.[36]

The final three books of the *Elements,* XI-XIII, are devoted to solid geometry. Book XI deals largely with parallelepipeds. Book XII applies the method of exhaustion, that is, the inscription of successive figures in the body to be evaluated, in order to prove that circles are to one another as the squares on their diameters, that pyramids of the same height with triangular bases are in the ratio of their bases, that the volume of a cone is one-third of the cylinder which has the same base and equal height, that cones and cylinders having the same height, are in the ratio of their bases, that similar cones and cylinders are to one another in the triplicate ratio of the diameters of their bases, and that spheres are in the triplicate ratio of their diameters. The method can be shown for the circle. Euclid inscribes a square in the circle and shows that it is more than half the circle. He bisects each arc and shows that each triangle so obtained is greater than half the segment of the circle about it. (In Figure 6, for example, triangle *EAB* is greater than half the segment of the circle *EAB* standing on *AB*.) If the process is continued indefinitely, according to X.I, we shall be left with segments of the circle smaller than some assigned magnitude, that is, the circle has been exhausted. (A little later Archimedes was to refine the method by also circum

scribing a polygon, and so compressing the figure, as it were, between inscribed and circumscribed polygons.) Euclid refrains from saying that as the process is continued indefinitely, the area of the polygon will in the limit approach the area of the circle, and rigorously proves that if his proposition is not granted, impossible conclusions would follow.

After some preliminary propositions book XIII is devoted to the construction in a sphere of the five regular solids: the pyramid (proposition 13), the octahedron (14), the cube (15), the icosahedron (16), and the dodecahedron (17). These five regular solids had been a prime subject of investigation by the Greek mathematicians, and because of the use made of them by Plato in the *Timaeus* were known as the Platonic figures.[37] The mathematical problem is to determine the edge of the figure in relation to the radius of the circumscribing sphere. In the case of the pyramid, octahedron, and cube, Euclid actually evaluates the edge in terms of the radius, and in the case of the icosahedron and dodecahedron he shows that it is one of the irrational lines classified in book X—a minor in the case of the icosahedron and an apotome in the case of the dodecahedron. In a final splendid flourish (proposition 18), Euclid sets out the sides of the five figures in the same sphere and compares them with each other, and in an addendum he shows that there can be no other regular solids. In Figure 7, *AC* = *CB*, *AD* = 2*DB*, *AG* = *AB*, *CL* = *CM*, and *BF* is divided in extreme and mean ratio at *N(BF:BN = BN:NF)*. He proves that *AF* is the side of the pyramid, *BF* is the

side of the cube, *BE* is the side of the octahedron, *BK* is the side of the icosahedron, and *BN* is the side of the dodecahedroon; their values, in terms of the radius *r*, are respectively , , , , .

Proclus, as already noted, regarded the construction of the five Platonic figures as the end of the *Elements*, in both senses of that ambiguous word. This is usually discounted on the ground that the stereometrical books had to come last, but Euclid need not have ended with the construction of the five regular solids; and since he shows the influence of Plato in other ways, this splendid ending could easily be a grain of incense at the Platonic altar.

Proclus sums up Euclid's achievement in the *Elements* in the following words:[38]

He deserves admiration preeminently in the compilation of his *Elements of Geometry* on account of the order and selection both of the theorems and of the problems made with a view to the elements. For he included not everything which he could have said, but only such things as were suitable for the building up of the elements. He used all the various forms of deductive arguments,[39] some getting their plausibility from the first principles, some starting from demonstrations, but all irrefutable and accurate and in harmony with science. In addition he used all the dialectical methods, the *divisional* in the discovery of figures, the *definitive* in the existential arguments, the *demonstrative* in the passages from first principles to the things sought, and the *analytic* in the converse process from the things sought to the first principles. And the various species of conversions,[40] both of the simpler (propositions) and of the more complex, are in this treatise accurately set forth and skillfully investigated, what wholes can be converted with wholes, what wholes with parts and conversely, and what as parts with parts. Further, we must make mention of the continuity of the proofs, the disposition and arrangement of the things which precede and those which follow, and the power with which he treats each detail.

This is a fair assessment. The *Elements* is on the whole a compilation of things already known, and its most remarkable feature is the arrangement of the matter so that one proposition follows on another in a strictly logical order, with the minimum of assumption and very little that is superfluous.

If we seek to know how much of it is Euclid's own work, Proclus is again our best guide. He says, as we have seen, that Euclid "put together the elements, arranging in order many of Eudoxus' theorems, perfecting many of Theaetetus,' and also bringing to irrefutable demonstration the things which had been only loosely proved by his predecessors."[41] According to a scholiast of book V, "Some say that this book is the discovery of Eudoxus, the disciple of Plato."[42] Another scholiast confirms this, saying, "This book is said to be the work of Eudoxus of Cnidus, the mathematician, who lived about the times of Plato."[43] He adds, however, that the ascription to Euclid is not false, for although there is nothing to prevent the discovery from being the work of another man, "The arrangement of the book with a view to the elements and the orderly sequence of theorems is recognized by all as the work of Euclid." This is a fair division of the credit. Eudoxus also, as we can infer from Archimedes,[44] is responsible for the method of exhaustion used in book XII to evaluate areas and volumes, based upon X.*l*, and Archimedes attributes to Eudoxus by name the theorems about the volume of the pyramid and the volume of the cone which stand as propositions 7 and 10 of book XII of the *Elements*. Although Greek tradition credited Hippocrates with discovering that circles are to one another as the squares on their diameters,[45] we can be confident that the proof as we have it in XII.2 is also due to Eudoxus.

The interest of Theaetetus in the irrational is known from Plato's dialogue,[46] and a commentary on book X which has survived in Arabic[47] and is attributed by Heiberg[48] to Pappus credits him with discovering the different species of irrational lines known as the medial, binomial, and apotome. A scholium to X.[49] (that squares which do not have the ratio of a square number to a square number have their sides incommensurable) attributes this theorem to Theaetetus. It would appear in this case also that the fundamental discoveries were made before Euclid but that the orderly arrangement of propositions is his work. This, indeed, is asserted in the commentary attributed to Pappus, which says:

As for Euclid he set himself to give rigorous rules, which he established, relative to commensurability and incommensurability in general; he made precise the definitions and the distinctions between rational and irrational magnitudes, he set out a great number of orders of irrational magnitudes, and finally he clearly showed their whole extent.[50]

Theaetetus was also the first to "construct" or "write upon" the five regular solids,[51] and according to a scholiast[52] the propositions concerning the octahedron and the icosahedron are due to him. His work therefore underlies book XIII, although the credit for the arrangement must again be given to Euclid.

According to Proclus,[53] the application of areas, which, as we have seen, is employed in 1.44 and 45, 11.5, 6, and 11, and VI.27, 28, and 29, is "ancient, being discoveries of the muse of the Pythagoreans."

A scholiast to book IV[54] attributes all sixteen theorems (problems) of that book to the Pythagoreans. It would appear, however, that the famous proof of what is universally known as "Pythagoras' theorem," 1.47, is due to Euclid himself. It is beyond doubt that this property of right-angled triangles was discovered by Pythagoras, or at least in his school, but the proof was almost certainly based on proportions and therefore not applicable to all magnitudes. Proclus says:

If we give hearing to those who relate things of old, we shall find some of them referring this discovery to Pythagoras and saying that he sacrificed an ox upon the discovery. But I, while marveling at those who first came to know the truth of this theorem, hold in still greater admiration the writer of the *Elements*, not only because he made it secure by a most clear proof, but because he compelled assent by the irrefutable seasonings of science to the still more general proposition in the sixth book. For in that book he proves generally that in right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly situated figures described on the sides about the right angle.[55]

On the surface this suggests that Euclid devised a new proof, and this is borne out by what Proclus says about the generalization. It would be an easy matter to prove VI.31 by using I.47 along with VI.22, but Euclid chooses to prove it independently of 1.47 by using the general theory of proportions. This suggests that he proved 1.47 by means of book I alone, without invoking proportions in order to get it into his first book instead of his sixth. The proof certainly bears the marks of genius.

To Euclid also belongs beyond a shadow of doubt the credit for the parallel postulate which is fundamental to the whole system. Aristotle had censured those "who think they describe parallels" because of a *petitio principii* latent in their theory.[56] There is certainly no *petitio principii* in Euclid's theory of parallels, and we may deduce that it was post- Aristotelian and due to Euclid himself. In nothing did he show his genius more than in deciding to treat postulate 5 as an indemonstrable assumption.

The significance of Euclid's *Elements* in the history of thought is twofold. In the first place, it introduced into mathematical reasoning new standards of rigor which remained throughout the subsequent history of Greek mathematics and, after a period of logical slackness following the revival of mathematics, have been equaled again only in the past two centuries. In the second place, it marked a decisive step in the geometrization of mathematics.[57] The Pythagoreans and Democritus before Euclid, Archimedes in some of his works, and Diophantus afterward showed that

Greek mathematics might have developed in other directions. It was Euclid in his *Elements*, possibly under the influence of that philosopher who inscribed over the doors of the Academy "God is for ever doing geometry," who ensured that the geometrical form of proof should dominate mathematics. This decisive influence of Euclid's geometrical conception of mathematics is reflected in two of the supreme works in the history of thought, Newton's *Principia* and Kant's *Kritik der reinen Vernunft*. Newton's work is cast in the form of geometrical proofs that Euclid had made the rule even though Newton had discovered the calculus, which would have served him better and made him more easily understood by subsequent generations; and kant's belief in the universal validity of Euclidean geometry led him to a transcendental aesthetic which governs all his speculations on knowledge and perception.

It was only toward the end of the nineteenth century that the spell of Euclidean geometry began to weaken and that a desire for the "arithmetization of mathematics" began to manifest itself; and only in the second quarter of the twentieth century, with the development of quantum mechanics, have we seen a return in the physical sciences to a neo-Pythagorean view of number as the secret of all things. Euclid's reign has been a long one; and although he may have been deposed from sole authority, he is still a power in the land.

The Data (Δεδμένα). The *Data*, the only other work by Euclid in pure geometry to have survived in Greek, is closely connected with books I-VI of the *Elements*. It is concerned with the different senses in which things are said to be given. Thus areas, straight lines, angles, and ratios are said to be "given in magnitude" when we can make others equal to them. Rectilineal figures are "given in species" or "given in form" when their angles and the ratio of their sides are given. Points, lines, and angles are "given in position" when they always occupy the same place, and so on. After the definitions there follow ninety-four propositions, in which the object is to prove that if certain elements of a figure are given, other elements are also given in one of the defined senses.

The most interesting propositions are a group of four which are exercises in geometrical algebra and correspond to *Elements* 11.28, 29. Proposition 58 reads: “If a given area be applied to a given straight line so as to be deficient by a figure given in form, the breadths of the deficiency are given.” Proposition 84, which depends upon it, runs: “If two straight lines contain a given area in a given angle, and if one of them is greater than the other by a given quantity, then each of them is given.” This is equivalent to solving the simultaneous equations

$$y - x = \alpha$$

$$xy = b^2,$$

and these in turn are equivalent to finding the two roots of

$$ax + x^2 = b^2.$$

Propositions 59 and 85 give the corresponding theorems for the excess and are equivalent to the simultaneous equations

$$y + x = \alpha$$

$$xy = b^2$$

and the quadratic equation

$$ax - x^2 = b^2.$$

A clue to the purpose of the *Data* is given by its inclusion in what Pappus calls the *Treasury of Analysis*.[58] The concept behind the *Data* is that if certain things are given, other things are necessarily implied, until we are brought to something that is agreed. The *Data* is a collection of hints on analysis. Pappus describes the contents of the book as known to him;[59] the number and order of the propositions differ in some respects from the text which has come down to us.

Marinus of Naples, the pupil and biographer of Proclus, wrote a commentary on, or rather an introduction to, the *Data* It is concerned mainly with the different senses in which the term “given” was understood by Greek geometers.

On Divisions of Figures. (Περί δίαιρεσεων βιβλίον). Proclus preserved this title along with the titles of other works of Euclid,[60] and gives an indication of its contents: “For the circle is divisible into parts unlike by definition, and so is each of the rectilineal figures; and this is indeed what the writer of the *Elements* himself discusses in his *Divisions*, dividing given figures now into like figures, now into unlike.”[61] The book has not survived in Greek, but all the thirty-six enunciations and four of the propositions (19, 20, 28, 29) have been preserved in an Arabic translation discovered by Woepcke and published in 1851; the remaining proofs can be supplied from the *Practica geometriae* written by Leonardo Fibonacci in 1220, one section of which, it is now evident, was based upon a manuscript or translation of Euclid’s work no longer in existence. The work was reconstructed by R. C. Archibald in 1915.

The character of the book can be seen from the first of the four propositions which has survived in Arabic (19). This is “To divide a given triangle into two equal parts by a line which passes through a point situated in the interior of the triangle.” Let *D* be a point inside the triangle *ABC* and let *DE* be drawn parallel to *CB* so as to meet *AB* in *E*. Let *T* be taken on *BA* produced so that $TB \cdot DE = 1/2\ AB \cdot BC$ (that is, let *TB* be such that when a rectangle having *TB* for one of its sides is applied to *DE*, it is equal to half the rectangle $AB \cdot BC$). Next, let a parallelogram be applied to the line *TB* equal to the rectangle $TB \cdot BE$ and deficient by a square, that is, let *H* be taken on *TB* so that

$$(TB - HT)\cdot HT = TB \cdot BE.$$

HD is drawn and meets *BC* in *Z*. It can easily be shown that *HZ* divides the triangle into two equal parts and is the line required.

The figures which are divided in Euclid’s tract are the triangle, the parallelogram, the trapezium, the quadrilateral, a figure bounded by an arc of a circle and two lines, and a circle. It is proposed in the various cases to divide the given figure into two equal parts, into several equal parts, into two parts in a given ratio, or into several parts in a given ratio. The propositions may be further classified according to whether the dividing line (transversal) is required to be drawn from a vertex, from a point within or without the figure, and so on.[62]

In only one proposition (29) is a circle divided, and it is clearly the one to which Proclus refers. The enunciation is “To draw in a given circle two parallel lines cutting off a certain fraction from the circle.” In fact, Euclid gives the construction for a fraction of one-third and notes a similar construction for a quarter, one-fifth, “or any other definite fraction.”[63]

Porisms (Πορίσματα). It is known both from Pappus[64] and from Proclus[65] that Euclid wrote a three book work called *Porisms*. Pappus, who includes the work in the *Treasury of Analysis*, adds the information that it contained 171 theorems and thirty-eight lemmas. It has not survived—most unfortunately, for it appears to have been an exercise in advanced mathematics;[66] but the account given by Pappus encouraged such great mathematicians as Robert Simson and Michel Chasles to attempt reconstructions, and Chasles was led thereby to the discovery of anharmonic ratios.

The term "porism" commonly means in Greek mathematics a corollary, but that is not the sense in which it is used in Euclid's title. It is clearly derived from πορίζω, "I procure," and Pappus explains that according to the older writers a porism is something intermediate between a theorem and a problem: "A theorem is something proposed with a view to the proof of what is proposed, and a problem is something thrown out with a view to the construction of what is proposed, [and] a porism is something proposed with a view to the finding [πορισμόν] of the very thing proposed."[67] Proclus reinforces the explanation. The term, he says, is used both for "such theorems as are established in the proofs of other theorems, being windfalls and bonuses of the things sought, and also for such things as are sought, but need discovery, and are neither pure bringing into being nor pure investigation."[68] As examples of a porism in this sense, Proclus gives two: first, the finding of the center of a circle, and, second, the finding of the greatest common measure of two given commensurable magnitudes.

Pappus says that it had become characteristic of porisms for the enunciation to be put in shortened form and for a number of propositions to be comprehended in one enunciation. He sets out twenty-nine different types in Euclid's work (fifteen in book I, six in book II, and eight in book III). His versions suggest that the normal form of Euclid's porisms was to find a point or a line satisfying certain conditions. Pappus says that Euclid did not normally give many examples of each case, but at the beginning of the first book he gave ten propositions belonging to one class; and Pappus found that these could be comprehended in one enunciation, in this manner:

If in a quadrilateral, whether convex or concave, the sides cut each other two by two, and the three points in which the other three sides intersect the fourth are given, and if the remaining points of intersection save one lie on straight lines given in position, the remaining point will also lie on a straight line given in position.[69]

Pappus proceeds to generalize this theorem for any system of straight lines cutting each other two by two. In modern notation, let there be n straight lines, of which not more than two pass through one point and no two are parallel. They will intersect in $1/2n(n–1)$ points. Let the $(n–1)$ points in which one of the lines is intersected be fixed. This will leave $1/2n(n–1)-(n–1)= 1/2(n–1)(n–2)$ other points of intersection. If $(n–2)$ of these points lie on straight lines given in position, the other $1/2(n–1)(n–2)–(n–2) = 1/2(n–2)(n–3)$ points of intersection will also lie on straight lines given in position, provided that it is impossible to form with these points of intersection any triangle having for sides the sides of the polygon.[70]

Pappus adds: "It is unlikely that the writer of the *Elements* was unaware of this result, but he would have desired only to set out the first principle. For he appears in all the porisms to have laid down only the first principles and seminal ideas of the many important matters investigated."[71]

Pappus' remarks about the definition of porisms by the "older writers" have been given above. He—or an interpolator—censures more recent writers who defined a porism by an incidental characteristic: "a porism is that which falls short of a locus theorem in respect of its hypotheses."[72] What this means is far from clear, but it led Zeuthen[73] to conjecture that Euclid's porisms were a by-product of his researches into conic sections—which, if true, would be a happy combination of the two meanings of porism. Zeuthen takes the first proposition of Euclid's first book as quoted by Pappus: "If from two points given in position straight lines be drawn so as to meet on a straight line given in position, and if one of them cuts off from a straight line given in position a segment measured toward a given point on it, the other will also cut off from another straight line a segment having to the first a given ratio."[74] He notes that this proposition is true if a conic section, regarded as a "locus with respect to four lines" (see below), is substituted for the first given straight line, with the two given points as points on it.[75] It will be convenient to turn immediately to Euclid's investigations into conic sections and the "three- and four-line locus," noting that, from one point of view, his *Porisms* would appear to have been the earliest known treatise on projective geometry and transversals.

Conics . We know from Pappus that Euclid wrote a four-book work on conic sections, but it has not survived even in quotation. The relevant passage in the *Collection* reads: "Apollonius, having completed Euclid's four books of conics and added four others, handed down eight volumes of conics."[76] The work was probably lost by Pappus' time, for in the next sentence he mentions as still extant the five books of Aristaeus on "solid loci." Aristaeus preceded Euclid, for Euclid, according to Pappus or an interpolator, thought that Aristaeus deserved the credit for the discoveries in conics he had already made, and neither claimed originality nor wished to overthrow what he had already done. (It is at this point that Pappus contrasts Euclid's character with that of Apollonius, noted above.) In particular, Euclid wrote as much about the three- and four-line locus as was possible on the basis of Aristaeus' conics without claiming completeness for his proofs.[77]

Euclid doubtless shared the early Greek view that conic sections were generated by the section of a cone by a plane at right angles to a generator, and he would have used the names "section of a right-angled cone," "section of an acute-angled cone," and "section of an obtuse-angled cone," which were in use until Apollonius established the terms "parabola," "ellipse," and "hyperbola"; but he was aware that an ellipse can be obtained by any section of a cone or cylinder not parallel to the base, for in his *Phaenomena* he says: "If a cone or cylinder be cut by a plane not parallel to the base, the section is a section of an acute-angled cone which is like a shield [θυρεός]."[78]

Furthermore, Euclid was aware of the focusdirectrix property (that a conic section is the locus of a point whose distance from a fixed point bears a constant relation to its distance from a fixed straight line), even though it is nowhere mentioned by Apollonius: Pappus cites the property as a lemma to Euclid's *Surface Loci*,[79] from which it is clear that it was assumed in that book without proof. It is likely, therefore, that it was proved either in Euclid's *Conics* or by Aristaeus.

Euclid was also aware that a conic may be regarded as the locus of a point having a certain relationship to three or four straight lines. He discussed this locus in his *Conics*, and he may be the original author to whom Pappus thinks Apollonius should have deferred.[80] The locus is thus defined by Pappus:

If three straight lines be given in position, and from one and the same point straight lines be drawn to meet the three straight lines at given angles, and if the ratio of the rectangle contained by two of the straight lines toward the square on the remaining straight line be given, then the point will lie on a solid locus given in position, that is, on one of the three conic sections. And if straight lines be drawn to meet at given angles four straight lines given in position, and the ratio of the rectangle contained by two of the straight lines so drawn toward the rectangle contained by the remaining two be given, then likewise the point will lie on a conic section given in position.[81]

The three-line locus is clearly a special case of the four-line locus in which two of the straight lines coincide. The general case is the locus of a point whose distances x, y, z, u from four straight lines (which may be regarded as the sides of a quadrilateral) have the relationship $xy: zu = k$, where k is a constant.

From the property that the ratio of the rectangles under the segments of any intersecting chords drawn in fixed directions in a conic is constant (being equal to the ratio of the squares on the parallel semidiameters), it is not difficult to show that the distances of a point on a conic from an inscribed trapezium bear the above relationship $xy: zu = k$, and it is only a further step to prove that this is true of any inscribed quadrilateral. It is rather more difficult to prove the converse theorem—that the locus of a point having this relationship to the sides, first of a trapezium, then of any quadrilateral, is a conic section—but it would have been within Euclid's capacity to do so.

Apollonius says of his own *Conics*:

The third book includes many remarkable theorems useful for the synthesis of solid loci and for determining limits of possibility. Most of these theorems, and the most elegant, are new, and it was their discovery which made me realize that Euclid had not worked out the synthesis of the locus with respect to three and four lines, but only a chance portion of it, and that not successfully; for the synthesis could not be completed without the theorems discovered by me.[82]

In the light of this passage Zeuthen conjectured that Euclid (and the other predecessors of Apollonius) saw that a point on a conic section would have the four-line property with respect first to an inscribed trapezium and then to any inscribed quadrilateral, but failed to prove the converse, even for a trapezium; they failed because they did not realize that the hyperbola is a curve with two branches.[83] It is an attractive suggestion.

Pappus exonerates Euclid from blame on the ground that "he wrote so much about the locus as was possible by means of the *Conics* of Aristaeus but did not claim finality for his proofs" and that "neither Apollonius himself nor anyone else could have added anything to what Euclid wrote, using only those properties of conics which had been proved up to Euclid's time."[84] Since Apollonius implies that he had worked out a complete theory, it is curious that he does not set it out in his treatise; but book III, propositions 53–56 of his *Conics*, when taken together, give what is in effect the converse of the three-line locus: "If from any point of a conic there be drawn three straight lines in fixed directions to meet respectively two fixed tangents to the conic and their chord of contact, the ratio of the rectangles contained by the first two lines so drawn to the square on the third is constant."

Apollonius in his first preface claims originality for his book IV and for parts of book III. He regarded the first four books as an introduction concerned with the elements of the subject. Since Pappus says that Apollonius completed the first four books of Euclid's *conics*, we may infer from the two passages taken together that Euclid's work covered the same ground as Apollonius' first three books, but not so completely. It would appear that Euclid's work was no advance on that of Aristaeus, which would account for the fact that the latter's *Conics* was still extant, although that of Euclid had been lost, by the time of Pappus.

What Pappus calls "those properties of conics which had been proved up to Euclid's time" can be conjectured from references by Archimedes to propositions not requiring demonstration "because they are proved in the elements of conics" or simply "in the *Conics*."[85] This would imply that the proofs were given by Aristaeus or Euclid or both. In addition, Archimedes assumes without proof the fundamental properties of the parabola, ellipse, and hyperbola in the form, for the ellipse, of

$PN^2:AN.A'N= CB^2:CA^2$,

where AA' is the major axis, BB' the minor axis, C the center, P any point on the curve, and N the foot of the perpendicular from P to AA'. More generally he assumes that if QV is an ordinate of the diameter PCP of an ellipse (with corresponding formulas for the parabola and hyperbola), the ratio QV^2: $PV\,P'V$ is constant. It would appear that Euclid must have treated the fundamental characteristics of the curves as proportions, and it was left to Apollonius to develop, by means of the application of areas, the fundamental properties of curves as equations between areas.[86]

Surface Loci (Τόποι πϱòς ἐπιφανεία). *Surface Loci*, a work in two books, is attributed to Euclid by Pappus and included in the *Treasury of Analysis*.[87] It has not survived, and its contents can be conjectured only from remarks made by Proclus and Pappus about loci in general and two lemmas given by Pappus to Euclid's work.

Proclus defines a locus as "the position of a line or surface having one and the same property,"[88] and he says of locus theorems (τοπικά) that "some are constructed on lines and some on surfaces." It would appear that loci on lines are loci which are lines and loci on surfaces are loci which are surfaces. But Pappus says that the equivalent of a quadratrix may be obtained geometrically "by means of loci on surfaces as follows," and he proceeds to use a spiral described on a cylinder (the cylindrical helix).[89] The possibility that loci on surfaces may be curves, of a higher order than conic sections, described on surfaces gets some support from an obscure passage in which Pappus divides loci into fixed, progressive, and reversionary, and adds that linear loci are "demonstrated" (δείκνυνται) from loci on surfaces.[90]

Of the two lemmas to the *Surface Loci* which Pappus gives, the former[91] and the attached figure are unsatisfactory as they stand; but if Tannery's restoration[92] is correct, one of the loci sought by Pappus contained all the points on the elliptical parallel sections of a cylinder and thus was an oblique circular cylinder; other loci may have been cones.

It is in the second lemma that Pappus states and proves the focus-directrix property of a conic, which implies, as already stated, that Euclid must have been familiar with it. Zeuthen, following an insight by Chasles,[93] conjectures that Euclid may have used the property in one of two ways in the *Surface Loci*: (1) to prove that the locus of a point whose distance from a given straight line is in a given ratio to its distance from a given plane is a cone; or (2) to prove that the locus of a point whose distance from a given point is in a given ratio to its distance from a given plane is the surface formed by the revolution of a conic about an axis. It seems probable that Euclid's *Surface Loci* was concerned not merely with cones and cylinders (and perhaps spheres), but to some extent with three other second-degree surfaces of revolution: the paraboloid, the hyperboloid, and the prolate (but not the oblate) spheroid. If so, he anticipated to some extent the work that Archimedes developed fully in his *On Conoids and Spheroids*.

Book of Fallacies (Ψενδαϱια). Proclus mentions a book by Euclid with this title which has not survived but is clearly identical with the work referred to as *Pseudographemata* by Michael Ephesius in his commentary on the *Sophistici elenchi* of Aristotle[94]. It obviously belonged to elementary geometry and is sufficiently described in Proclus' words:

Do you, adding or subtracting accidentally, fall away unawares from science and get carried into the opposite error and into ignorance? Since many things seem to conform with the truth and to follow from scientific principles, but lead astray from the principles and deceive the more superficial, he has handed down methods for the clear-sighted understanding of these matters also, and with these methods in our possession we can train beginners in the discovery of paralogisms and avoid being misled. The treatise in which he gave this machinery to us he entitled [the *Book of*] Fallacies, enumerating in order their various kinds, exercising our intelligence in each case by theorems of all sorts, setting the true side by side with the false, and combining the refutation of the error with practical illustration. This book is therefore purgative and disciplinary, while the Elements contains an irrefutable and complete guide to the actual scientific investigation of geometrical matters.[95]

Astronomy: Phaenomena (Φαινόμενα). This textbook of what the Greeks called *sphaeric*, intended for use by students of astronomy, survives in two recensions, of which the older must be the nearer to Euclid's own words.[96] It was included in the collection of astronomical works which Pappus calls 'O', Αστϱονομούμενος, *The Treasury of Astronomy*, alternatively known as *The Little Astronomy*, in contrast with Ptolemy's *Syniaxis*, or *Great Astronomy*. In the older, more authentic recension it consists of a preface and sixteen propositions.

The preface gives reasons for believing that the universe is a sphere and includes some definitions of technical terms. Euclid in this work is the first writer to use "horizon" absolutely—Autolycus had written of the "horizon (i.e., bounding) circle"—and he introduces the term "meridian circle." The propositions set out the geometry of the rotation of the celestial sphere and prove that stars situated in certain positions will rise or set at certain times. Pappus comments in detail on certain of the propositions.[97]

It is manifest that Euclid drew on Autolycus, but both of them cite without proof a number of propositions, which suggests that they had in their hands a still earlier textbook of sphaeric, which Tannery conjectured to have been the composition of Eudoxus. Many of the propositions are proved in the *Sphaerica* of Theodosius, written several centuries later. He naturally uses the theorems of Euclid's *Elements* in his proofs. By examining the propositions assumed by Autolycus, and by further considering what other propositions are needed to establish them, it is thus possible to get some idea of how much of Euclid's *Elements* was already known in the fourth century before Christ.[98]

Optics: Optica ('οπτικά). The *Optica*, which is attributed to Euclid by Proclus, is also attested by Pappus, who includes it, somewhat curiously, in the *Little Astronomy*.[99] It survives in two recensions; there is no reason to doubt that the earlier one is Euclid's own work, but the later appears to be a recension done by Theon of Alexandria in the fourth century, with a preface which seems to be a pupil's reproduction of explanations given by Theon at his lectures.[100]

The *Optica*, an elementary treatise in perspective, was the first Greek work on the subject and remained the only one until Ptolemy wrote in the middle of the second century.[101] It starts with definitions, some of them really postulates, the first of which assumes, in the Platonic tradition, that vision is caused by rays proceeding from the eye to the object. It is implied that the rays are straight. The second states that the figure contained by the rays is a cone which has its vertex in the eye and its base at the extremities of the object seen. Definition 4 makes the fundamental assumption that "Things seen under a greater angle appear greater, and those under a lesser angle less, while things seen under equal angles appear equal." When he comes to the text, Euclid makes a false start in proposition L—"Of things that are seen, none is seen as a whole"—because of an erroneous assumption that the rays of light are discrete; but this does not vitiate his later work, which is sound enough. From proposition 6 it is easy to deduce that parallel lines appear to meet. In the course of proposition 8 he proves the equivalent of the theorem

where α and β are two angles and $\alpha \angle \beta \angle 1/2\pi$. There are groups of propositions relating to the appearances of spheres, cones, and cylinders. Propositions 37 and 38 prove that if a straight line moves so that it always appears to be the same size, the locus of its extremities is a circle with the eye at the center or on the circumference. The book contains fifty-eight propositions of similar character. It was written before the *Phaenomena*, for it is cited in the preface of that work. Pappus adds twelve propositions of his own based on those of Euclid.

Catoptrica (κατοπτρικά) . Proclus also attributes to Euclid a book entitled *Caroptrica*, that is, on mirrors. The work which bears that name in the editions of Euclid is certainly not by him but is a later compilation, and Proclus is generally regarded as having made a mistake. If the later compilation is the work of Theon, as may well be the case, it would have been quite easy for Proclus to have assigned it to Euclid inadvertently.

Music: Elements of Music (ΑΑί κατά μουστκήν στοιχειώσεις) . Proclus[102] attributes to Euclid a work with this title and Marinus,[103] in his preface to Euclid's *Data*, refers to it as. Μονσικής στοιχεῖα. Two musical treatises are included in the editions of Euclid's works, but they can hardly both be by the same author, since the *Sectio canonis*, or *Division of the Scale* (κατατομή κανόνοs), expounds the Pythagorean doctrine that the musical intervals are to be distinguished by the mathematical ratio of the notes terminating the interval, while the *Introduction to Harmony* (Είσαγωγή άρμονική) is based on the contrary theory of Aristoxenus, according to which the scale is formed of notes separated by a tone identified by the ear. It is now universally accepted that the *Introduction to Harmony* is the work of Cleonides, the pupil of Aristoxenus, to whom it is attributed in some manuscripts; but there is no agreement about the *Sectio canonis*, except that such a trite exposition of the Pythagorean theory of musical intervals is hardly worthy to be dignified with the name *Elements of Music*. The strongest argument for its authenticity is that Porphyry in his commentary on Ptolemy's *Harmonica* quotes almost the whole of it except the preface and twice, or perhaps three times, refers to Euclid's *Sectio canonis* as though it is the work he is quoting[104] but the passages cited by Porphyry differ greatly from the text in dispute. Gregory, who was the first to question the attribution to Euclid, would have assigned it to Ptolemy,[105] along with the *Introduction to Harmony*; but his main reason, that it is not mentioned before Ptolemy, is not sufficiently strong to outweigh the primitive character of the work. Tannery thinks that the two last propositions, 19 and 20, which specially justify the title borne by the treatise, may have been added by a later editor who borrowed from Eratosthenes, but that the rest of the work must have been composed before 300 b.c. and would attribute it to the school of Plato.[106] This is not convincing, however, for we have seen how Euclid perpetuated arithmetical theories that had become outmoded, and he could have done likewise for the Pythagorean musical theory. (It is of no significance that there are three arithmetical propositions in the *Sectio canonis* not found in the *Elements*.) As for Platonism, Euclid was himself a Platonist. Jan, who included the book under Euclid's name in his *Musici scriptores Graeci*, takes the view that it was a summary of a longer work by Euclid himself.[107] Menge, who edited it for *Euclidis opera omnia*, considers that it contains some things unworthy of Euclid and is of the opinion that it was extracted by some other writer from the authentic *Elements of Music*, now lost.[108] All that it seems possible to say with certainty is that Euclid wrote a book entitled *Elements of Music* and that the *Sectio canonis* has some connection with it.

Mechanics. No work by Euclid on mechanics is extant in Greek, nor is he credited with any mechanical works by ancient writers. According to Arabic sources, however, he wrote a *Book on the Heavy and the Light*, and when Hervagius was about to publish his 1537 edition there was brought to him a mutilated fragment, *De levi et ponderoso*, which he included as one of Euclid's works. In 1900 Curtze published this side by side with a *Liber Euclidis de gravi et levi et de comparatione corporum ad invicem* which he had found in a Dresden manuscript. It is clearly the same work expressed in rather different language and, as Duhem observed, it is the most precise exposition that we possess of the Aristotelian dynamics of freely moving bodies. Duhem himself found in Paris a manuscript fragment of the same work, and in 1952 Moody and Clagett published a text, with English translation, based chiefly on a manuscript in the Bodleian Library at Oxford. A little earlier Sarton had expressed the view that "It contains the notion of specific gravity in a form too clear to be pre-Archimedean"; but it is not all that clear, and there is no reason to think that Archimedes was the first Greek writer to formulate the notion of specific gravity. It is no objection that the dynamics is Aristotelian, for, as Clagett points out, "The only dynamics that had been formulated at all, in the time in which Euclid lived, was the dynamics of Aristotle." In Clagett's judgment, "No solid evidence has been presented sufficient to determine the question of authenticity one way or the other."[109]

In 1851 Woepcke published under the title *Le livre d'Euclide sur la balance* an Arabic fragment that he had discovered in Paris. The fact that the letters used in the figures follow each other in the Greek order suggests a Greek origin. It contains a definition, two axioms, and four propositions and is an attempt to outline a theory of the lever, not on the basis of general dynamical considerations, as in Aristotle, but on the basis of axioms which may be regarded as selfevident and are confirmed by experience. It is therefore Euclidean in character; and since it falls short of the finished treatment of the subject by

Archimedes, it could very well be an authentic work of Euclid, although it owes its present form to some commentator or editor. Woepcke found confirmation of its authenticity in a note in another Paris manuscript, *Liber de canonio*. After citing the proposition that the lengths of the arms of a lever parallel to the horizon are reciprocally proportional to the weights at their extremities, the author adds: “Sicut demonstratum est ab Euclide et Archimede et aliis.” Heiberg and Curtze were unwilling to ascribe to the *Book on the Balance* an earlier origin than the Arabs, but Duhem accepted its authenticity. Clagett, after first allowing as “quite likely that the text was translated from the Greek and that in all probability there existed a Greek text bearing the name of Euclid,” has more recently expressed the opinion that it “may be genuine and is of interest because, unlike the statement of the law of the lever in the Aristotelian *Mechanics*, its statement on the subject is proved on entirely geometrical grounds.”[110]

Duhem’s researches among the Paris manuscripts led him to discover a third mechanical fragment attributed to Euclid under the title *Liber Euclidis de ponderibus secundum terminorum circumferentiam*. It contains four propositions about the circles described by the ends of the lever as it rises and falls. As it stands, it is unlikely to be a direct translation of a Euclidean original, but it could derive from a work by Euclid. Duhem noticed how these three fragmentary works fill gaps in each other and conjectured that they might be the debris of a single treatise. This indeed seems probable, and although Duhem was inclined to identify the treatise with Ptolemy’s lost work *On Turnings of the Scale (περί ϱοπῶ̂ν)* the ultimate author from whom all three fragments spring could have been Euclid. In view of the Arabic traditions, the high probability that the work on the balance is derived from Euclid, the way in which the fragments supplement each other, and the fact that Euclid wrote on all other branches of mathematics known to him and would hardly have omitted mechanics, this is at least a hypothesis that can be countenanced.

An amusing epigram concerning a mule and an ass carrying burdens that the ass found too heavy is attributed to Euclid. It was first printed by Aldus in 1502 and is now included in the appendix to the *Palantine Anthology*, which Melancthon rendered into Latin verse.[111]

NOTES

1. The identification began in ancient times, for Aelian (second/third century), *On the Characteristics of Animals* VI.57, Scholfield ed., II (London-Cambridge, Mass., 1959), 76.26–78, 10, notes that spiders can draw a circle and “lack nothing of Euclid”(Εὔκλαδον δέονται ούςέν).

2. The reference is Archimedes, *On the Sphere and the Cylinder* 1.2, Heiberg ed., I, 12.3: δίά τό β τον α τῶ̂ν Εύκλαίδον—“by the second [proposition] of the first of the [books] of Euclid.” This is the proposition “To place at a given point a straight line equal to a given straight line.” Johannes Hjelmslev, “Über Archimdes’ Grössenlehre,” in *Kongelige Danske Videnskabernes Selskabs Skrifter*, Matematisk-fysiske Meddelelser, **25** , 15 (1950), 7, considers that the reference should have been to Euclid I.3—“Given two unequal straight lines, to cut off from the greater a straight line equal to the less”—but the reference to Euclid I.2 is what Archimedes needed at that point. Hjelmslev also argues that the reference is inappropriate because Archimedes is dealing with magnitudes, but for Archimedes magnitudes (in this instance, at any rate) can be represented by straight lines to which Euclid’s propositions apply. He is on stronger ground, however, in arguing that “Der Hinweis ist aber jedenfalls vollkommen naiv und muss von einem nicht sachkundigen Abschreiber eingesetzt worden sein.” Hjelmslev receives some encouragement from E. J. Dijksterhuis, *Archimedes*, p. 150, note, who justly observes: “It might be argued against this that, all the same, Euclidean constructions can be applied to these line segments functioning as symbols. For the rest, the above doubt as to the genuineness of the reference is in itself not unjustified. Archimedes never quotes Euclid anywhere else; why should he do it all at once for this extremely elementary question?” Jean Itard, *Les livres arithméetiques d’Euclide*, pp. 9–10, accepts Hjelmslev’s contentions wholeheartedly, and concludes, “II y a certainement interpolation par quelque scoliaste ou copiste obscur.”

The reference was certainly in Proclus’ text, for Proclus says that Archimedes mentions Euclid, and nowhere else does he do so. If the reference were authentic, it would be relevant that *On the Sphere and the Cylinder* was probably the fourth of Archimedes’ works (T. L. Heath, *The Works of Archimedes*. p.xxxii).

3. It is possible that when Archimedes says in *On the Sphere and the Cylinder* I.6, Heiberg ed., I, 20.15–16, ταύτα λϱ έντή Στοιχειώοει παϱδέδοται, “for these things have been handed down in the *Elements*,” he may be referring to Euclid’ he may be referring to Euclid”s *Elements*, particularly XII.2 and perhaps also X. I; but since there were other *Elements*, and the term was also applied to a general body of doctrine not attributable to a particular author, the reference cannot be regarded as certain.

4. Pappus, *Collection* VII. 35, Hultsch ed., II, 678.10–12: σχολάσας (Hultsch συσκολάσας) τοîs σαν παθηταίς έν ’αλξανδϱ πξαίοτον χϱότον.

5. Proclus, *In primum Euvlidis*, Friedlein ed., p. 68.6–23.

6. The word is γέγονε. It literally means “was born”; but E. Rohde in the article “Γέγονε in den Biographica des Suidas,” in *Rheinisches Museum für Philologie*, n.s. **33** (1878), 161–220, shows that out of 129 instances in the *Suda* it is certainly

equivalent to “flourished” in eighty-eight cases, and probably in another seventeen. This must be the meaning in Proclus, for his anecdote implies that Euclid was not younger than Ptolemy.

7. This was Ptolemy I, commonly called Ptolemy Soter, who was born in 367 or 366 b.c., became ruler of Egypt in 323, declared himself king in 304, effectively abdicated in 285, and died in 283 or 282.

8. The Greek text as printed by Friedlein, p. 68.11–13 from the surviving manuscript M (Monacensis) is γέγονε δέ ούτος ό άνήϱ ’επί τού πϱώτου πτολεηαίον’ καί γάϱ ό Αϱχιήδςης έμιβαλών καί τώ πϱώτω μνεύεύει τού Εύκλείδον. The second καί is clearly superfluous, or else a miscopying of some other word (to substitute έν would ease the problem of interpretation); and έπβαλών is not easy to understand. Grynaeus and August printed the words as αϱχμήδης καί τώ πϱώτΨ in their editions (1533; 1826), and the manuscript Z, which is the basis of Zamberti’s Latin translation (1539) did not have έπιβαγών. Since Heiberg’s discussion in *Litterärgeschichtliche Studien über Euklid*, pp. 18–22, the words ό Αϱχμήδης έπιβαλών καί τώ πϱώτώ have generally been understood to mean “Archimedes, following closely on the first [Ptolemy],” but Peter Fraser in *Alexandria*, I, 386–388 and II, note 82, offers a new interpretation. He interprets έπιβαλών as meaning “overlapping” and thinks it refers not to Ptolemy but to Euclid, with αύτ understood; he sees τΨ πϱύτύ, understood as έν τώ πϱώτΨ, as a reference to the first work in the Archimedean corpus, that is, *On the Sphere and the Cylinder*. *His translation is therefore “This man flourished under the first Ptolemy; for Archimedes, who overlapped with him, refers to him in his first book [?].” The theory is attractive, but I do not agree with Fraser that there is any awkwardness in τώ πϱώ* referring to Ptolemy so soon after nor do I see any difficulty in saying that Archimedes (b. 287) followed closely on Ptolemy I (abdicated 285, d. 283/282). On the whole, therefore, I prefer Heiberg’s intepretation, but Fraser’s full discussion merits careful study.

9. Archimedes died in the siege of Syracuse in 212 b.c., according to Tzetzes, at the age of seventy-five; if so, he was born in 287. Eratosthenes, to whom Archimedes dedicated *The Method*, was certainly a contemporary, but the work in which he said so has not survived. The *Suda* records that he was born in the 126th olympiad (276–273 b.c.).

10. Stobacus, *Eclogues* II, 31.115, *Anthologium*, Wachsmuth and Hense, eds., II (Berlin, 1884); 228.30–33.

11. See T.L. Heath, *The Thirteen Books of Euclid’s Elements*, 2nd ed., I, 117–124, 146–151. “On the whole I think it is from Aristotle that we get the best idea of what Euclid understood by a postulate and an axiom or common notion” (*ibid*., p. 124). See also T. L. Heath, *Mathematics in Aristotle* (Oxford, 1949), pp. 53–57.

12. Aristotle, *Prior Analytics* I, 24, 4IbI3–22.

13. Hultsch, in Pauly-Wissowa, VI, col. 1004, also gives 295 b.c. as the date of Euclid’s bign). The latest and most thorough discussion is by Peter Fraser, in *Alexandria*, I 386–388, with notes in II, especially note 82. He concludes that Euclid may have been born about 330–320 b. c. and did not live much, if at all, after about 270 b. c. This would give him a middle date of 300–295. The round figure of 300 b. c. is given by Hankel, Gow, Zeuthen, Cantor, Loria, Hoppe, Heath, and van der Waerden. Michel gives the same date “au plus tard.” Thaer puts Euclid’s productive period (Wirksamkeit) in the last decade of the fourth century b.c. Heinrich Vogt, “Die Entdeckungsgeschichte des Irrationalen nach Plato usw.,” in *Biblioteca mathematica*, 3rd ser., **10** (1910), 155, and—in greater datil—in “Die Lebenzeit Euclids,” *ibid*, **13** (1913), 193–202, puts Euclid’s at about 365 and the composition of the Elements at 330–320. Similar dates are given by Max Steck in his edition of P. L. Schönberger, *Proklus Diadochus*(Halle, 1945), and in *Forschungen und Forschritte*, **31** (1957), 113; but these authors, as Fraser rightly notes, do not pay sufficient attention to the links of Euclid with Ptolemy Soter and of his pupils with Apollonius. The latest date suggested for Euclid’s *floruit* is 280 b. c., given in the brief life prefixed by R. N. Adams to the twenty-first and subsequent editions of Robert Simson’s *Elements* (London, 1825), but it rests on no reasoned argument.

14. It would hardly be necessary to mention this confusion, of which the first hint is found in Valerius Maximus VIII. 12, Externa I, in the reign of Tiberius (14–37), were it not common in the Middle Ages and repeated in all the printed editions of Euclid from 1482 to 1566. Karl R. Popper, *Conjectures and Refutations, and Refutatuons*, 3rd ed. (London, 1969), p. 306, has revived it (“Euclid the Geometrician... you don’t mean the man from Megara, I presume”), but only, it must be assumed, in jest.

Jean Itard has recently advanced the theory that Euclid may not have been an individual but a school. In *Les liores arithmetiques d’Euclide*, p. II, he advances three hypotheses: (1) that Euclid was a single individual who composed the various works attributed to him; (2) that Euclid was an individual, the head of a school which worked under him and perhaps continued after his death to produce books to which they gave his name; (3) that a group of Alexandrian mathematicians issued their works under the name of Euclid of Megara, just as (he alleges) the chemists of the same period attributed their works to Democritus. Of these speculations he thinks the second “paraît être la plus raisonnable.” Itard exaggerates “les difficultes qui surgissent à chaque instant dans la chronologie Iorsque l’on admet l’existence d’un seul Euclide”; there is a lack of precise information but no difficulty about Euclid’s chronology. No one has hitherto seen any reason for thinking that the author of the *Elements* could not also have been the author of the other books attributed to him. There are differences within the books of the *Elements* themselves, notably the difference between books VII-IX, with which Itard is particularly concerned, and books V and X; but these are explicable by less drastic suggestions, as will be shown later. The reason why the name of Euclid the Geometer ever came to be confused with Euclid of Megara, who lived a century earlier, is clear from the passage of Valerius Maximus cited above. Valerius says that Plato, on being asked for a solution to the problem of making an altar double a cubical altar, sent his inquiries to “Euclid the geometer.” One early commentator wished to alter this to “Eudoxus,” which is probably right. The first specific identification of Euclid the Geometer with Euclid of Megara does not occur until Theodorus Metochita (*d*. 1332), who writes of "Euclid of Megara, the Socratic philosopher, contemporary with Plato" as the author of works on plane and solid geometry, data, optics, and so on. Euclid was a common Greek name; Pauly-Wissowa lists no fewer than eight Eukleides and twenty Eukleidas.

15. The idea that he was born at Gela in Sicily springs from the same confusion. Diogenes Laertius II. 106 says that he was "of Megara, or according to some, of Gela, as Alexander says in the *Diadochai*."

16. The passage is bracketed by Hultsch, but he brackets with frequency and not always with convincing reason.

17. Stobaeus, *Eclogues* II, 31.114, Wachsmuth and Hense eds., II, 228.25–29.

18. The so-called books XIV and XV are not by Euclid. Book XIV is by Hypsicles, probably in the second century b.c.; book XV, by a pupil of Isidore of Miletus in the sixth century.

19. Proclus, *In primum Euclidis*, Friedlein ed., p. 72.6–13.

20. This is one of two mathematical uses of the term διορισμός, the other being a determination of the conditions of possibility. In the present sense it is almost part of the particular enunciation. Proclus, Friedlein ed., pp. 203.1–205.12, explains these formal divisions of a proposition.

21. The fullest discussion is in Pappus, *Collection* VII, Pref. 1–3, Hultsch ed., II, 634.3–636.30. It was James Gow, *A Short History of Greek Mathematics*, p. 211, note 1, who first recognized that the correct translation of τόπος ἀναλυόμενος "storehouse (or treasury) of analysis."

22. Aristotle, *Posterior Analytics* I, 10, 76a31–77a4.

23. The equidistance theory was represented in antiquity by Posidonius, as quoted by Proclus, *In primum Euclidis*, Friedlein ed., p. 176.7–10; Geminus, also as quoted by Proclus, *ibid*., p. 177.13–16; and Simplicius as quoted by al-Nayrizr, Curtze ed., pp. 25.8–27.14. (The "philosopher Aganis" also quoted in this passage must be Geminus.) The direction theory is represented by Philoponus in his comment on Aristotle, *Posterior Analytics* II, 16, 65a4 and was probably held by Aristotle himself.

24. For Ptolemy's attempt, see Proclus, *In primum Euclidis*, Friedlein ed., pp. 365.5–367.27; for Proclus' own attempt, *ibid*., pp. 368.24–373.2.

25. Proclus, *In primum Euclidis*, Friedlein ed., pp. 419.15–420.23, explains at some length what is meant by the application of areas, their exceeding, and their falling short.

26.*Ibid*., p. 269.8–21.

27. Isaac Barrow, in lecture VIII of 1666, *Lectiones habitae in scholis publicis academiae cantabrigiensis* (Cambridge, 1684), p. 336, states, "Cum hoc elegio praefixum hanc disputationem claudo, nihil extare (me judice) in toto Elementorum opere proportionalitatum doctrinam subtilius inventum, solidius stabilitum, accuratius pertractatum." The English translation is that of Robert Simson, at the end of his notes to book V of *The Elements of Euclid*, 21st ed. (London, 1825), p. 294. Simson "most readily" agrees with Barrow's judgment.

28. Augustus De Morgan, "Proportion," in *The Penny Cyclopaedia*, XIX, 51. Oskar Becker's theory (see Bibl.) that there was an earlier general theory of proportion hinted at by Aristotle is discussed in the article on Theaetetus.

29. H.G. Zeuthen, *Lehre von den Kegelschnitten im Altertum*, p.2.

30. Heath, *The Thirteen Books of Euclid's Elements*, I, 124–126.

31. Plutarch, *Quaestiones conviviales* VIII, 2, 4.720a—compare *Non posse suaviter vivi secundum Epicurum*, 11, 1094b—says that the discovery of this proposition, rather than the one about the square on the hypotenuse of a right-angled triangle, was the occasion of a celebrated sacrifice by Pythagoras.

32. According to VII, definition 21 (20), "Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second as the third is of the fourth." H.G. Zenthen, *Histoire des mathematiques*, p. 128, comments; "Sans doute, en ce qui concerne l'egality des rapports, cette definition ne renferme rien d'autre que ce qu'impliquait déjà la cinquième définition du cinquième Livre." The same author, in his article "Sur la constitution des livres arithmétiques des Élements d'Euclide," in *Oversigt over det K. Danske Videnskabernes Selskabs Forhandlinger* (1910), 412–413, sees a point of contact between the special and general theories in VII.19, since it shows that the definition of proportion in V, definition 5,

has, when applied to numbers, the same significance as in VII, definition 21 (20); and we can henceforth borrow any to the propositions proved in book V.

33. “This book has a completeness which none of the others (not even the fifth) can boast of: and we could almost suspect that Euclid, having arranged his materials in his own mind, and having completely elaborated the tenth Book, wrote the preceding books after it, and did not live to revise them thoroughly” (Augustus De Morgan, “Eucleides,” in *Dictionary of Greek and Roman Biography and Mythology*, William Smith, ed., II [London, 1846], 67). See also W. W. Rouse Ball, *A Short Account of the History of Mathematics*, 3rd ed. (London, 1901), pp. 58–59.

34. After defining a point as “that which has no part” (I, definition 1), Euclid says, “The extremities of a line are points” (I, definition 3), which serves to link a line with a point but is also a concession to an older definition censured by Aristotle as unscientific. So for lines and surfaces (I, definition 6). Among unused definitions are “oblong,” “rhombus,” and “rhomboid,” presumably taken over from earlier books.

35. Paul-Henri Michel, *De pythagore a Euclide*, p. 92.

36. See Augustus De Morgan, “Eucleides,” in *Smith’s Dictionary of Greek and Roman Biography*, p. 67; and “Irrational Quantity,” in *The Penny Cyclopaedia*, XIII, 35–38.

37. See Eva Sachs, *Die funf Platonischen Kö;rper* (Berlin, 1917).

38. Proclus, *In primum Euclidis*, Friedlein ed., p. 69.4–27.

39. In Greek, σνλλογισμοί but the word can hardly be used here in its technical sense. Two attempts have been made to turn the *Elements* into syllogisms!

40. Geometrical conversion is discussed by Proclus, *In primum Euclidis*, Friedlein ed., pp. 252.5–254.20.

41.*Ibid*., p. 68.7–10.

42. Scholium 1, book V, *Euclidis opera omnia*, Heiberg and Menge, eds., V, 280.7–9.

43. Scholium 3, book V, *ibid*., p. 282.13–20.

44. Archimedes, *On the Sphere and the Cylinder* I, preface, *Archimedis opera omnia*, Heiberg ed., I, 2–13; compare *Quadrature of the Parabola*, preface, *ibid*., 262–266.

45. Simplicius, *Commentary on Aristotle’s Physics* A2, 185a14, Diels ed. (Berlin, 1882), 61.8–9.

46. Plato, *Theaetetus*, 147D ff.

47. Franz Woepcke, in *Mémoires présentés à l’Académie des sciences*, **14** (1856), 658–720; W. Thomson, *Commentary of Pappus on Book X of Euclid’s Elements*, Arabic text and trans., remarks, notes, glossary by G. Junge and Thomson (Cambridge, Mass., 1930; repr., 1968).

48. J. L. Heiberg, *Litterärgeschichtliche Studien über Euklid*, pp. 169–171.

49. Scholium 62, book X, *Euclidis opera omnia*, Heiberg and Menge, eds., V, 450.16.

50. T. L. Health, *The Thirteen Books of Euclid’s Elements*, III, 3–4; for Thomson’s trans., *see op. cit*., pp. 63–64.

51.*Suda Lexicon, s.v*., Adler ed., I.2 (Leipzig, 1931), Θ93, p. 689.6–8.

52. Scholium 1, book XIII, *Euclidis opera omnia*, Heiberg and Menge, eds., V, 654.5–6.

53. Proclus, *In primum Euclidis*, Friedlein ed., p. 419.15–18.

54. Scholium 4, book IV, *Euclidis opera omnia*, Heiberg and Menge, eds., V, 273.13–15.

55. Proclus, *In primum Euclidis*, Friedlein ed., p. 426.6–18.

56. Aristotle, *Prior Analytics* II, 16, 65a4.

57. Compare Karl Popper, *Conjectures and Refutations*, 3rd ed., pp. 88–89: “Ever since Plato and Euclid, but not before, geometry (rather than arithmetic) appears as the fundamental instrument of all physical explanations and descriptions, in the theory of matter as well as in cosmology.” Popper has no doubt that Euclid was a Platonist and that the closing of the *Elements* with the construction of the Platonic figures is significant.

58. Pappus, *Collection* VII.3, Hulsch ed., II, 636.19.

59.*Ibid*., pp. 638.1–640.3.

60. Proclus, *In primum Euclidis*, Friedlein ed., p. 69,4.

61.*Ibid*., p.644.22–26.

62. A detailed classification is made in R. C. Archibald, *Euclid’s Book on Divisions of Figures*, pp. 15–16.

63. Hero, *Metrica* III. 8, Schöne ed., III, *Heronis Alexandrini opera quae supersunt omnia* (Leipzig, 1903), 172.12–174.2, considers the related problem “To divide the area of a circle into three equal parts by two straight lines.” “That this problem is not rational,” he notes, “is clear”; but because of its utility he proceeds to give an approximate solution.

64. Pappus, *Collection* VII.13, Hultsch ed., II, 648.18–19.

65. Proclus, *In primum Euclidis, Friedlein* ed., p. 302.12–13.

66. If it had survived, it might have led B. L. van der Waerden to modify his judgments in *Science Awakening*, 2nd ed. (Groningen, undated [1956?]), p. 197: “Euclid is by no means a great mathematician... Euclid is first of all a pedagogue, not a creative genius. It is very difficult to say which original discoveries Euclid added to the work of his predecessors.”

67. Pappus, *Collection* VII.14, Hultsch ed., II, 650.16–20.

68. Proclus, *In primum Euclidis*, Friedlein ed., p. 301.22–26.

69. Pappus, *Collection* VII.16, Hultsch ed., II, 652.18–654. Some words have been added in the translation for the sake of clarity.

70. Robert Simson, *De porismatibus tractatus in Opera quaedam reliqua* (Glasgow, 1776), pp. 392–393, elucidated this passage in elegant Latin, which Gino Loria, in *Le scienze esatte nell’antica Grecia*, 2nd ed. (Milan, 1914), pp. 256–257, first put into modern notation.

71. Pappus, *Collection* VII.17, Hultsch ed., II, 654.16–19.

72.*Ibid*., VII.14, Hultsch ed., II, 652.2.

73. H. G. Zeuthen, *Die Lehre von den Kegelschnitten im Altertum*, pp. 165–184.

74. Pappus, Collection VII.18, Hultsch ed., II, 656.1–4.

75. H. G. Zeuthen, *op. cit*., p. 152.

76. Pappus, *Collection* VII.30, Hultsch ed., II, 672.18–20.

77.*Ibid*., pp. 676.25–678.15.

78. Euclid, *Phaenomena*, preface, in *Euclidis opera omnia*, Heiberg and Menge, eds., VIII, 6.5–7.

79. Pappus, *Collection* VII.312, Hultsch ed., II, 1004.23–1006.2.

80.*Ibid*., p. 678.14.

81.*Ibid*., p. 678.15–24.

82. Apollonius, Conics I, preface, *Apollonii Pergaei quae Graece exstant*, Heiberg ed., I (Leipzig, 1891), 4.10–17.

83. H. G. Zeuthen, *Die Lehre von den Kegelschnitten im Altertum*, pp. 136–139.

84. Pappus, *Collection* VII.35, Hultsch ed., II, 678.4–6; *ibid*., VII.33, p. 676.21–24.

85. Archimedes, *Quadrature of a Parabola*, proposition 3, in *Archimedes opera omnia*, II, 2nd Heiberg ed. (Leipzig, 1910–1915), 268.3; *On Conoids and Spheroids*, proposition 3, Heiberg ed., I, 270.23–24; *ibid*., p. 274.3. But when the Latin text of On Floating Bodies, II.6, Heiberg ed., II, 362.10–11, says of a certain proposition, "Demonstratum est enim hocper sumpta," it probably refers to a book of lemmas rather than to Euclid's it probably refers to a book of lemmas rather than to Euclid's Conics.

86. For a full discussion of the propositions assumed by Archimedes, the following works may be consulted: J. L. Heiberg, "Die Kenntnisse des Archimedes über die Kegelschnitte," in *Zeitschrift fur Mathemalik und Physik*, Jahrgang 25, Hist.-lit. Abt. (1880), 41–67; and T. L. Heath, Apollonius of Perga, pp. l-lxvi; *The Works of Archimedes*, pp. lii-liv; A History of Greek Mathematics, II (Oxford, 1921), 121–125.

87. Pappus, *Collection* VII.3, Hultsch ed., II, 636.23–24.

88. Proclus, *In primum Euclidis*, Friedlein ed., p. 394.17–19.

89. Pappus, *Collection* IV.51, Hultsch ed., I, 258.20–262.2.

90.*Ibid*., VII.21, Hultsch ed., II, 660.18–662.22. A large part of the passage is attributed to an interpolator by Hultsch, but without reasons.

91.*Ibid*., VII.312, p. 1004.17–22.

92. Paul Tannery, review of J. L. Heiberg's *Litterärgeschichtliche Studien uber Euklid*, in *Bulletin des sciences mathematiques*, 2nd ser., **6** (1882), 149–150; reprinted in *Memoires scientifiques*, XI (Toulouse-Paris, 1931), 144–145.

93. Michel Chasles, "Apercu historique," pp. 273–274; H. G. Zeuthen, *Die Lehre von den Kegelschnitten im Altertum*, pp. 423–431. J. L. Heiberg takes a different view in his *Litterargeschichtliche Studien uber Euklid*, p. 79.

94. Alexander (?), *Commentary on Aristotle's Sophistici elenchi*, Wallies ed., (Berlin, 1898), p. 76.23.

95. Proclus, *In primum Euclidis*, Friedlein ed., pp. 69.27–70.18.

96. The older recension is, however, best illustrated in a Vienna manuscript of the twelfth century; the later recension is found in a Vatican manuscript of the tenth century.

97. Pappus, *Collection* VI.104–130, Hultsch ed., II, 594–632.

98. The task was attempted by Hultsch, *Berichte uber die Verhandlungen der Kgl. Sachsischen Gesellschaft der Wissenschaften zu Leipzig*, Phil.-hist. Classe, **38** (1886), 128–155. The method definitely establishes as known before Euclid the following propositions: I.4, 8, 17, 19, 26, 29, 47; III.1–3, 7, 10, 16 (corollary), 26, 28, 29; IV.6; XI.3, 4, 10, 11, 12, 14, 16, 19, and 38 (interpolated). But Hultsch went too far in adding the whole chain of theorems and postulates leading up to these propositions, for in some cases (e.g., I.47) Euclid worked out a novel proof.

99. Proclus, *In primum Euclidis*, Friedlein ed., p. 69.2; Pappus, *Collection* VI.80–103, Hultsch ed., II, 568.12–594.26.

100. Only the later recension was known until the end of the nineteenth century, but Heiberg then discovered the earlier one in Viennese and Florentine manuscripts. Both recensions are included in the Heiberg-Menge *opera omnia*.

101.*See* A. Lejeune, *Euclide et Ptolémée: Deux stades de l'optique géométrique grecque*.

102. Proclus, *In primum Euclidis*, Friedlein ed., p. 69.3.

103. Marinus, *Commentary on Euclid's Data*, preface, in *Euclidis opera omnia*, Heiberg and Menge, eds., VIII, 254.19.

104.Καί αὐτὸς ὁ Στοικειωτής Εὐκλείδης έν τῇ τοῦ Κανὸνος κατατομῇ, Porphyry, *Commentary on Ptolemy's Harmonies*, Wallis ed., *Opera mathematica*, III (Oxford, 1699), 267.31–32; έν τῇ τοῦ Κανυόνος Κατατομῇ Εὐκλείδου, 272.26–27; Καί αὐτΨ͂ τΨ͂ Στὸικεωτῇ καί αλλοις κανονικοίς, *ibid*., 269.5–6.

105. David Gregory, *Euclidis quae supersunt omnia*, preface.

106. Paul Tannery, *"Inauthenticité de la Division du canon* attribuee a Euclide," in *Comptes rendus des seances de l'Academie des inscriptions et belles-lettres*, **4** (1904), 439–445; also in his *Memoires*, III, 213–219.

107.*Excerpta potius dicas quam ipsa uerba hominis sagacissimi*, in C. Jan, ed., *Musici scriptores Graeci*, p. 118.

108.*Euclidis opera omnia*, Heiberg and Menge, eds., VIII, xxxvii-xlii.

109. J. Hervagius, ed., "Euclidis de levi et ponderoso fragmentum," in *Euclidis Megarensis mathematici clarissimi Elementorum geometricorum libri xv* (Basel, 1537), pp. 585–586, and foreword; M. Curtze, "Zwei Beiträge zur Geschichte der Physik im Mittelalter," in *Biblioteca mathematica, 3rd ser*.,**1** (1900),51–54; p. Duhem. *Les origines de la statique*,1, 61–97; and George Sarton, *Introduction to the History of Science*, **1** , 156.

110. F. Woepcke,"Notice sur des traductions arabes de deux ouvrages perdus d'Euclide," in *Journal asiatique*, 4th set., **18** (1851), 217–232; M. Curtze, "Das angebliche Werk des Eukleides uber die Wagge," in *Zeitschrift fur Mathematik und Physik*, **19** (1874), 262–263; P. Duhem, *op. cit*., pp. 61–97; Marshall Clagett, *The Science of Mechanics in the Middle Ages, p. 28; and Greek Science in Antiquity* (London, 1957), p. 74.

111.*Anthologia palatina, Appendix nova epigrammaturm*, Couguny ed., (Paris, 1890), 7.2; *Euclidis opera omnia*, Heiberg and Menge, eds., VIII. 285, with Melancthon's rendering on p. 286.

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