Eutocius of Ascalon | Encyclopedia.com

(*b*. Palestine, *ca*. a.d. 480)

mathematics.

Eutocius was the author of commentaries on three works by Archimedes. He also edited and commented on the first four books of the *Conics* of Apollonius.

His commentary on the first book of Archimedes' *On the Sphere and Cylinder* was dedicated to Ammonius, who was a pupil of Proclus and the teacher of Simplicius and many other sixth-century philosophers, and who could not have lived long after 510. Eutocius' four commentaries on the *Conics* are dedicated to Anthemius of Tralles, the architect of Hagia Sophia in Constantinople, who died about 534. For these reasons the central point of Eutocius' activities may be put about 510, and it has become conventional to date his birth about 480.

The old belief that Eutocius flourished about fifty years later arose from a note at the end of three of his Archimedean commentaries—on *On the Sphere and Cylinder*, Books I and II, and on the *Measurement of a Circle*—to the effect that each of them was "edited by Isidorus, the mechanical engineer, our teacher." These words, bracketed by Heiberg, cannot refer to Eutocius because they are not compatible with his relationship to Ammonius, for Isidorus of Miletus continued the construction of Hagia Sophia after the death of Anthemius about 534 and could not have been Eutocius' teachers. The words are best understood as an interpolation by a pupil of Isidorus and contain the interesting information that Isidorus revised the commentaries in question. Similarly, a reference in the commentary on *On the Sphere and Cylinder*, Book II (Archimedes, Heiberg ed., III, 84.8–11) to an instrument for drawing parabolas invented by "Isidorus, the mechanical engineer, our teacher" is also best understood as an interpolation. Tannery mentions the possibility that the Isidorus in question may have been a nephew of the successor of Anthemius, who supervised the reconstruction of Hagia Sophia after an earthquake in 557. Ascalon (now Ashkelon), Where Eutocius was born, lay between Azotus (now Ashdod) and Gaza on the coast of Palestine; it is the city made famous in the lament of David over Saul and Jonathan: "Tell it not in Gath, publish it not in the streets of Askelon" (II Samuel 1:20). The *Suda Lexicon* relates an unedifying story of a Thracian mercenary named Eucotius who made a lot of money and tried to buy himself into society, first at Eleutheropolis (now Beyt Guvrin, Israel), then at Ascalon, but few have followed Tannery in seeing an ancestor; it seems more probable that Ascalon has been introduced into this story by reason of the mathematician's name and fame.

In his preface to his interpretation of the *Measurement of a Cricle* Eutocius refers to his earlier commentaries on *On the Sphere and Cylinder*, and in the commentary on Book I he asks Ammonius to bear with him if he should have erred through youth (Archimedes, Heiberg ed., III, 2.13). He explains that he has found no satisfactory commentaries on Archimedes before his own time and promises further elucidation of the master if his work should meet with the approval of Ammonius. Apart from the *Measurement of a Circle*, he later wrote commentaries on both books of Archimedes' treatise *On Plane Equilibria*. The commentary on Book I was dedicated to an otherwise unknown Peter, whose name reveals him to have been a Christian. It is a fair inference that Eutocius did not know the works of Archimedes entitled *Quadrature of a Parabola* and *On Spirals*, for if he had, he would have referred to them at certain points of his commentary (Archimedes, Heiberg ed., III, 228.25; 278.10; 280.4; 286.13) instead of making less suitable references. Presumably the commentaries on Apollonius' *Conics* were written later than those on Archimedes' works, but there is no direct evidence. All these commentaries have survived. It has been debated whether he also wrote a commentary on Ptolemy's *Syntaxis*, but there is no suggestion of one in a passage of his commentary on the *Measurement of a Circle* (Archimedes, Heiberg ed., III, 232.15–17), where he mentions "Pappus and Theon and many others" as having interpreted that work.

Eutocius is not known to have done any original mathematical work, and his elucidations of Archimedes and Apollonius do not add anything of mathematical significance. Nevertheless, the examples of long multiplication in his commentary on the *Measurement of a Circle* are the best available evidence of the way in which the Greeks handled such operations, and he preserves solutions of mathematical problems by the earlier Greek geometers that are sometimes the sole evidence for their existence and are therefore of major importance for the historian of mathematics.

It is through Eutocius that we have a valuable collection of solutions by Greek geometers of the problem of finding two mean proportionals to two given straight lines, that is, if a and b are two given straight lines, to find two other straight lines x and y such that $a:x = x:y = y:b$. It was to this that a problem which attracted the best Greek mathematicians for several centuries—how to find a cube double another cube—had been reduced by Hippocrates, for if $a:x = x:y = y:b$, then $a^3:x^3 = a:b$, and if $b = 2a$, then x is the side of a cube double a cube of side a. From that time the problem appears to have been attacked exclusively in this form.

The first proposition of Archimedes' *On the Sphere and Cylinder*, Book II, is "Given a cone or cylinder, to find a sphere equal to the cone or cylinder." He shows as analysis that this can be reduced to the problem of finding two mean proportionals and then, in the synthesis, says: "Between the two straight lines, let two mean proportionals be found." It is at this point that Eutocius begins an extended comment (Archimedes, Heiberg ed., III, 54.26–106.24). After noting that the method of finding two mean proportionals is in no way explained by Archimedes, he observes that he had found the subject treated by many famous men, of whom he omits Eudoxus because in his preface he said he had solved the problem by curved lines but had not used them in the proof and had, moreover, treated a certain discrete proportion as though it were continuous, which a mathematician of his caliber would not have done. In order that the thinking of those men whose solutions have been handed down might be manifest, Eutocius sets out the manner of each discovery. He gives a solution attributed to Plato (but almost certainly wrongly attributed), followed by solutions given by Hero in his *Mechanics* and *Belopoeïca*, by Philo of Byzantium, by Apollonius, by Diocles in his work *On Burning Mirrors*, by Pappus in his *Introduction to Mechanics*, by Sporus of Nicaea, by Menaechmus (two solutions), by Archytas as related by Eudemus, by Eratosthenes, and by Nicomedes in his book *On Conchoidal Lines*. (This is not a chronological order; chronologically the order would probably be Archytas, Eudoxus, Menaechmus, the pseudo-Plato, Eratosthenes, Nicomedes, Apollonius, Philo, Diocles, Hero, Sporus, and Pappus. There is, indeed, no discernible order in Eutocius' list.)

Hero's solution is given in his *Mechanics* I, 11, which has survived only in an Arabic translation, and in his *Belopoeïca*, and is reproduced by Pappus, *Collection* III, 25–26. Pappus' solution is given in *Collection* III, 27 and VIII, 26; it is the latter passage that Eutocius has in mind. The conchoid is described by Pappus, *Collection* IV, 39–40, and he mentions that it was used by Nicomedes for finding two mean proportionals but does not give a proof. The other solutions would not be known but for their preservation by Eutocius. It is a pity that he did not include what purported to be Eudoxus' solution despite the obvious errors in transmission, but for what he has preserved he deserves the gratitude of posterity. The solution ascribed to Eratosthenes is prefaced by a letter, allegedly from Eratosthenes to Ptolemy Euergetes, giving the history of the problem of doubling the cube and its reduction to the problem of finding two mean proportionals; the letter is not authentic, but it closes with a genuine condensed proof and an epigram that Eratosthenes put on a votive monument. The solution attributed to Plato is probably not authentic because, among other reasons, it is mechanical, but the solutions of Eudoxus and Menaechmus show that the problem was studied in the Academy and may be Platonic. According to Eutocius, Nicomedes was exceedingly vain about his solution and derided that of Eratosthenes as impractical and lacking in geometrical sense. The solutions of Diocles, Sporus, and Pappus are substantially identical and so are those of Apollonius, Hero, and Philo.

It is only a little later, in commenting on the fourth

ptoposition of *On the Sphere and Cylinder*, Book II, that Eutocius gives a further precious collection of solutions that would not otherwise be known. Proposition 4 is the problem "To cut a given sphere by a plane so that the volumes of the segments are to one another in a given ratio." In Proposition 2 Archimedes had shown that a segment of a sphere is equal to a cone with the same base as the segment and height $h(3r - h)/(2r - h)$, where r is the radius of the sphere and h is the height of the segment (*LA* in the figure). In Proposition 4 he proves geometrically that if h, h' are the heights of the two segments, so that $h + h' = 2r$, and they stand in the ratio $m:n$, then

By the elimination of h' this becomes the cubic equation

The problem is thus reduced (in modern notation) to finding the solution of a cubic equation that can be written

Archimedes preferred to treat this as a particular case of a general equation

$x^2(a - x) = bc^2$,

where b is a given length and c^2 a given area. For a real solution it is necessary that

In the particular case of II, 4, there are always two real solutions.

Before proceeding to the synthesis of the main problem, Archimedes promised to give the analysis and synthesis of this subsidiary problem at the end, but Eutocius could not find this promise kept in any of the texts of Archimedes. He records that after an extensive search he found in an old book a discussion of some theorems that seemed relevant. They were far from clear because of errors and the figures were faulty, but they seemed to give the substance of what he wanted. The language, moreover, was in the Doric dialect and kept the names for the conic sections that had been used by Archimedes. Eutocius was therefore led to the conclusion, as we also must be, that what he had before him was in substance the missing text of Archimedes, and he proceeded to set it out in the language of his own day. The problem is solved, in modern notation, by the intersection of the parabola and rectangular hyperbola

Others before Eutocius had noticed the apparent failure of Archimedes to carry out his promise, and Eutocius also reproduced solutions by Dionysodorus and Diodes. Dionysodorus solved the particular case of the cubic equation to which II, 4 reduces, that is,

His solution is the intersection of the parabola and rectangular hyperbola

Diocles solved not the subsidiary equation but the original problem, II, 4, by means of the intersection of an ellipse

and the rectangular hyperbola

$(x+a)(y+b)= 2ab$.

It is clear that the *Measurement of a Circle* was already reduced to three propositions, with the second and third in the wrong order, when Eutocius had it before him. The chief value of his commentary is that he works out in detail the arithmetical steps where Archimedes merely gives the results. Archimedes requires a number of square roots. Eutocius excuses himself from working them out, on the ground that the method is explained by Hero and by Pappus, Theon, and other commentators on Ptolemy's *Syntaxis*, but he multiplies the square root by itself to show how close the approximation is. At the end Eutocius reveals that Apollonius in a work called 'Ωκντόκίον(*Formula for Quick Delivery*) found a closer approximation to the ratio of a circumference of a circle to its diameter than did Archimedes; and he exculpates Archimedes from the censure of Sporus of Nicaea, whose own teacher, Philo of Gadara, also found a more exact value, on the ground that Archimedes was looking for a figure useful in daily life.

Apart from what has been noted above, Eutocius' comments on Archimedes do not add much of value to the text, and occasionally he errs, as in saying that two parabolic segments in Proposition 8 of *On Plane Equilibria* are similar (Archimedes, Heiberg ed., III, 290.23–24). In a commentary on the difficult lemma that is Proposition 9 of the same book and leads to the location of the center of a gravity of portion of a parabola cut off by parallel chords, he admits himself forced to paraphrase.

The commentaries on the *Conics* display more mathematical acumen. In his preface to Book I, Apollonius explains how uncorrected copies came to be in circulation before he had completed his revision. It is therefore probable that there were variant readings and alternative proofs in the manuscripts from earliest days. It is clear that when Eutocius came to comment on Apollonius, he had before him differing versions, and he found it necessary to prepare a recension for his own purposes; in two manuscripts the four books of his comment have the heading "A Commentary of Eutocius of Ascalon on the First (Second, Third, Fourth) of the *Cinics* of Apollonius as Edited by Himself." Eutocius' edition suffered at the hands of interpolators, probably in the ninth century, when mathematics had a renaissance at Constantinople under Leo the Mathematician. The best manuscript of the commentary (W, Cod. Vat. gr. 204) was copied in the tenth century, and at a number of points Eutocius' citations from Apollonius are clearly nearer to the original than is the text of the *Conics* as we have it today. In commenting on Apollonius, Eutocius had been preceded by Serenus and Hypatia. The most interesting features of the commentary are in the early pages, where Eutocius emphasizes the generality of Apollonius' method of producing conic sections from any cone.

All the books by Archimedes on which Eutocius commented have survived, and his elucidations may have contributed to their survival. There must also be some significance in the fact that the four books of the *Conics* on which he commented have survived in Greek, whereas Books V-VII have survived only in Arabic and Book V-VII is entirely lost. His commentaries on Archimedes were translated into Latin along with the parent works by William of Moerbeke in 1269. The commentaries have usually been printed with the editions of Archimedes and Apollonius and have never been printed separately. The definitive text is to be found in Heiberg's editions of Archimedes and Apollonius with a Latin translation and valuable prolegomena and notes.

BIBLIOGRAPHY

I. Original Works. Eutocius' commentaries can be found in *Comrnentarii in libros Archimedis De sphaera et cylindro, in Dimensionem circuli et in libros De planorum aequilibris, in Archimedis opera omnia*, J. L. Heiberg, ed., 2nd ed., III (Leipzig, 1915), 1–448; and *Commentaria in Conica, in Apollonii Pergaei quae graece exstani*, J. L. Heiberg, ed., II (Leipzig, 1893), 168–361.

II. Secondary Literature. See Paul Tannery, "Sur l'histoire des lignes et surfaces courbes dans l'antiquité," in *Bulletin des sciences mathématiques*, 2nd ser., 7(1883), 278–291; and "Eutocius et ses contemporains," *ibid*.,**8** (1884), 315–329, repr. in *Mémoires scientifiques*, II (Toulouse-Paris, 1912), 1–47, 118–136; and Sir Thomas Heath, *A History of Greek Mathematics* (Oxford, 1921), I, 52, 57–58; II, 25, 45, 126, 518, 540–541.

Ivor Bulmer-Thomas