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Fibonacci, Leonardo, or Leonardo of Pisa | Encyclopedia.com

(*b*. Pisa, Italy, ca. 1170; *d*. Pisa, after 1240)

mathematics.

Leonardo Fibonacci, the first great mathematician of the Christian West, was a member of a family named Bonacci, whose presence in Pisa since the eleventh century is documented. His father's name is known to have been Guilielmo. It is thus that Fibonacci is to be understood as a member of the Bonacci family and not as "son of a father of the name of Bonacci," as one might suppose from the words "filio Bonacij" or "de filiis Bonacij," which appear in the titles of many manuscripts of his works. The sobriquet "Bigollo" (from *bighellone*, loafer or ne'er-de-well), used by Leonardo himself, remains unexplained. Did his countrymen wish to express by this epithet their disdain for a man who concerned himself with questions of no practical value, or does the word in the Tuscan dialect mean a much-traveled man, which he was?

Life. Leonardo himself provides exact details on the course of his life in the preface to the most extensive and famous of his works, the book on calculations entitled *Liber abbaci* (1202). His father, as a secretary of the Republic of Pisa,[1] was entrusted around 1192 with the direction of the Pisan trading colony in Bugia (now Bougie), Algeria. He soon brought his son there to have him learn the art of calculating, since he expected Leonardo to become a merchant. It was there that he learned methods "with the new Indian numerals," and he received excellent instruction (*exmirabili magisterio*). On the business trips on which his father evidently soon sent him which took him to Egypt, Syria, Greece (Byzantium), Sicily, and Provence, he acquainted himself with the methods in use there zealous study and in disputations with native scholars. All these methods, however—so he reports— as well as "algorismus" and the "arcs of Pythagoras" (apparently the abacus of Gerbert) appeared to him as in "error" in comparison with the Indian methods.[2] It is quite unclear what Leonardo means here by the "algorismus" he rejects; for those writings through which the Indian methods became known, especially after Sacrobosco, a younger contemporary of Leonardo, bear that very name. Could he mean the later *algorismus linealis*, reckoning with lines, the origin of which is, to be sure, likewise obscure?

Around the turn of the century Leonardo returned to Pisa. Here for the next twenty-five years he composed works in which he presented not only calculations with Indian numerals and methods and their application in all areas of commercial activity, but also much of what he had learned of algebraic and geometrical problems. His inclusion of the latter in his own writings shows that while the instruction of his countrymen in the solution of the problems posed by everyday life was indeed his chief concern, he nevertheless also wished to provide material on theoretical arithmetic and geometry for those who were interested in more advanced questions. He even speaks once of wanting to add the "subtleties of Euclid's geometry";[3] these are the propositions from books **II** and **X** of the *Elements*, which he offers to the reader not only in proofs, in Euclid's manner, but in numerical form as well. His most important original accomplishments were in indeterminate analysis and number theory, in which he went far beyond his predecessors.

Leonardo's importance was recognized at the court of the Hohenstaufen emperor Frederick **II** . Leonardo's writings mention the names of many of the scholars of the circle around the emperor, including Michael Scotus, a court astrologer whom Dante(*Inferno*, **XX,** 115 ff.) banished to hell; the imperial philosopher, Master Theodorus; and Master Johannes of Palermo. Through a Master Dominicus, probably the Dominicus Hispanus mentioned by Guido Bonatti (see Boncompagni, Intorno ad alcune opere di Leonardo Pisano, p. 98, n.), Leonardo was presented to the emperor, who evidently desired to meet him, when Frederick held court in Pisa about 1225.[4] After 1228 we know almost nothing more concerning Leonardo's activity in Pisa. Only one document has survived, from 1240, in which the republic of Pisa awards the "serious and learned Master Leonardo Bigollo"(discretus et sapiens)a yearly *salarium* of "libre **XX** denariorem"in addition to the usual allowances, in recognition of his usefulness to the city and its citizens through his teaching and devoted services. He evidently had advised the city and its officials, without payment, on matters of accounting, a service the city expected him to continue. This decree of the city, which was inscribed on a marble tablet in a Pisa city archives in the nineteenth century,[5] is the last information we have on Leonardo's life.

Writings. Five works by Leonardo are preserved:

1. The Liber abbaci (1202, 1228);
2. The Practica *geometriae* (1220/1221);
3. A Writing entitled *Flos* (1225);
4. An undated letter to Theodorus, the imperial philosopher;

5. The *Liber quadratorum* (1225). We know of further works, such as a book on commercial arithmetic, *Di minor guisa;*[6] especially unfortunate is the loss of a tract on book X of the Elements, for which Leonardo promised a numerical treatment of irrationals instead of Euclid's geometrical presentation.[7]

Leonardo's works have been collected in the edition by Boncompagni; in 1838 Libri edited only one chapter of the *Liber abbaci*. Boncompagni, however, provides only the Latin text without any commentary. Hence, despite much specialized research on the *Flos* and on the *Liber quadratorum*, which Ver Eecke has translated into French, there is still no exhaustive presentation of Leonardo's problems and methods. The most detailed studies of the substance of the works are those by Cantor, Loria, and Youschkevitch.

Liber abbaci. The word abacus in the title does not refer to the old abacus, the sand board; rather, it means computation in general, as was true later with the Italian masters of computation, the *maestri d'abbaco*. Of the second treatment of 1228, to which "new material has been added and from which superfluous removed," there exist twelve manuscript copies from the thirteenth through the fifteenth centuries; but only three of these from the thirteenth and the beginning of the fourteenth centuries are complete. Leonardo divided this extensive work, which is dedicated to Michael Scotus, into fifteen chapters; it will be analyzed here in four sections.

Section 1 (chapter 1-7; *Scritti*, I, 1-82). Leonardo refers to Roman numerals and finger computation, which the student still needs for marking intermediate results.[8] Then the Indian numerals are introduced; following the Arabic manner, the units stand "in front" (on the right), and the fractions are on the left of the whole numbers. In addition, he introduces the fraction bar. All the computational operations are taught methodically through numerous examples and the results are checked, mostly by the method of casting out nines (seven and eleven are also used in this way). Rules are developed for the factoring of fractions into sums of unit factors. Various symbols are introduced for the representation of fractions.

Thus, for example, is to be read as means ; and is to be understood as . Finally, signifies . The first—and the most frequently employed —of these representational methods corresponds to the ascending continued fraction . Numerous tables (for multiplication, prime numbers, factoring numbers, etc.)complete the text.

Section 2 (chapters 8-11; *Scritti*, I, 83-165). This section contains problems of concern to merchants, such as the price of goods, calculation of profits, barter, computation of interest, wages, calculations for associations and partnerships, metal alloys, and mixture calculations; the computations of measurements and of currency conversions in particular reflect the widespread trade of the medieval city with the lands bordering the Mediterranean. One of the mixture problems included is known from Chinese mathematics, the "problem of the 100 birds;" a problem in indeterminate analysis, it requires that one purchase for 100 units of money 100 birds of different sorts, the price of each sort being different.

Section 3 (chapters 12 and 13; *Scritti*, I, 176-351). This is the most extensive section and contains problems of many types, which are called *erraticae questiones*. They are mostly puzzles, such as are found in the mathematical recreations of all times. Among them are the "cistern problems" (A spider climbing the wall of a cistern advances so many feet each day and slips back so many feet each night. How long will it rake it to climb out?) and from Egyption mathematics, the famous so-called "hau calculation," which can be expressed in the form.$ax \pm b/c.\ x = s$. Leonardo calls them *questiones arborum* after the first example, in which a tree is supposed to stand twenty-one ells above the ground with 7/12 of its lenght in the earth; therefore, x-$7/12x = 21$. Another group are "motion problems," involving either pursuit (as in the famous "hare and hound" problem, in which one must determine how long it will take a hound chasing a hare at a proportional speed to catch the hare) or opposite movements. In both cases the motions can be delayed through backward movements. Since in many problems the speed is not constant, but increases arithmetically, rules for the summation of series are given at the beginning of chapter 12. A group of problems that had already appeared in the epigrams of *The Greek Anthology* (a recent deition is W.R. Paton, ed. [Cambridge, Mass., 1953], V, 25 ff.) and can be designated as "giving and taking," is called by Leonardo *de personis habentibus denarios;* in these there are two or more people, each of whom demands a certain sum from one or several of the others and then states what the proportion now is between his money and that of the others. A simple example is (1)$x + 7 = 5.(y - 7)$;(2)$y + 5 = 7.\ (x - 5)$. In the problem of "the found purse" (*de inventione bursarum*) two or more people find a purse, and we are told for each individual what ratio the sum of his money and the total money in the purse has to the sum of the remaining individuals' monies; for example, with three people the modern arrangement would be (1) $x + b = 2.(y + z)$;(2) $y + b = 3.\ (x + z)$; (3) $z + b = 4(x + y)$. They are, therefore, problem in indeterminate analysis.

Another very extensive group, "one alone cannot buy," takes the from of "horse buying" (*de hominibus equum emare volentibus*). In this case it is given that one of those concerned can buy an object only if he receives from the other (or others) a portion of his (or their) cash.[9] Variations are also given that involve up to seven people and five horses; in these cases, if the price of the horse is not known, the problem is indeterminate. A problem of this type involving three people, where the equation would be

corresponds to Diophantus II, 24. A further group treats the business trips of a merchant, which are introduced as *de viagiis*. These are the famous problems of the "gate-keeper in the apple garden." It is here that the problems involving mathematical nesting of the form$<[(a_1x - b_1)\cdot a_2 - b_2]\cdot\ a_3 - \ldots>\cdot\ a_n - b_n = s$ are to be solved.[10] Of the multitude of other problems treated in the *Liber abbaci*, the following should be mentioned: numerous remainder problems, in which, for example, a number n is sought with the property $n \equiv 1$ (mod.2,3,4,5, and 6) $\equiv 0$ (mod.7); the Chinese remainder problem Tayen,[11] the finding of perfect numbers; the summation of a geometric series; the ancient Egyptian problem of the "seven old women"(to find, the seven wives of St. Ives); the Bachet weight problem; the chess problem (to fine); and the rabbit problem. This last problem assumes that a pair of rabbits requires one month to mature and thereafter reproduces itself once each month. If one starts with a single pair, how many pairs will one have after n months? The answer leads to the famous Fibonacci series, the first recurrent series. Its general form for any term k_n is $k_n = k_{n-1} + k_{n-2}$; and, in this case, it can be expressed as

Leonardo demonstrates an astonishing versatility in the choice of methods of solution to be used in particular instances; he frequently employs a special procedure, for which he usually has no specific name and which has been tailored with great skill to fit the individual problem. He also shows great dexterity in the introduction of an auxiliary unknown; in this he is like Iamblichus, who demonstrated the same talent in his explanation of the *Epanthema* of Thymaridas of Paros. At other times Leonardo makes use of definitely general methods. These include the simple false position, as in the"hau calculations"and the *regula versa*, in which the calculation is made in reverse order in the nesting problems in, *de viagiis;* there is also the double false position, to which the whole of chapter 13 is devoted and which is called—as in Leonardo7rsquo;s Arabic models—*regula elchatayn*. With this rule, linear and pure quadratic problems can be solved with the aid of two arbitrarily chosen quantities, a_1 and a_2, of unknown magnitude and the resulting errors,f_1 and f_2. Leonardo knew this procedure, but he generally used a variation, The latter consists in ascertaining from the two errors how much closer one has come to the true answer (*veritati appropinquinare)*[12] in the second attempt and then determining the number that one must now choose in order to obtain the correct solution. A special solution for an indeterminate problem is provided by the *regula proportionis*. If, for example, in the final equation of a problem $63/600x = 21/200b$, then, according to this rule, $x = 21/200$ and $b = 63/600$ or, in whole numbers,$x = 63$ and $b = 63$.

Leonardo also employed, as easy mechanical solutions, formulas (especially in the"horse buying,""found purse,"and"journey"problems) that can have been obtained only by means of algebra. He knew the algebraic methods very well; he called them *regulae rectae* and stated that they were used by the Arabs and could be useful in many ways. He called the unknown term *res* (Arabic Shai,"thing") and since he used no operational symbols and no notations for further unknowns here (see, however, under *Practica geometriae*), he had to designate them as *denarii secundi* or, as the case might be, *denarii tercii hominis* and take the trouble to carry them through the entire problem. For most of the problems Leonardo provided two or more methods of solution.

An example of the "giving and taking" type is the following, in which the system named above—(1)$x + 7 = 5.(y - 7)$;(2) $7.(x - 5) = y + 5$—is involved. First,$x + y$ is presented as a line segment; at the point of contact $y = 7$ and $x = 5$ is marked off along both sides. Then the segment $y - 7$(or $x - 5$) is equal to 1/6 (or 1/8) of the whole segment $x + y$, and together, therefore, the two segments equal $7/24. (x + y)$. The further solution is achieved by means of the simple false position $x + y = 24$. There follows still another algebraic solution, this one using the *regulae rectae*. First,$y - 7$ is designated as *res*; then (1) $x = 5$ *res* $- 7$ and (2) *res* $+ 12 = 7.(5$ *res* $- 12)$.[13]

In some cases the problem is not solvable because of mutually contradictory initial conditions. In other cases the problem is called *insolubilis, incongruum*, or *inconveniens*,[14] unless one accepts a "debit" as a solution. Leonardo is here thinking of a negative number, with which he also makes further calculations. Our operations $22 + (-9) = 22 - 9$ and $-1 + 11 = + 10$ he represents with *adde denarios*, as 22 *cum debito secudi (= 9) scilicet extrahe 9 de 22 and debitum primi (= -1) cum bursa (= 11) erunt* 10.[15]

Section 4 (chapters 14 and 15; Scribitti,**I** 352-387). Leonardo here shows himself to be a master in the application of algebraic methods and an outstanding student of Euclid. Chapter 14, which is devoted to calculations with radicals, begins with a few formulas of general arithmetic. Called "keys" (claves), they are taken from book II of Euclid's Elements. Leonardo explicitly says that he is forgoing any demonstrations of his own since they are all proved there. The fifth and sixth propositions of book II are especially important; from them, he said, one could derive all the problems of the *Aliebra*, and the *Almuchabala*. Square and cube roots are taught numerically according to the Indian-Arabic algorithm, which in fact corresponds to the modern one.

Leonardo also knew the procedure of adding zeros to the radicands in order to obtain greater exactness; actually, this had already been done by Johannes Hispalensis(*fl*. 1135-1153) and al-Nasawī(*fl.ca*. 1025). Next, examples are given that are illustrative of the ancient methods of approximation. For the first approximation is $a_1 = a + r/2a$. with $r_1 = a^2_1 - A$, the second approximation is then $a_2 = a_1 - r_1/2a^1$. With the cube root the first approximation is .

For a second approximation Leonardo now set $r_1 = A - a_1^3$ and

He was no doubt thinking if this further approximation when he spoke of his own achievement,[16] for the first approximation was already known to al-Nasawī. The chapter then goes on systematically to carry out complete operations with Euclidean irrationals. There are expressions such as

The proof, which is never lacking, of the correctness of the calculations is presented geometrically. On one occasion the numbers are represented as line segments, for example, in the computation of

where proposition 4 of book II of the *Elements* is used as a "key" On the other hand, the proof is made by means of rectangular surfaces. An example is Here is conceived as the area of a square, to which at one corner, through the elongation of the two intersecting sides, a square of area is joined. Thus is the side of a larger square, which consists of the squares and and the two rectangles each equal to .

With respect to mathematical content Leonardo does not surpass his Arab predecessors. Nevertheless, the richness of the examples and of their methodical arrangement, as well as the exact proofs, are to be emphasized. At the end of chapter 15, which is divided into three sections, one sees particularly clearly what complete control Leonardo had over the geometrical as well as the algebraic methods for solving quadratic equations and with what skill he could use them in applied problems. The first section is concerned with proportions and their multifarious transformations. In one problem, for example, it is given that (1) $6:x = y:9$ and (2) $x + y = 21$. From (1) it is determined that $xy = 54$; then, using Euclid II, 5,

and

$x - y = 15$.

From this follow the solutions 3 and 18.[17] The end points of the segment (abcd...or abcd...); for example, *a.b* signifies a segment. Leonardo, however, also speaks about the numbers *a.b.c.d.*, by which he means (ab). (cd). Sometimes, through, only a single letter is given for the entire segment.

The second section first presents applications of the Pythagorean theorem, such as the ancient Babylonian problem of a pole leaning against a wall and the Indian problem of two towers of different heights. On the given line joining them (i.e., their bases) there is a spring which shall be equally distant from the tops of the towers. The same problem was solved in chapter 13 by the method of false position. Many different types of problems follow, such as the solution of an indeterminate equation $x^2 + y^2 = 25$, given that $3^2 + 4^2 = 25$; or problems of the type *de viagiis*, in which the merchant makes the same profit on each of his journeys. Geometric and stereometric problems are also presented; thus, for example, the determination of the amount of water running out of a receptacle when various bodies, including a sphere (with $\pi = 3^1/_7$), are sunk in.

The third section contains algebratic quadraic problems (*questiones secundum modum algebre*). First, with reference made to "Maumeht," i.e., to al-Khwārizmī, the six normal forms $ax^2 = bx$, $ax^2 = c$, $bx = c$, $ax^2 + bx = c$, $ax^2 + c = bx$ (here Leonardo is acquainted with both solutions), and $ax^2 = bx + c$ are introduced; they are then exactly computed in numerous, sometimes complicated, examples. Frequently what is sought is the factorization of a number, usually 10, for example,

Another problem is ,and still another is Here Leonardo represents x^2 as a square divided into three rectangular parts: , $2x$, and 4. With the aid of Euclid II, 6 he then obtains . Since in the problem a "fortune" (x^2 = avere) was sought, the final solution is . Leonardo also includes equations of higher degrees that can be reduced to quadratics. For example, it is given that(1)$y = 10/x$; (2)$z = y^2/x$; and (3)$z^2 = x^2 + y^2$. This leads to $x^8 + 100x^4 = 10{,}000$. The numerical examples are taken largely from the algebra of al-Khwārizmī and al-Karajī,[18] frequently even with the same numerical values. In this fourth section of the *Liber abbaci* there also appear further names for the powers of the unknowns.

When several unknowns are involved, then (along with *radix* and *res* for x)a third unknown is introduced as *pars*("part," Arabic, *qasm)*; and sometimes the sum of two unknowns is designated as *res*. For x^2, the names *quadratus, census, and aver*("wealth," Arabic,*māl*)[19] are employed; for x^3, *cubus*; for x^4*census de censu* and *censuum census*; and for x^6, *cubus cubi*. The constant term is called *numerus, denarius, or dragma*.

Practica geometriae (Scritti, II, 1-224). This second work by Leonardo, which he composed in 1220 or 1221, between the two editions of the *Liber abbaci*, is dedicated to the Magister Dominicus mentioned above. Of the nine extant manuscripts one is in Rome, which Boncompagni used, and two are in Paris.[20] In this work Leonardo does not wish to present only measurement problems for the layman; in addition, for those with scientific interests, he considers geometry according to the method of proof. Therefore, the models are, on the one hand, Hero and the *Agrimensores*, and Euclid and Archimedes on the other. Leonardo had studied the *Liber embadorum* of Plato of Tivoli(1145) especially closely and took from it numerical values. This work by Plato was a translation of the geometry of Savassorda(Abraham bar Hiyya), written in Hebrew, which in turn reproduced Arabic knowledge of the subject.

The *Practica* is divided into eight chapters(distinctiones), which are preceded by an introduction. In the latter the basic concepts are explained, as are the postulates and axioms of Euclid(including the spurious axioms 4, 5, 6, and 9) and the linear and surface measures current in Pisa.[21] The first chapter presents, in connection with the surfaces of rectangles, examples of the multiplication of segments, each of which is given in a sum of various units(rod, foot, ounce, etc.). The propositions of book II of the *Elements* are also recalled. The second chapter and the fifth chapter treat, as a preparation for the following problems, square and cube roots and calculation with them in a manner similar to that of the *Liber abbaci*. Next, the duplications of the cube by Archytas, Philo of Byzantium, and Plato, which are reported by Eutocius, are demonstrated, without reference to their source. The solutions of Plato and Archytas, Leonardo took from the *Verba filiorum* of the Banü Müsä, a work translated by Gerard of Cremona. That of Philo appears also in Jordanus de Nemore's *De triangulis*, and probably both Leonardo and Jordanus took it from a common source.(See M. Clagett,*Archimedes in the Middle Ages, I*, 224, 658-660.) The third chapter provides a treatment with exact demonstrations of the calculation of segments and surfaces of plane figures: the triangle, the square, the rectangle, rhomboids (*rumboides*), trapezoids(*figurae quae habent capita abscisa*), polygons, and the circle; for the circle, applying the Archimedean polygon of ninety-six sides, π is determined as 864:275 ~ 3.141818.... In addition, Leonardo was acquainted with quadrilaterals possessing a reentrant angle(figura barbata)in which a diagonal falls outside the figure.

Many of the problems lead to quadratic equations, for which the formulas of the normal forms are used. They are given verbally. Hence, for example, in the problem $4x - x^2 = 3$, we are told: If from the sum of the four sides the square surface is subtracted, then three rods remain. Attention is also drawn here to the double solution. Along with this, Leonardo gives practical directions for the surveyor and describes instrumental methods, such as can be used in finding the foot of the altitude of a triangular field or in the computation of the projection of a field lying on a hillside. Among the geodetic instruments was an archipendulum. With the help of it and a surveyor's rod, the horizontal projections of straight lines lying inclined on a hillside could be measured. For the surveyor who does not understand the Ptolemaic procedure of determining half-chords from given arcs, appropriate instructions and a table of chords are provided. This is the only place where the term *sinusversus arcus*, certainly borrowed from Arabic trigonometry, appears. The fourth chapter is devoted to the division of surfaces; it is a reworking of the *Liber embadorum*, which ultimately derives from Euclid's lost *Book on Divisions of Figures*; the latter can be reconstructed(see Archibald) from the texts of Plato of Tivoli and of Leonardo and from that of an Arabic version. In the sixth chapter Leonardo discusses volumes, including those of the regular polyhedrons, in connection with which he refers to the propositions of book **XIV** of Euclid. The seventh chapter contains the calculation of the heights of tall objects, for example, of a tree, and gives the rules of surveying based on the similarity of triangles; in these cases the angles are obtained by means of a quadrant.

The eighth chapter presents what Leonardo had termed "geometrical subtleties"(subtilitates) in the preface to the *Liber abbaci*. Among those included is the calculation of the sides of the pentagon and the decagon from the diameter of circumscribed and inscribed circles; the inverse calculation is also given, as well as that of the sides from the surfaces. There follow two indeterminate problems: $a^2 + 5 = b^2$ and $c^2 - 10 = d^2$. The *Liber quadratorum* treats a similar problem: $a^2 + 5 = b^2$, together with $a^2 - 5 = c^2$ Finally, to complete the section on equilateral triangles, a triangle and a square are inscribed in such a triangle and their sides are algebraically calculated, with the solution given in the sexagesimal system.

Flos (Scritti, II, 227–247). The title of this work, which–like two following ones–is preserved in a Milanese manuscript of 1225, is INCIPIT *flos Leonardibigoli pisani super solutionibus quarmdam questionum ad numerum et ad geometriam vel ad utrumque pertintium*. Sent to Frederick II, it contains the elaboration of questions that Master Johannes of Palermo posed in the emperor's presence in Pisa. The work had been requested by Cardinal Raniero Capocci da Viterbo; Leonardo, moreover, provided him with additional problems of the same type. For the first problem (involving the equations $x^2 + 5 = y^2$ and $x^2 -5 =z^2$) only the solution is presented; it is treated in the *Liber quadratorum*. The second question that Master Johannes had posed $x^3 + 2x^2 + 10x = 20$. Leonardo, who knew book X of the *Elements thoroughly*, demonstrates that the solution can be neither a whole number, nor a fraction, nor one of the Euclidean irrational magnitudes. Consequently, he seeks an approximate solution. He gives it in sexagesimal form as $1° \, 22'' \, 7'' \, 42''' \, 33^{IV} \, 4^{V} \, 40^{VI}$, the 40 being too great by about 1 1/2. We know only that the same problem appears in the algebra of alKhayyāmī,[22] where it is solved by means of the intersection of a circle and a hyperbola. One may suppose that the solution follows from the Horner method, which was known to the Chinese and the Arabs.

Next Leonardo presents a series of indeterminate liner problems. If the first of these (*tres homines pecuniam communem habentes*),[23] which had already been solved by various methods in the *Liber abbaci*, was really posed by Master Johannes, then he must have taken it from the algebra of al-Karjī.[24] The following examples are well- known from the *Liber abbaci* as "the found purse" and "one alone cannot but" problems. Here, too, negative solutions are given. In one problem with six unknowns, one of them is chosen arbitrarily, while *causa* and *res* are taken for two of the others.

Letter to Mater Theodours (undated; Scritti, II, 247-252). The principal subject of the letter is the "problem of the 100birds", which Leonardo had already discussed in the *Liber abbaci*. This time, however, Leonardo develops a general method for the solution of indeterminate problems. A geometrical problem follows that is reminiscent of the conclusion of the *Practica geometriae*. A regular pentagon is to algebra in a model for the early application of algebra in geometry. The solution is carried through too the point where a quadratic equation is reached, and then an approximate value is determined—again sexagesmally. The letter concludes with a liner problem with five unknows; instead of a logically constructed calculation, however, only a machanical formula is given.

Liber quadratorum (Scritti, II, 253-279). This work, composed in 1225, is a first-rate scientific achievement and shows Leonardo as a major number theorist. Its subject, which had already appeared among the Arabs[25] and was touched upon at the end of the *Practica geometriae* and in the introduction to the *Flos*, is the question, proposed by Master Johannes, of finding the solution of two simultaneous equations $x^2 + 5 = y^2$ and $x^2 - 5 = z^2$, or $y^2 - x^2 = x^2 - z^2 = 5$. The problem itself does not appear until late in the text; before that Leonardo develops propositions for the determination of Pythagorean yields a square. He first considers the odd numbers from 1 to $(a^2 - 2)$ for odd a; the sum is . If a^2 is added to this expression, then another square results, . For even a the corresponding relation is

Leonardo was acquainted with still further number triples, such as the Euclidean: $2pq, p^2 - q^2, p^2 + q^2$; and he had already given another one in the *Liber abbaci*[26].

He obtains still more triples in the following manner: if $(a^2 + b^2)$ and $(x^2 + y^2)$ are squares and if, further, a:b ≠ x:y and a:b ≠ y:x, then it is true that $(a^2 + b^2)\cdot(x^2 + y^2) = (ax + by)^2 + (bx - ay)^2 = (ay + bx)^2 + (by + ax)^2$. The problem was known to

Diophantus, and a special case exists in a cuneiform text from Susa. Next Leonardo introduces a special class of numbers: $n = ab\cdot(a + b)\cdot(a - b)$ for even $(a + b)$, and $n = 4ab\cdot (a + b)\cdot (a - b)$ for odd $(a + b)$. He names such a number *congruum* and demonstrates that it must be divisible by 24. He finds that $x^2 + h$ and $x^2 - h$ can be squares simultaneously only if h is a *congruum*. For $a = 5$ and $b = 4$, $h = 720 = 5 \cdot 12^2$. The problem now, therefore, is to obtain two differences of squares $y^2 - x^2 = x^2 — z^2 = 720$. He determines that $2401 — 1681 = 1681 — 961$, or $49^2 - 41^2 = 41^2 - 31^2$. Following division by 12^2 he gets

and

One does not learn how Leonardo obtains the squares 961, 1681, and 2401; however, one can ascertain it from a procedure in Diophantus.[27] Leonardo then proves a further series of propositions in number theory, such as that a square cannot be a *congruum*, that $x^2 + y^2$ and $x^2 — y^2$ cannot simultaneously be squares, that $x^4 — y^4$ cannot be a square, etc. Next Leonardo considers expressions such as the following; $x + y + z + x^2$, $x + y + z + x^2 + y^2$, and $x + y + z + x^2 + y^2 + z^2$. They are all to be squares and they are to hold simultaneously. This was another of Master Theodorus' questions. In the questions treated in the *Liber quadratorum*, Leonardo was long without a successor.

In surveying Leonardo's activity, one sees him decisively take the role of a pioneer in the revival of mathematics in the Christian West. Like no one before him he gave fresh consideration to the ancient knowledge and independently furthered it. In arithmetic he showed superior ability in computations. Moreover, he offered material to his readers in a systematic way and ordered his examples from the easier to the more difficult. His use of the chain rule in the "Rule of Three" is a new development; and in the casting out of nines he no longer finds the remainder solely by division, but also employs the sum of digits. His rules for factoring numbers and the formation of perfect numbers are especially noteworthy, as is the recurrent series in the "rabbit problem." He treated indeterminate equations of the first and second degrees in a manner unlike that of anyone before him; ordinarily he confines himself to whole-number solutions—in contrast with Diophantus— where such are required. In geometry he demonstrates, unlike the *Agrimensores*, a thorough mastery of Euclid, whose mathematical rigor he is able to recapture, and he understands how to apply the new methods of algebra to the solution of geometric problems. Moreover, in his work a new concept of number seems to be emerging, one that recognizes negative quantities and even zero as numbers. Thus, on one occasion[28] he computes $360 — 360 = 0$ and $0:2 = 0$. Especially to be emphasized is his arithmetization of the Euclidean propositions and the employment of letters as representatives for the general number.

Leonardo's Sources. Early in his youth Leonardo already possessed the usual knowledge of a merchant of his time, as well as that preserved from the Roman tradition (abacus, surveying, formulas, etc.). Then came his journeys. What he absorbed on them cannot in most cases be determined in detail. The knowledge of the Greeks could have reached him either from the already existing Latin translations of the Arabic treatments or in Constantinople, where he had been. One can, to be sure, establish where individual problems and methods first appear, but one cannot decide whether what is involved is the recounting of another's work or an original creation of Leonardo's. The only clear cases are those in which a problem is presented with the same numerical values or when the source itself is named.

Leonardo is fully versed in the mathematics of the Arabs; for example, he writes mixed numbers with the whole numbers on the right. Algebra was available to him in the translations of the works of al-Khwarizml by Adelard of Bath, Robert of Chester, and Gerard of Cremona or in the treatment by Johannes Hispalensis. The numerical examples are frequently taken directly from the algebra of al-Khwarizmi or from the *Liber embadorum* of Plato of Tivoli, e.g., the paradigm $x^2 + 10x = 39$. The calculation with irrationals and the relevant examples correspond to those in the commentary on Euclid by al-Nayrīzī (Anaritius), which Gerard of Cremona had translated. Countless problems are taken, in part verbatim, from the writings of Abu-Kāmil and of al-Karajī. The cubic equation in the *Flos* stems from al-Khayyāmī. Leonardo readily refers to the Arabs and to their technical words, such as *regula elchatayn* (double false position), *numerus asam* (the prime number), and *figura cata* (which he uses in connection with the chain rule); this is the "figure of transversals" in the theorem of Menelaus of Alexandria.

The geometry of the Greeks had become known through the translations from the Arabic of Euclid's *Elements* by Adelard of Bath, Hermann of Carinthia, and Gerard of Cremona, through al-Nayrīzī's commentary, and perhaps to some extent through the anonymous twelfth-century translation of the *Elements* from the Greek (see *Harvard Studies in Classical Philology*, 71 [1966], 249-302); for the measurement of circles there existed the translations of Archimedes' work by Plato of Tivoli and Gerard of Cremona [29] For the geometric treatment of the cone and sphere, of the measurement of the circle and triangle (with Hero's formula), and of the insertion of two proportional means, the *Verba filiorum* of the Banū Mūsā was available in Gerard of Cremona's translation and was used extensively by Leonardo. On the other hand, problems from the arithmetic of Diophanuts could have come only from Arabic mathematics or from Byzantium. On this subject Leonardo had obtained from the "most learned Master Mucus" a complicated problem of the type "one alone cannot buy", which is also represented in Diophantus. (That Leonardo actually had access to the Greek is shown by his rendering of ῥητοί as "riti".) Other problems that point to Byzantium are those of the type "giving and taking" and the "well problems", which had already appeared in the arithmetical epigrams of *The Greek Anthology*.

Leonardo also includes problems whose origin lies in China and India, such as the *Ta yen* rules, remainder problems, the problem of the "100 birds", and others. Concerning the course of their transmission, nothing definitive can be said. Nevertheless, they were most likely (like the "100 birds" problem found in Abū Kāmīl) transmitted through the Arabs. Problems that appeared in ancient Babylonia (quadratic equations, Pythagorean number triples) or in Egypt (unit fraction calculations, "the seven old women") had been borrowed from the Greeks.

Influence. With Leonardo a new epoch in Western mathematics began; however, not all of his ideas were immediately taken up. Direct influence was entered only by those portions of the *Liber abbaci* and of the *Practice* that served to introduced Indian-Arabic numerals and methods and contributed to the mastering of the problems of daily life. Here Leonardo became the teacher of the masters of computation (*the mastoid' abbaco*) and of the surveyors, as one learns from the *Summa* of Luca Pacioli, who often refers to Leonardo. These two chief works were copied from the fourteenth to the sixteenth centuries. There are also extracts of the *Practuica*, but they are confined to the chapters on plane figures and surveying problems; they dispense with extract proofs and with the *subtilitaes* of the eight chapter.

Lenardo was also the teacher of the "Consists", who took their name from the word *causa*, which was used for the first time in the West by Leonardo in place of *res* or *radix*. His alphabetical designation for the general number or coefficient was first improved by Viète(1591), who used consonants for the known quantities and vowels for the unknowns.

Many of the problems treated in the *Liber abbaci*, especially some of the puzzle problems of recretional arithmetic, reappeared in manuscripts and then in printed arithmetics of later times; e.g, the problem types known as "giving and taking", "hare and hound", "horse buying", "the found purse", "number guessing", "the twins' inheritance", and the indeterminate problem of the "100 birds", which reappread as the "rule of the drinkers" (*regular coeds, regular potato rum*) and whose solution Euler established in detail in his algebra (1767). Cardano, in his *Artis arithmeticae tracatus de integris*, mentions appreciatively Leonardo's achievements when he speaks of Pacioli's *Summa*. One may suppose, he states, that all our knowledge of non-Greek mathematics owes its existence to Leonardo, who, long before Pacioli, took it from the Indians and Arabs.[30]

In his more advanced problems of number theory, especially in the *Liber quadratorum*, Leonardo at first had no successor. This situation lasted until the work of Diophantus became available in the original text and was studied and edited by Bachet de Meziriac(1621); he, and then Format, laid the foundation for modern number theory. Leonardo, however, remained forgotten. Commanding's plan to edit the *Practica* was not carried out. While the historians Heilborner (1742) and Montucla (1758) showed their ignorance of Leonardo's accomplishments, Cossali (1797)placed him once more in the proper light; however, since the texts themselves could not be found, Cossali had to rely on what was available in Pacioli. It is thanks to Libri and Boncompagni that all five of Leonardo's works are again available.

NOTES

1. In Italian his title is depute della patria pubblico (Bibliotheca Meglibechiana, Florence, Palchetto III, no. 25) and publicocancelliere (BibliotecaComunale, Siena, L. IV.21).

2. Scitti, I, 1; "quasi errorem computavi respect modi indordum".

3.*Ibid*,: "quedam etiam ex subtilitatibus euclidis geometrice artis apponens".

4. On the dating, See Cantor, Vorlesungen über Geschichte der Mathematik, II,41.

5. IIlustration in Arrighi, *Leonardo Fibonacci*, p. 15.

6. Boncompagni, *Introno ad alcune opera di Leonardo Piano, p* 248.

7.*Ibid*., p.246: "ideo ipsum X^{m} librum glosare incepi, reduces intellect um ipsius ad numerum qui in eo peer lines et superficies demonstratur".

8. He should "hold the numbers in his hand" ("retinere in manu", Scritti, I, 7). Thus the student "should bring memory and understanding into harmony with the hands and the numerals" (Scritti, I, 1).

9. Youschkevitch, *Geschichte der Mathematik im Mittelater*, p. 377; Vogal, "Zur Geschichte der linearen Gleichungen mit mehreren Unbekannten."

10. See Vogel, *En byantinisches Rechenbuch des fruhen 14. Jahrhunderts*, p. 157.

11. See Canor, *op. cit*, p. 26.

12.*Scritti*, I, 318: "adpropinquacio veritati."

13.*Ibid*., 191: "una res et denarii 12 sunt septuplum quinque rerum et de denariis 12."

14.*Ibid*., 228, 351.

15.*Ibid*., 228, 352.

16. *Ibid.*, 378: “inveni hunc modum reperiendi radices.”

17. *Ibid.*, 396.

18. A list of the common examples is in Woepke, *Extrait du Fakhrī, Traité d’algèbre par Aboū Bekr Mohammed ben Alhaçan Alkarkhī*, p. 29.

19. *Scritti*, I, 442 ff.

20. See Libri, *Histoire des sciences mathematiques en Italie*, II, 305.

21. See Arrighi, op. cit., p. 18.

22. See Woepke, *L’algèbre d’Omar Alkayyāmī*, p. 78.

23. See Cantor, *op. cit.*, pp. 48 f.

24. See Woepke, *Extrait du Fakhrí...*, pp. 141 ff.

25. See Youschkevitch, *op. cit.*, p. 235.

26. *Scritti*, I, 402. The example there involves $x^2 + y^2 = 41$.

27. See Ver Eecke, *Léonard de Pise*, p. 44.

28. *Scritti*, I, 296.

29. See Clagett, *Archimedes in the Middle Ages*, I, *The Arabo-Latin Tradition*, chs. 1, 2.

30. G. Cardano, *Opera*, X (Lyons, 1663), 118, col. 2.

31. See Loria, *Storia delle matematiche*, I, 383.

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