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(*b*. Moschin, near Posen, Germany [now Poznan, Poland], 5 May 1833; *d*. Berlin, Germany, 26 April 1902)

mathematics.

Fuchs was a gifted analyst whose works form a bridge between the fundamental researches of Cauchy, Riemann, Abel, and Gauss and the modern theory of differential equations discovered by Poincaré, Painlevé, and Picard.

By the time Fuchs was a student at the Friedrich Wilhelm Gymnasium, his unusual aptitude in mathematics had awakened a corresponding interest in the discipline. At the University of Berlin he studied with Ernst Eduard Kummer and Karl Weierstrass, and it was the latter who introduced him to function theory, an area that was to play an important role in his own researches. Fuchs received the doctorate from Berlin in 1858, then taught first at a Gymnasium and later at the Friedrich Werderschen Trade School. In 1865 he went to the University of Berlin as a *Privatdozent*, and there he began the study of regular singular points. In 1866 he became an extraordinary professor at the university. From 1867 to 1869 Fuchs was professor of mathematics at the Artillery and Engineering School and then went as ordinary professor to Greifswald. He remained there five years, spent one year at Göttingen and then, in 1875, went to Heidelberg. In 1882 Fuchs returned to Berlin, where he became professor of mathematics, associate director of the mathematics seminars, and a member of the Academy of Sciences. He remained at Berlin until his death. For the last ten years of his life he was editor of the *Journal für die reine und angewandte Mathematik*.

Except for a few early papers in higher geometry and number theory, all of Fuchs's efforts were devoted to differential equations.

In his monumental 1812 work, *Disquisitionesgenerales circa seriem infinitam*, Gauss investigated the hypergeometric series and noted that, for appropriately chosen parameters, most known functions could find representation through this series. In 1857 Riemann conjectured that functions so expressed, which satisfy a homogeneous linear differential equation of the second order with rational coefficients, might be employed in the solution of any linear differential equation. This provided an alternate approach to the power series development which had been presented by Cauchy and extended by Briot and Bouquet. With Riemann's work as inspiration these methods were synthesized and extended by Fuchs in a series of papers that began to appear in 1865.

In the real domain the method of successive approximations was first applied by Liouville (1838) to homogeneous linear differential equations of the second order and later (1864) by M. J. Caqué to the n th-order case. A second method of proving the existence of solutions derives from a method suggested by Euler (1768), developed by Cauchy (1820–1830), and refined by Lipschitz in 1876.

For a first-order equation, $dw/dz = f(z,w)$, the Cauchy-Lipschitz method can be extended to the complex domain, as can the method of limits. It was Fuchs, however, who provided the proof of the existence of solutions, satisfying initial conditions for $z = z_0$, for the linear differential equation of order n. The general homogeneous linear differential equation of order n has the form

and it is assumed that the $p_i\ (z)$ are analytic throughout a domain D in the Z plane. With z_0 and z in D, c_r are chosen so that the Taylor series,

formally satisfies the differential equation. The c_r are shown to be finite as long as the initial values are finite. Furthermore, if M_ν is the upper bound of $|p^\nu|$ on $|z - z_0| = a$, then , so that if

then

within $|Z-Z_0| = a$ and on the circumference.

If now w is replaced by W in equation (1), where, as above,

where $C_i = |c_i|$, then

and, hence

But the circle of convergence of the dominant series can, as Fuchs showed, be readily found to be $|z - z_0| = a$. Consequently there is a solution to the differential equation, satisfying the given initial conditions, when $z=z_0$, which is expressible by a Taylor series that is absolutely and uniformly convergent within any circle, that has its center at z_0, in which the $p_i (z)$ are analytic.

The singularities of the solution are precisely the singularities of equation (1), so that, because of the linearity of this equation, there are neither movable singularities nor movable poles. To determine whether the point at infinity is a singular point, it is necessary only to effect the substitution $z = I/Z$ and reduce the equation to the form given by (1). If the equation in Z has singularities at the origin, then the equation in z has a singular point at infinity. Thus, in all cases the singular points can be found simply by inspecting the equation itself.

Fuchs introduced the term "fundamental system" to describe n linearly independent solutions of the linear differential equation $L (u) = 0$. It is clear that for any nonsingular point such a fundamental set of solutions exists. The so-called Fuchsian theory is concerned with the same existence question in relation to an arbitrarily given singular point and, once the existence problem is solved, an investigation of the behavior of the solutions in the neighborhood of the singular point.

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Jerome H. Manheim