Hopf, Heinz | Encyclopedia.com

(*b*. Breslau, Germany [now Wrocław, Poland], 19 November 1894; *d*. Zollikon, Switzerland, 3 June 1971)

mathematics.

Hopf attended school and started his university study of mathematics in his birthplace, but his studies were soon interrupted by a long period of military service during World War I. A fortnight's leave in the summer of 1917 determined his mathematical future: he ventured into Erhard Schmidt's set theory course at the University of Breslau and became fascinated by Schmidt's exposition of L. E. J. Brouwer's proof of the dimension invariance by means of the degree of continuous mappings.

In 1920 Hopf followed Schmidt to Berlin where, with topological research, he earned his Ph.D. in 1925 and his *Habilitation* in 1926. At Göttingen in 1925 he became acquainted with Emmy Noether and met the Russian mathematician P. S. Alexandroff, with whom he formed a lifelong friendship. Rockefeller fellowships enabled the two friends to spend the academic year 1927–1928 at Princeton University, where topology was fostered by O. Veblen, S. Lefschetz, and J. W. Alexander. In 1931 Hopf was appointed a full professor at the Eidgenössische Technische Hochschule in Zurich, assuming the chair of Weyl, who had gone to Göttingen.

The greater part of Hopf's work was algebraic topology, motivated by vigorous geometric intuitions. Although the number of his papers was relatively small, no topologist of that period inspired as great a variety of important ideas, not only in topology, but also in quite varied domains. He was awarded many honorary degrees and memberships in learned societies. From 1955 to 1958 he was the president of the International Mathematical Union.

Hopf was a short, vigorous man with cheerful, pleasant features. His voice was well modulated, and his speech slow and strongly articulated. His lecture style was clear and fascinating; in personal conversation he conveyed stimulating ideas. With his wife Anja, he extended hospitality and support to persecuted people and exiles.

After Brouwer created his profound "mixed" method in topology, Hopf was the first to continue Brouwer's work on a large scale. He focused on the mapping degree and the mapping class (homotopy class), which had been mere tools in Brouwer's work. Hopf set out to prove that Brouwer's mapping degree was a sufficient homotopy invariant for mappings of spheres of equal dimension (2, nos. 5, 11, 14) and in this context he studied fixed points (2, no. 8) and singularities of vector fields (2, no. 6). His initially crude and too directly geometric methods underwent gradual refinement, first by Emmy Noether's abstract algebraic influence, then through the combinatorial ideas of the American school. In 1933 his efforts culminated in the development of a complete homotopy classification by homology means of mappings of n-dimensional polytopes into the n-dimensional sphere S^n (2, no. 24).

Hopf's study of vector fields led to a generalization of and a formula about the integral curvature (2, no. 2), as a mapping degree of normal fields (1925). An extension of Lefschetz' fixed point formula (2, no. 9) was the result of work done in 1928. As a new and powerful tool to investigate mappings of manifolds, Hopf defined the inverse homomorphism (2, no. 16) using the Cartesian product of the related manifolds—a device he took from Lefschetz. In fact Hopf's 1930 paper on this subject goes back to his stay at Princeton. Not until the arrival of cohomology and the cohomology products was the inverse homomorphism better understood and more firmly integrated into algebraic topology (5).

Hopf's next great topological feat was the 1931 publication (2, no. 18) on an infinity of homotopy classes of S^3 into S^2, and the definition of the "Hopf invariant" for these mappings. As early as 1927 Hopf conjectured that the "Hopf fiber map" was homotopically essential, but the tool to prove this conjecture had still to be created: the idea of considering inverted mappings. Hopf's work on this subject was influential in W. Hurewicz' shaping the concept of homotopy groups, and in particular in his investigation (1935–1936) of homotopy groups of fiber spaces (4). H. Freudenthal, by a synthesis of Hopf's and Hurewicz' work, proved the completeness of Hopf's classification and discovered the suspension (6). From these beginnings homotopy of spheres developed after World War II into a growing field of research, to which Hopf himself had contributed (1935) the investigation of the case of mappings of S^{2n-1} into S^n (2, no. 26).

Vector fields and families of vector fields remained a concern of Hopf's. He stimulated Stiefel's work (7), which led to the discovery of what is now called Stiefel-Whitney classes, and that of B. Eckmann as well (8, 9). Hopf's 1941 paper on bilinear forms (2, no. 38) fits into the same context, as does his influential discovery (1948) of the concept of almost complex manifolds (2, no. 52), which, among other things, led to his 1958 paper with F. Hirzebruch (2, no. 66). Hopf's most important contribution to this area of mathematics is his paper, published in 1941, but begun in 1939, on the homology of group manifolds (2, no. 40), in which he proved the famous theorem that compact manifolds with a continuous multiplication with unit (now called H-manifolds) have a polynomial cohomology ring with all generators of odd dimension. The theorem had already been known by Lie groups methods for the four big classes. Hopf formulated the theorem in terms of homology; his tool was again the inverse homomorphism—Hopf did not like and never became fully acquainted with cohomology. He wrote a few more papers on this subject (2, nos. 41, 46) and instigated H. Samelson's 1942 investigations (10).

In 1936 Hurewicz (4) had proved that in polytopes with trivial higher homotopy groups the fundamental group uniquely determines the homology groups, raising the question of how the one determines the others. Hopf tackled the problem in his papers (2, nos. 40, 45, 49) of 1942 and 1944, which led to independent investigations of Eckmann (11), S. Eilenberg and S. MacLane (12), and Freudenthal (13). The result was the cohomology of groups, the first instance of cohomological algebra, which has since developed into a broad new field of mathematics.

Another beautiful idea of Hopf's was transferring Freudenthal's concept of ends of topological groups to spaces possessing a discontinuous group with a compact fundamental domain (2, no. 47), which Freudenthal in turn converted into a theory on ends of discrete groups with a finite number of generators (14).

Hopf was also interested in global differential geometry. With W. Rinow he contributed the concept of a complete surface (2, no. 20); with Samelson (2, no. 32) a proof of the congruence theorem for convex surfaces; with H. Schilt (2, no. 34) a paper on isometry and deformation; and he studied relations between the principal curvatures (2, no. 57).

Two beautiful papers of Hopf's that should be mentioned are one on the turning around of the tangent of a closed plane curve (2, no. 27) and one on the set of chord lengths of plane continua (2, no. 31), published in 1935 and 1937 respectively. Hopf was also interested in number theory, which he enjoyed teaching and to which he devoted a few papers.

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