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(*b*. Turin, Italy, 25 January 1736; *d* Paris, France, 10 April 1813)

mechanics, celestial mecahanics, astronomy, mathematics.

Lagrange's life divides very naturally into three periods. The first comprises the years spent in his native Turin (1736–1766). The second is that of his work at the Berlin Academy, between 1766 and 1787. The thirds finds him in Paris, from 1787 until his death in 1813.

The first two periods were the most fruitful in terms of scientific activity, which began as early as 1754 with the discovery of the calculus of variations and continued with the application of the latter to mechanics in 1756. He also worked in celestial mechanics in this first period, stimulated by the competitions held by the Paris Academy of Sciences in 1764 and 1766.

The Berlin period was productive in mechanics as well as in differential and integral calculus. Yet during that time Lagrange distinguished himself primarily in the numerical and algebraic solution of equations, and even more in the theory of numbers.

Lagrange's years in Paris were dedicated to didactic writings and to the composition of the great treatises summarizing his mathematical conceptions. These treatise, while closing the age of eighteenth-century mathematics, prepared and in certain respects opened that of the nineteenth century.

Lagrange's birth and baptismal records give his name as Lagrangia, Giuseppe Lodovico, and declare him to be the legitimate son of Giuseppe Francesco Lodovico Lagrangia and Teresa Grosso. But from his youth he signed himself Lodovico LaGrange or Luigi Lagrange, adopting the French spelling of the patronymic. His first published work, dated 23 July 1754, is entitled "Lettera di Luigi De la Grange Tournier," Until 1792 he and his correspondents frequently employed the particle, very common in three words: de la Grange. The contract prepared for his second marriage (1792) is the name of Monsieur Joseph-Louis La Grange, without the particle. In 1814 the *éloge* written for him by Delambre, the permanent secretary of the mathematics section of the Institut de France, was entitled "Notice sur la vie et les ouvrages de M. le Comte J. L. Lagrange' and his death certificate designated him Monsieur Joseph Louis Lagrange, sénateur. As for the surname Tournier, he used it for only a few years, perhaps to distinguish himself from his father, who held office in Turin.

His family was, through the male members, of French origin, as stated in the marriage contract of 1792. His great-grandfather, a cavalry captain, had passed from the service of France to that of Charles Emmanuel II, duke of Savoy, and had married a Conti, from a Roman family whose members included Pope Innocent XIII.

His grandfather—who married Countess Bormiolo—was Treasurer of the Office of Public Works and Fortifications at Turin. His father and later one of his brothers held his office, which remained in the family until its suppression in 1800. Lagrange's mother, Teresa Gros, or Grosso, was the only daughter of a physician in Cambiano, a small town near Turin. Lagrange was the eldest of eleven children, most of whom did not reach adulthood.

Despite the official position held by the father—who had engaged in some unsuccessful financial speculations—the family lived very modestly. Lagrange himself declared that if he had had money, he probably would not have made mathematics his vocation. He remained with his family until his departure for Berlin in 1766.

Lagrange's father destined him for the law—a profession that one of his brothers later pursued—and Lagrange offered no objections. But having begun the study of physics under the direction of Beccaria and of geometry under Filippo Antonio Revelli, he quickly became aware of his talents and henceforth devoted himself to the exact sciences. Attracted first by geometry, at the age of seventeen he turned to analysis, then a rapidly developing field.

In 1754 Lagrange had a short essay printed in the form of a letter written in Italian and addressed to the geometer Giulio da Fagnano. In it he developed a formal calculus based on the analogy between Newton's binomial theorem and the successive differentiations of the product of two functions. He also communicated this discovery to Euler in a letter written in Latin slightly before the Italian publication. But in August 1754, while glancing through the scientific correspondence between Leipniz and Johann Bernoulli, Lagrange observed that his "discovery" was in fact their property and feared appearing to be a plagiarist and imposter.

This unfortunate start did not discourage Lagrange. He wrote to Fagnano on 30 October 1754 that he had been working on the tautochrone. The first essay is lost, but we know of two later memoirs on the same subject. The first was communicated to the Berlin Academy on 4 March 1767.[1] Criticized by the French Academician Alexis Fontaine des Bertins, Lagrange responded in "Nouvelles réflexions sur les tautochrones."[2]

At the end of December 1755, in a letter to Fagnano alluding to correspondence exchanged before the end of 1754, Lagrange speaks of Euler's *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solution problematis isoperimetrici latissimo sensu accepti*, published at Lausanne and Geneva in 1744. The same letter shows that as early as the end of 1754 Lagrange had found interesting results in this area, which was to become the calculus of variations (a term coined by Euler in 1766).

On 12 August 1755 Lagrange sent Euler a summary, written in Latin, of the purely analytical method that he used for this type of problem. It considered, he wrote in 1806, in varying the *y*'s in the integral formula in *x* and *y*, which should be a maximum or a minimum by ordinary differentiations, but relative to another characteristic δ, different form the ordinary characteristic *d*. It was further dependent on determining the differential value of the formula with respect to this new characteristic by transporting the sign δ after the signs *d* and ∫when it is placed before. The differentials of δy under the ∫signs are then eliminated through integration by parts.

In a letter to d'Alembert of 2 November 1769 Lagrange confirmed that this method of maxima and minima was the first fruit of his studies—he was only nineteen when he devised it—and that he regarded it as his best work in mathematics.

Euler replied to Langrange on 6 September 1755 that he was very interested in the technique. Lagrange's merit was likewise recognized in Turin; and he was named, by a royal decree of 28 September 1755, professor at the Royal Artillery School with an annual salary of 250 crowns—a sum never increased in all the years he remained in his native country.

In 1756, in a letter to Euler that has been lost, Lagrange applied the calculus of variations to mechanics. Euler had demonstrated, at the end of his *Methodus*, that the trajectory described by a material point subject to the influence of central forces is the same as that obtained by supposing that the integral of the velocity multiplied by the element of the curve is either a maximum or a minimum. Lagrange extended "this beautiful theorem" to an arbitrary system of bodies and derived from it a procedure for solving all the problems of dynamics.

Euler sent these works of Lagrange to his official superior Maupertuis, then president of the Berlin Academy. Finding in Lagrange an unexpected defender of his principle of least action, Maupertuis arranged for him to be offered, at the earliest opportunity, a chair of mathematics in Prussia, a more advantageous position than the one he held in Turin. This proposition, transmitted through Euler, was rejected by Lagrange out of shyness; and nothing ever came of it. At the same time he was offered a corresponding membership in the Berlin Academy, and on 2 September 1756 he was elected an associate foreign member.

In 1757 some young Turin scientists, among them Lagrange, Count Saluzzo (Giuseppe Angelo Saluzzo di Menusiglio), and the Physician Giovanni Cigna, founded a scientific society that was the origin of the Royal Academy of Sciences of Turin. One of the main goals of this society was the publication of a miscellany in French and Latin, *Miscellanea Taurinensia ou Mélanges de Turin*, to which Lagrange contributed fundamentally. The first three volumes appeared at the beginning of the summers of 1759 and 1762 and in the summer of 1766, during which time Lagrange was in Turin. The fourth volume, for the years 1766-1769, published in 1733, included four of his memoirs, written in 1767, 1768, and 1770 and sent from Berlin.

The first three volumes contained almost all the works Lagrange Published while in Turin, with the following exceptions; the courses he gave at the Artillery School on mechanics and differential and integral calculus, which remained in manuscript and now appear to have been lost; the two memories for the competitions set by the Paris Academy of Sciences in 1764 and 1766; and his contribution to Louis Dutens's edition of Leibniz' works.

In volume 1 of the *Mélanges de Turin* are Lagrange's "Recherches sur la Méthode de maximis et minimis,"[3] really an introduction to the memoir in volume 2 on the calculus of variations (dating, as noted above, from the end of 1754).

Another short memoir, "Sur l'intégration d'une équation différentielle à différences finies, qui contient la théorie des suites récurrentes," [4] was cited by Lagrange in 1776 as an introduction to investigations on the calculus of probabilities that he was unable to develöp for lack of time. There is also his unfinished and unpublished translation of Abraham de Moivre's *The Doctrine of Chances,* the third edition of which appeared in 1756. Lagrange mentioned this translation—which seems to have been lost—in a letter to Laplace of December1776.[5]

Recherches sur la nature et la propagatin du son[6] constitutes a thorough and extensive study of a question much discussed at the time. In it Lagrange displays an astonishing erudition. He had read and pondered the writings of Newton, Daniel Benoulli, Taylor. Euler, and Alembert; and his own contribution to the problem of vibrating strings makes him the equal of his predecessors.

Work of the same order is presented in “Nouvelles recherches sur la nature et la propagation du son”[7] and “Additions aux premiéres recherches,”[8] both published in volume, though, is “Essai d’une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies,”[9] a rather brief memoir in which Lagrange published his analytic techniques of the calculus of variations. Here he developed the insights contained in his Latin letter to Euler of 1755 and added two appendixes, one dealing with minimal surfaces and the other with polygons of maximal area. Although published in 1762, the memoir and its first appendix were written before the end of 1759.

“Application de la méthode précédente à la solution de différens problèmes de dynamique”[10] made the principle of least action, joined with the theorem of *forces vives* (or *via viva*), the very foundation of dynamics. Rather curiously, Lagrange no longer used the expression “least action”, which he had employed until then, a minor failing due, perhaps, to the death of Maupertuis. This memoir heralded the *Mécanique ana;lytique* of 1788 in its style and in the breadth of the author’s views.

Volume 3 of the *Mélanges de Turin* contains “Solution de différens problèmes de calcul intégral,” [11] An early section treats the integration of a general affine equation of arbitrary order. Lagrange here employed his favorite tool, integration by parts. He reduced the solution of the equation with second member to that of the equation without second member. This discovery dates—as we know from the correspondence with d’Alembert[12]—from about the end of 1764.

Lagrange’s research also encompassed Riccati’s equations and a functional equation, which he treated in a very offhand manner. Examining some problems on fluid motion, he outlined a study of the function later called Laplacian. He was following Euler, but with the originality that marked his entire career.

The consideration of the movement of a system of material points making only infinitely small oscillations around their equilibrium position led Lagrange to a system of linear differential equations. In integrating it he presented for the first time—explicity—the notion of the characteristic value of a linear substitution.

Lagrange finally arrived at applications to the theory of Jupiter and Saturn. In September 1765 he wrote on this subject that, due to lack of time, he was contenting himself with applying the formulas he had just discovered to the variations in the eccentricity and position of the aphelia of the two planets and to those in the inclination and in the position of the nodes of their orbits. These were inequalities that “no one until now has undertaken to determine with all the exactitude” demanded.

Investigations of this kind were related to the prize questions proposed by the Paris Academy of Sciences. In 1762 it established a competition, for 1764, based on the question “Whether it can be explained by any physical reason why the moon always presents almost the same face to us’ and how, by observations and by theory, it can be determined whether the axis of this planet is subject to some proper movement similar to that which the axis of the earth is known to perform, producing precession and nutation”.

In 1763 Lagrange sent to Paris “Recherches sur la libration de la lune dans lesquelles on tâche de résoudre la question proposée par l’Académie royale des sciences pour le prix de l’année 1764.” [13] In this work he provided a satisfactory explanation of the equality of the mean motion of translation and rotation but was less successful in accounting for the equality of the movement of the nodes of the lunar equator and that of the nodes of the moon’s orbit on the ecliptic.

Lagrange also fruitfully applied the principle of virtual velacities, which is intimately and necessarily linked with his techniques in the calculus of variations. He also made it the basis of his *Mécanique analytique* of 1788. This principle has the advantages, over that of least action, of including the latter principle as well as the principle of *forces vives* and thus of giving mechanics a unified foundation. He had not yet achieved a unified point of view in the memoir published in 1762. Arriving at three differential equations, he demonstrated that they are identical to those relating to the precession of the equionoxes and the nutation of the earth’s axis that d’Alembert presented in the *Mémoires* of the Paris Academy for 1754. Lagrange returned to this question and gave a more complete solution of it in figure non sphérique de cette planéte,” included in the *Mémoires* of the Berlin Academy for 1780 (Published in 1782).[14] Laplace wrote to him on this subject on 10 February 1783. “The elegance and the generality of your analysis, the fortunate choice of your coordinates, and the manner in which you treat your differential equation, especially those of the movement of the equinoctial points and of the sublimity of your results, has filled me with admiration.”

In 1763 d’Alembert, then on his way to Berlin, was not a member of the jury that judged Lagrange’s entry. He had already been in correspondence with Lagrange but did not know him Personally. Nevertheless, he had been able to judge of his ability through the *Mélanges de Turin*. In the meantime the Marquis Caraccioli, ambassador from the kingdom of Naples to the court of Turin, was transferred by his government to London. He took along the young Lagrange, who until then seems never to have left the immediate vicinity of Turin.

Lagrange departed his native city at the beginning of November 1763 and was warmly received in Paris, where he had been preceded by his memoir on lunar libration. He may perhaps have been treated too well in the Paris scientific community, where austerity was not a leading virtue. Being of a delicate constitution, Lagrange fell ill and had to interrupt his trip. His mediocre situation in Turin aroused the concern of d’Alembert, who had just returned from Prussia. D’Alembert asked Mme. Geoffrin to intercede with the ambassador of Sardinia at the court of Turin:

Monsieur de la Grange, a young geometer from Turin, has been here for six weeks. He has become quite seriously ill and he needs, not financial aid, for M[r] le marquis de Caraccioli directed upon leaving for England that he should not lack for anything, but rather some signs of interest on the part of his native country… In him Turin possesses a treasure whose worth it perhaps does not know.

In the spring of 1765 Lagrange returned to Turin by way of Geneva and, without attempting to visit Basel to see Daniel Bernoulli, went on d'Alembert's advice to call on Voltaire, who extended him a cordial welcome. Lagrange reported: "He was in a humorous mood that day and his jokes, as usual, were at the expense of religion, which greatly amused the gathering. He is, in truth, a character worth seeing".

D'Alembert's intervention had had some success in Turin, where the king and the ministers held out great hopes to Lagrange—in which he placed little trust.

Meanwhile the Paris Academy of Sciences had proposed for the prize of 1766 the question "what are the inequalities that should be observed in the movement of the four satellites of Jupiter as a result of their mutual attractions…" D'Alembert publicly objected to this subject, which he considered very poorly worded and incorrect, since the actions of the sun on these satellites were completely ignored. His stand on this matter led to a very sharp correspondence between him and Clairaut.

In August 1765 Lagrange sent to the Academy of Sciences "Recherches sur les inégalités des satellites de Jupiter…,[15] which won the prize. He wrote to d'Alembert on 9 September 1765: "what I said there concerning the equation of the center and the latitude of the satellites appears to me entirely new and of very great importance in the theory of the planets, and I am now preparedo apply it to Saturn and Jupiter." He was alluding to the works published in volume 3 of the *Mélanges de Turin*.

The fine promises of the court of Turin had still not been fulfilled. In the autumn of 1765 d'Alembert, who was on excellent terms with Frederick II of Prussia, suggested to Lagrange that he accept a position in Berlin. He replied, "It seems to me that Berlin would not be at all suitable for me while M[r] Euler is there." On 4 March 1766 d'Alembert notified him that Euler was going to leave Berlin and asked him to accept the latter's post. It seems quite likely that Lagrange would gladly have remained in Turin had the king been willing to improve his material and scientific situation. On 26 April, d'Alembert transmitted to Lagrange the very precise and advantageous propositions of the king of Prussia; and on 3 May, Euler, announcing his departure for St. Petersburg, offered him a place in Russia. Lagrange accepted the proposals of the Prussian king and, not without difficulties, obtained his leave at the beginning of July through the intercession of Frederick II with the king of Sardinia.

Lagrange left for Berlin on 21 August 1766, travelling first to Paris and London. After staying for two weeks with d'Alembert, on 20 September he arrived in the English capital, summoned there by Caraccioli. He then embarked for Hamburg and finally reached Berlin on 27 October. On 6 November he was named director of the mathematics section of the Berlin Academy. He quickly became friendly with Lambert and Johann Castillon, who stood sullenly aloof from the Academy when it passed him over for a colleague young enougyh "to be his son."

Lagrange's duties consisted of the monthly reading of a memoir, which was sometimes published in the Academy's *Mémoires* (sixty-three such memoirs were published there), and supervising hte Academy's mathematical activities. He had no teaching duties of the sort he had had in Turin and would have again, although more episodically, in Paris. His financial compensation was excellent, and he never sought to improve on it during the twenty years he was there.

In September 1767, eleven months after his arrival, Lagrange married his cousin Vittoria Conti. "My wife", he wrote to d'Alembert in July 1769, "who is one of my cousins and who even lived for a long time with my family, is a very good housewife and has no pretensions at all". He also declared in this letter that he had no children and, moreover, did not want any. He wrote to his father in 1778 or 1779 that his wife's health had been poor for several years. She died in 1783 after a long illness.

The paris Academy of Sciences had become accustomed to including Lagrange among the competitors for its biennial prizes, and d'Alembert constantly importuned him to participate. The question for 1768, like the one for 1764, concerned the theory of the moon. D'Alembert wrote to him: "This, it seems to me, is a subject truly worthy of yours efforts." But Lagrange replied on 23 February 1767: "The king would like me to compete for your prize, because he thinks that Euler is working on it; that, it seems to me, is one more reason for me not to work on it".

The prize was postponed until 1770. Lagrange excused himself on 2 June 1769: "The illness that I have had these past days, and from which I am still very weakened, has completely upset my work schedule, so that I doubt whether I shall be able to compete for the prize concerning the moon as I had Planned."

Lagrange did, however, participate in the competition of 1772 with his "Essai sur le Probléme des trois corps."[16] The subject was still the theory of the moon. In 1770 half the prize had been awarded to a work composed jointly by Euler and his son Johann Albecht. The question was proposed again for 1772, and the prize was shared by Lagrange and Euler. On 4 April 1771 Lagrange wrote to d'Alembert: "I intend to send you something for the prize. I have considered the three-body problem in a new and general manner, not that I believe it is better than the one previously employed, but only to approach it *allio modo;* I have applied it to the moon, but I doubt very much that I shall have the time to complete the arithmetical calculations."

On 25 March 1772 d'Alembert announced to Lagrange: "You are sharing with M^r Euler the double prize of 5,000 livres,… by the unanimous decision of the five judges MM de Condorcet, Bossut, Cassini, Le Monnier, and myself. We believe we owe this recognition to the beautiful analysis of the three-body problem contained in your piece." In a note on this memoir J. A. Serret wrote:

The first chapter deserves to be counted among Lagrange's most important works. The differential equations of the three-body problem…constitute a system of the twelfth order, and the complete solution required twelve integrations. The only knowns were those of the *force vive* and three from the principle of areas. Eight remained to be discovered. In reducing this number to seven Lagrange made a considerable contribution to the question, one not surpassed until 1873…[17]

For the prize of 1774, the Academy asked whether it were possible to explain the secular equation of the moon by the attraction of all the celestial bodies, or by the effect of the nonsphericity of the earth and of the moon. Lagrange, who was equal to the scope of the subject, felt very stale and at the end of August 1773 withdrew from the contest. At d'Alembert's request Condorcet persuaded him to persevere. He was granted an extension and thanked the jury for this favor in February 1774. He took the prize with "Sur l'équation séculaire de la lune." [18]

The topic proposed for 1776 was the theory of the perturbations that comets might undergo through the action of the planets. Lagrange found the subject unpromising, withdrew, and wrote to d'Alembert in theory of the variations of the elements of the planets resulting from their mutual action." These personal investigations resulted in three studies. One was presented in the *Mémoires* of the Paris Academy: "Recherches sur les équations séculaires des mouvements des noeuds et des inclinaisons des orbites des planétes."[19] Another appeared in the *Mémoires* of the Berlin Academy: "Sur le mouvement des noeuds des orbites planétaires."[20] The third was published in the Berlin *Ephemerides* for 1782. "Sur la diminution de l'obliquité de l'écliptique." [21]

It is understandable that Lagrange, having set out on his own path and being occupied with many other investigations, neglected to enter the competition on the comets. He excused himself by referring to his bad health and the inadequancy of the time allowed, and he pointed out to d'Alembert: "You now have young men in France who could do this work."

Only one entry, from St. Petersburg was submitted and the contest was adjourned until 1778. Lagrange "solemnly" promised to compete this time but sent nothing, and the prize was given to Nicolaus Fuss. The same subject was proposed for 1780, and in the summer of 1779 Lagrange submitted" Recherches sur la théorie des perturbations que les cométes peuvent éprouver par l'action des planétes," [22] which won the double prize of 4,000 livres. This was the last time that he participated in the competitions of the Paris Academy.

Lagrange's activity in celestial mechanics was not confined solely to these competitions: in Turin it has often taken an independent direction. In 1782 he wrote to d'Alembert and Laplace that he was working "a little and slowly" on the theory of the secular variations of the aphelia and of the eccentricities of all the planets. This research led to the *Théorie des variations séculaires des éléments des planètes*[23] and the memoir "Sur les variations séculaires des mouvements moyens des pllanétes," the latter published in 1785.[24] A work on a related subject is "Théorie des variations périodiques des mouvements des planétes," the first part of which, containing the general formulas, appeared in 1785.[25] The second, concerning the six principal planets, was published in 1786.[26]

Lagrange's work in Berlin far surpassed this classical aspect of celestial mechanics. Soon after his arrival he presented "Mémoire sur le passage de Vénus du 3 Juin 1769,"[27] an occasional work that disconcerted the professional astronomers and contained the first somewhat extended example of an elementary astronomical problem solved by the method of three rectangular coordinates. He later returned sporadically to questions of pure astronomy, as in the two-part memoir "Sur le problème de la détermination des comètes d'après trois observations," [28] published in 1780 and 1785, and in some articles for the Berlin *Ephemerides*. Furthermore, in 1767 he wrote "Re-cherches sur le mouvement d'un corps qui est attirévers deux centres fixes,"[29] which grneralized research analogous to that of Euler.

In October 1773 Lagrange composed *Nouvelle solution du problème du mouvement de rotation d'un corps de figure quelconque qui n'est animé par aucune force accélératrice*.[30] "It is," he wrote Condorcet, "a problem already solved by Euler and by d'Alembert … My method is completely different from theirs.. It is, moreover, based on formulas that can be useful in other cases and that are quite remarkable in themselves." His method was constructed, in fact, on a purely algebraic lemma. The formulas he provided—with no proof—in this llemma plertain today to the multiplication of determinants.

A by-product of this study of dynamics was Lagrange's famous *Solutions analytiques de quelques problèmes sur les pyramides triangularies*.[31] Starting from the same formulas as those of the lemma mentioned above, again asserted without proof, he expressed the surface, the volume, and the radii of circumdcribed, inscribed, and escribed spheres and located the center of gravity of every triangular pyramid as a function of the lengths of the six edges. Published in May 1775, this memoir must have been written shortly after the preceding one, perhaps in the fall of 1773. It displays a real duality. Today it would be classed in the field of pure algebra, since it employs what are now called determinants, the square of a determinant, an inverse matrix, an orthogonal matrix, and so on.

From about the same period and in the same vein is “Sur l’attraction des sphéroïdes elliptiques,”[32] in which, after praising the solutions obtained by Maclaurin and d’Alembert with “the geometric method of the ancients that is commonly, although very improperly, called synthesis,” Lagrange presented a purely analytic solution.

Lagrange had devoted several of his Turin memoirs to fluid mechanics. Among them are those on the propagation of sound. The study of the principle of least action, which appeared in volume 2 of the *Mélanges de Turin*, contained about thirty pages dealing with this topic; and “Solution de différens problèmes de calcul intégral” included another sixteen. He returned to this subject toward the end of his stay in Berlin with “Mémoire sur la théorie du mouvement des fluides,”[33] read on 22 November 1781 but not published until 1783. Laplace, before undertaking a criticism of this work, wrote to him on 11 February 1784: “Nothing could be added to the elegance and generality of you analysis”.

Lagrange submitted to the *Mémoires de Turin* for 1784–1785 “Percussion des fluides.”[34] In 1788 he published in the Berlin Academy’s *Mémoires* “Sur la maniére de rectifier deux endroits des principes de Newton relatifs à la propagation du son et au mouvement des ondes.”[35] These works are contemporary with or later than the composition of the *Mécanique analytique*.

Lagrange began works of a very different sort as soon he arrived in Berlin. They are inspired by Euler, whom he always read with the greatest attention.

First Lagrange presented in the *Mémoires* of the Berlin Academy for 1767 (published in 1769) “Sur la solution des problèmes indéterminés du second degré,”[36] in which he copiously cited his predecessor at the Academy and utilized the “Euler criterion.” On 20 September 1768 he sent “Solution d’un probléme d’arithmétique"[37] to the *Mélanges de Turin* for inclusion in volume 4. Through a series of unfortunate circumstances this second memoir was not published until October 1773. In it Lagrange alluded to the preceding memoir, and through a judicious and skillful use of the algorithm of continued fractions he demonstrated that Fermat’s equation $x^2 - ay^2 = 1$ can be solved in all cases where x, y and a are positive integers, a not being a perfect square and y being different from zero. This is the first known solution of its celebrated problem. The last part of this memoir was developed in “Nouvelle méthode pour résoudre les problèmes indéterminés en nombres entiers,”[38] presented in the Berlin *Mémoires* for 1768 but not completed until February 1769 and published in 1770.

On 26 August 1770 Lagrange reported to d’Alembert the publication of teh German edition of Euler’s *Algebra* (St. Petersburg, 1770): “Its containes nothing of interest except for a treatise on the Diophantine equations, which is, in truth, excellent. … If you have the time you could wait for the French translation that they hope to bring out, and to which I shall be able to add some brief notes.” The translation was done by Johann III Bernoulli and sent, and Lagrange’s additions,[39] to Lyhons for publication around May 1771. The entie work appeared in the summer of 1773.

In his additions Lagrange paid tribute to the works of Bachet de Méziriac on indeterminate first-degree equations and again considered the topics discussed in the memoirs cited above, at the same time simplifying the demonstrations. In particular he elaborated a great deal on continued fractions.

Meanwhile, in the Berlin *Mémoires* for 1770 (published 1772) he presented “Démonstration d’un théorème d’arithmétique.”[40] On the basis of Euler’s unsuccessful but nevertheless fruitful attempts, he set forth the first demonstration that every natural integer is the sum of at most four perfect squares.

On 13 June 1771 Lagrange read before the Berlin Academy “Démonstration d’un théorème nouveau concernant les nombres premiers.”[41] The theorem in question was one devolped by Wilson that had simply been stated in Edward Waring’s *Meditationes lagebraicae* (2nd ed., Cambridge, 1770). Lagrange was the first to prove it, along with the reciprocal proposition: “For *n* to be a prime number it is necessary and sufficient that 1.2.3…(n-1) + 1 be divisible by *n*.”

A fundamental memoir on the arithmetic theory of quadratic forms, modestly entitled “Recherches d’arithmétique,”[42] led the way for Gauss and Legendre. It appeared in two parts, the first in may 1775 in the Berlin *Mémoires* for 1773 and the second in June 1777, in the same periodical’s volume for 1775.

Always timid before d’Alembert, whom he knew to be totally alien to this kind of investigation, Lagrange wrote to him regarding his memoirs recently published in Berlin: “The Recherches d’arithmétique’ are the ones that caused me the most difficulty and are perhaps worth the least. I believe you never wished to find out very much about this material, and I don’t think you are wrong.…” The encouragement that he vainly sought from his old friend was perhaps given him by Laplace, to whom he declared, when sending him the second part of his memoir on 1 September 1777: “I hastened to have it published only because you have encouraged me by your approval.”

In any case, Lagrange was well aware of the value of his investigations—and posterity has agreed with his judgment. In the first part of the paper he stated: “No one I know of has yet treated this material in direct and general manner, nor provided rules for finding a priori the priori the principal properties of numbers that can be related to arbitrarily given formulas. As this subject is one of the most curious in arthimetic and particularly merits the attention of geometers because of the great difficulties it contains, I shall attempt to treat it more thoroughly than has previously been done.” It may be said that Lagrange, who in many of his works is the last great mathematician of the eighteenth century, here opens up maginificently the route to the abstract mathematics of the nineteenth century.

On 20 March 1777 Lagrange read another paper before the Academy: "Sur quelques problémes de l'analyse de Diophante."[43] It includes an exposition of "infinite descent" inspired by Fermat's comment on that topic, but this designation does not appear, since Fermat used it only in manuscripts that were unknown at that time. Lagrange writes: "The principle of Fermat's demonstration is one of the most fruitful in the entire theory of numbers nd above all in that of the whole numbers. M^r Euler has further developed this principle." This memoir also contains solutions to several difficult problems in indeterminate analysis.

Lagrange's known arithmetical works end at this point, while he was still in Berlin. Yet "Essai d'analyse numérique sur la transformation des fractions,"[44] published at Paris in the *Journal de l'école polytechnique* (1797–1798), shows that Lagrange did not lose interest in problems of this type. But the main portion of his work in this area is concentrated in the first ten years of his stay in Berlin (1767–1777). The fatigue mentioned in the letter of 6 July 1775 (cited above) was probably real, for this pioneering work was obviously exhausting.

During these ten years Lagrange also tackled algebraic analysis—or, more precisely, the solution of both numerical and literal equations. On 29 October 1797 he read "Sur l'élimination des inconnues dans les équations"[45] (published in 1771), in which he employed Cramer's method of symmetric functions but sought to make it more rapid by use of the series development of log $(1 + u)$. Nothing seems to remain of this "improvement"of Cramer's method.

Two important memoirs appeared in 1769 and 1770, respectively: "Sur la résolution des équations numérique" and "Addition au mémoire sur la résolution des équations numériques."[47] In them Lagrange utilized the algorithm of continued fractions, and in the "Addition" he showed that the quadratic irrationals are the only ones that can be expressed as periodic continued fractions. He returned to the questions in the additions to Euler's *Algebra*. The two memoirs later formed the framework of the *Traité de la résolution des équations numériques de tous les degrés,*[48] the first edition of which dates from 1798.

On 18 January and 5 April 1770 Lagrange read before the Academy his "Nouvelle méthode pour résoudre les équations littérales par le moyen des séries."[49] The method was probably suggested to him by a verbal communication from Lambert. The latter had presented, in the *Acta helvetica* for 1758, related formulas for trinomial equations but had not demonstrated them. Lagrange's formula was destined to make a great impact. He stated it in a letter to d'Alembert of 26 August 1770, as follows: "Given the equation $\alpha - + \phi(\chi) = 0$, $\phi(\chi)$ denoting an arbitrary function of x, of which p is one of the roots; I say that one will have $\psi(p)$ denoting an arbitrary function of p,

where

provided that in this series one replaces x by α, after having carried out the differentiations indicated, taking dx as a constant."

Euler, his disciple Anders Lexell, d'Alembert, and Condorcet all became extremely interested in this discovery as soon as they learne of it. The "demonstrations" of it that Lagrange and his emulators produced were hardly founded on anything more than induction. Laplace later presented a better proof. Lagrange's formula occupied numerious other mathematicians, including Arbogast, Parseval, Servois, Hindenburg, and Bürmann. Cauchy closely examined the conditions of convergence, which had been completely ignored by the inventor; and virtually every analyst of the nineteenth century considered the problem.

On 1 November 1770 Lagrange communicated to the Academy the application of his series to "Kepler's problem."[50] But the culmination of his research in the theory of equations was a memoir read in 1771:"Réflexions sur la résolution algébrique des équations."[51] In November 1770 Vandermonde read before the Paris Academy an analogous but independent and perhaps more subtle study, published in 1774. These two memoirs constituted the source of all the subsequent works on the algebraic solution of equations. Lagrange publicly acknowledged the originality and depth of Vandermonde's research. As early as 24 February 1774 he wrote to Condorcet: "Monsieur de Vandermonde seems to me a very great analyst and I was very delighted with his work on equations."

Whereas Lagrange started from discriminating critical-historical study of the writings of his predecessors—particularly Tschirnhausen, Euler, and Berzout—Vandermonde based his work directly on the principle that he analytic expression of the roots should be a function of these roots that can be determined from the coefficients alone. Yet each of these two memoirs reveals the appearance of the concept of the permutation group (without the term, which was coined by Galois), a concept which later played a fundamental role.

Two other memoirs on this subject should be mentioned. "Sur la forme des racines imaginaires des équations,"[52] ready for printing in October 1773, evoked the following response from d'Alembert: "Your demonstration on imaginary roots seems to me to leave nothing to be desired, and I am very much obliged to you for the justice you have rendered to mine, which, in fact, has the minor fault (perhaps more apparent than real) of not being direct, but which is quite simple and easy." D'Alembert was alluding to his *Cause des vents* (1747). According to the extremely precise testimony of Delambre in his biographical notice, the demonstration by François Daviet de Foncenex that appeared in the first volume of *Mélanges de Turin* was very probably at least inspired by Lagrange. It is known that in 1799 Gauss subjected these various attempts to fierce criticism.

The last memoir in this area that should be cited appeared in 1779: "Recherches sur la détermination du nombre des racines imaginaires dans les équations littérales."[53]

Lagrange’s works in infinitesimal analysis are for the most part later than those concerned with number theory and algebra and were composed at intervals from about 1768 and 1787. More in agreement with prevailing tastes, they assured Lagrange a European reputation during his lifetime.

Returning, without citing it, to his letter to Fagnano of 1754, Lagrange presented in the Berlin *Mémoires* for 1772 (published in the spring of 1774) “Sur une nouvelle espéce de calcul relatif à la différentiation et à l’intégration des quantités variables.”[54] This work, which is in fact on outline of his *Théorie des fonctions*, (1797), greatly impressed Lacroix, Condorcet, and Laplace. Based on the analogy between powers of bionials and differentials, it is one of the sources of the symbolic calculuses of the nineteenth century. A typical example of Lagrange’s thinking as an analyst is this sentence taken from the memoir: “Although the principle of this analogy [between powers and differentials] is not self-evident, nevertheless, since the conclusions drawn from it are not thereby less exact, I shall make use of it to discover various theorems. …”

On 20 September 1768 he sent to the *Mélanges de Turin*, along with “Mémoire d’analyse indéterminée,” the eassay “Sur l’intégration de quelques équations différentielles dont les indéterminées sont séparées, mais dont chaque membre en particulier n’est point intégrable.”[55] In it Lagrange drew inspiration from some of Euler’s works; and the latter wrote to him on 23 March 1775, when the essay finally came to his attention: “I was not sufficiently able to admire the skill and facility with which you treat so many thorny matters that have cost me much effort…in particular the integration of this differential equation:

in all cases where the two numbers *m* and *n* are rational.”

With this essay, as with certain works of Jakob Bernoulli, Fagnano, Euler, Landen, and others, we are in the prehistory of the theory of elliptic functions, to which period belongs one other memoir by Lagrange. Included in volume 2 of the miscellany of the Academy of Turin for 1784–1785—this academy was founded in 1783 and Lagrange was its honorawry president—it was entitled “Sur une nouvelle méthode de calcul intégral pour les différentielles affectées d’un radical carré sous lequel la variable ne passe pas le quatriéme degré.”[56] Lagrange here proposed to find convergent series for the integrals of this type of differential, which is frequent in mechanics. To this purpose he transformed these differentials in such a way that the fourth-degree polynomial placed under the radical separated into the factors $1 + px^2$ and $1 + qx^2$, the coefficients p and q being either very unequal or almost equal. Lagrange also utilized in this work the “arithmetico-geometric mean” and reduced the integration of the series

$(A + A'U + A'U^2 + A'U^3 + v\,dx)$

to that of the differential $Vdx/(1 - aU)$. This memoir, which is difficult to date precisely, was written in the last years of his stay in Berlin, after the death of his wife.

We shall now consider some earlier works on differential and partial differential equations. About March 1773 Lagrange read before the Berlin Academy his study “Sur l’intégration des équations aux différences partielles du premier ordre.”[57] The Berlin *Mémoires* for 1774 (published in 1776) contained the essay “Sur les intégrales particuliéres des équations différentielles.”[58] In these two works he considered singular integrals of differentials and partial differential equations. This problem had only been lightly touched on by Clairaut, Euler, d’Alembert, and Condorcet. Lagrange wrote: “Finally I have just read a memoir that M[r] de Laplace presented recently. … This reading awakened old ideas that I had on the same subject and resulted in the following investigations … [which constitute] a new and complete theory.” Laplace wrote on 3 February 1778 that he considered Lagrange’s essay “a masterpiece of analysis, by the important of the subject, by the beauty of the method, and by the elegant manner in which it is presented.”

The Berlin *Mémoires* for 1776 (published 1778) included the brief study “Sur l’usage des fractions continues dans le calcul intégral.”[59] The algorithm Lagrange proposed in it had, according to him, the advantage over series of giving, when it exists, the finite integral of a differential equation, while the other method can yield only approximations.

The memoir “Sur différentes questions d’analyse relatives à la théorie des intégrales particuliéres”[60] may have been written about 1780. In it Lagrange extended and deepended his studies of particular integrals. He demonstrated the equivalence of the integrations of the equations.

and the system

Finally, just as he was leaving Prussia, Lagrange presented in the Berlin *Mémoires* for 1785 (published in 1787) “Méthode générale pour intégrer les équations partielles du premier ordre lorsque ces différences ne sont que linéaires.”[61] This “general method” completed the preceding memoir.

Lagrange’s contribution to the calculus of probabilities, while not inconsiderable, is limited to a few memoirs. We have cited one of them written before 1759 and mentioned his translation of de Moivre. Two others are “Mémoire sur l’utilité de la méthode de prendre le milieu entre les résultats de plusieurs observations…,”[62] composed before 1774, and “Recherches sur les suites récurrentes dont les termes varient de plusieurs maniéres différentes …,”[63] read before the Berlin Academy in May 1776. The latter memoir was inspired by two essays of Laplace, the reading of which recalled to Lagrange his first writing on the question, which predated 1759. He proposed to add to this early work and to Laplace's essays, and to treat the same subject in a manner at once simpler, more direct, and above all more general. Last we may mention, from the Paris period, "Essai d'arithmétique politique sur les premiers besoins de l'intérieur de la République,"[64] written in *an* IV (1795–1796).

The considerable place that mechanics, and more particularly celestial mechanics, occupied in Lagrange's works resulted in contributions that we scattered among numerous memoirs. Thinking it proper to present his ideas in a single comprehensive work, on 15 September 1782 Lagrange wrote to Laplace: "I have almost completed a *Traité de mécanique analytique*, based uniquely on [principle of virtual velocities]; but, as I do not yet know when or where I shall be able to have it printed, I am not rushing to put the finishing touches on it."

The work was published at Paris. A. M. Legendre had assumed the heavy burden of correcting the proofs; and his former teacher, the Abbé Joseph-François Marie, was entrusted with the arrangements with the publishers, agreeing to buy up all the unsold copies. By the time the book appeared, at the beginning of 1788, Lagrange had settled in Paris.

About 1744 there was already talk of Lagrange's returning to Turin. In 1781, through the mediation of his old friend Caraccioli, then viceroy of Sicily, the court of Naples offered him the post of director of the philosophy section of the academy recently establish in that city. Lagrange, however, rejected the proposal. He was happy with his situation in Berlin and wished only to work there in peace. But the death of his wife in August 1783 left him very distressed, and with the death of Frederick II in August 1786 he lost his strongest support in Berlin. Advised of the situation, the princes of Italy zealously competed in attracting him to their courts.

In the meantime Mirabeau, entrusted with a semioffical diplomatic mission to the court of Prussia, asked the French government to bring Lagrange to Paris through an advantageous offer. Of all the candidates, Paris was victorious. France's written agreement with Lagrange was scrupulously respected by the public authorities through all the changes of regime. In addition, Prussia accorded him a generous pension that the was still drawing in 1792.

Lagrange left Berlin on 18 May 1787. On 29 July he became *pensionnaire vétéran* of the Paris Academy of Sciences, of which he had been a foreign associate member since 22 may 1772. Warmly welcomed in Paris, he experienced a certain lassitude and did not immediately resume his research. Yet he astonished those around him by his extensive knowledge of metaphysics, history, religion, linguistics, medicine, and botany. He had long before formulated a prudent rule of contact: "I believe that, in general, one of the first principles of every wise man is to conform stictly to the laws of the country in which he is living, even when they are unreasonable." In this frame of mind he experienced the sudden changes of the Revolution, which he observed with interest and sometimes with sympathy; but without the passion of his friends and colleagues Condorcet, Laplace, Monge, and Carnot.

In 1792 Lagrange married Renée-Francoise-AdélaÏde Le Monnier, the daughter of his colleague at the Academy, the astronomer Pierre Charles Le Monnier. This was a troubled period, about a year after the flight of the king and his arrest at Varennes. Nevertheless, on 3 June the royal family signed the marriage contract "as a sign of its agreement to the union." Lagrange had no children from this second marriage, which, like the first, was a happy one.

Meanwhile, on 8 May 1790 the Constituent Assembly had decreed the standardization of weights and measures and given the Academy of Sciences the task of establishing a system founded on fixed bases and capable of universal adoption. Lagrange was naturally a member of the commission entrusted with this work.

When the academies were suppressed on 8 August 1793 this commission was retained. Three months later Lavoisier, Borda, Laplace, Coulomb, Brisson, and Delambre were purged from its membership; but Lagrange remained as its chairman. In September of the same year the authorities ordered the arrest of all foreigners born within the borders of the enemy powers and the confiscation of their property. Lavoisier intervened with Joseph Lakanal to obtain an exception for Lagrange, and it was granted.

The Bureau des Longitudes was established by the National Convention on 25 June 1795, and Lagrange was a member of it from the beginning. In this capacity he returned to concerns that had been familiar to him since his participation, with Johann Karl Schulze and J. E. Bode, among others, in the editing of the Berlin *Ephemerides*.

A decree of 30 October established an *école normale*, designed to train teachers and to standardize education. This creation of the convention was short-lived. Generally known as the école Normale de l'An III, it lasted only three months and eleven days. Lagrange, with Laplace as his assistant, taught elementary mathematics there.

Founded on 11 March 1794 at the instigation of Monge, the école Centrale des Travaux Publics, which soon took the name École Polytechnique, still exists. Lagrange taught analysis there until 1799 and was succeeded by Sylvestre Lacroix.

The constitution of *an* III replaced the suppressed academics with the Institut National. On 27 December 1795 Lagrange was elected chairman of the provisional committee of the first section, reserved for the physical and mathematical sciences.

By the coup d'état of 18–19 Brumaire, *an* VIII (9–10 November 1799) Bonaparte replaced the Directory with the consulate. A Sénat Conservateur, which continued to exist under the Empire, was established and included among its members Lagrange, Monge, Berthollet, Carnot, and other scientists. In addition Lagrange, like Monge, became a grand officer of the newly founded Legion of Honor. In 1808 he was made count of the Empire by a law covering all the senators, ministers, state councillors, archbishops, and the president of the legislature. He was named *grand croix* of the Ordre Impérial de la Réunion—created by Nepoleon in 1811—at the same time as Monge, on 3 April 1813.

Lagrange was by now seriously ill. He died on the morning of 11 April 1813, and three days later his body was carried to the Panthéon. The funeral oration was given by Laplace in the name of the Senate and by Lacépède in the name of the Institute. Similar ceremonies were held in various universities of the kingdom of Italy; but nothing was done in Berlin, for Prussia had joined the coalition against France. Napoleon ordered the acquisition of Lagrange's papers, and they were turned over to the Institute.

With the appearance of the *Mécanique analytique* in 1788, Lagrange proposed to reduce the theory of mechanics and the art of solving problems in that field to general formulas, the mere development of which would yield all the equations necessary for the solution of every problem.

The *Traité* united and presented from a single point of view the various principles of mechanics, demonstrated their connection and mutual dependence, and made it possible to judge their validity and scope. It is divided into two parts, statics and dynamics, each of which treats solid bodies and fluids separately. There are no diagrams. The methods pesented require only analytic operations, subordinated to a regular and uniform development. Each of the four sections begins with a historical account which is a model of the kind.

Lagrange decided, however, that the work should have a second edition incorporating certain advances. In the *Mémoires de l'Institut* he had earlier published some essays that represented a last, brilliant contribution to the development of celestial mechanics. Among them were "Mémoire sur la théorie générale de la variation des constantes arbitraires dans tous les problèmes de la mécanique,"[65] read on 13 March 1809, and "Second mémoire sur la théorie de la variation des constantess arbitraires dans les problémes de mécanique dans lequel o simplifie l'application des formules générales à ces problémes,"[66] read on 19 February 1810. Arthur Cayley later deemed this theory "perfectly complete in itself."

It was necessary to incorporate the therory and certain of its applications to celestial mechanics into the work of 1788. Thr first volume of the second edition appeared in 1811.[67] Lagrange died while working on the second volume, which was not published until 1816.[68] Even so, a large portion of it only repeated the first edition verbatim.

"Les leçons élémentaires sur les mathématiques données à l'ècole nomale" (1795)[69] appeared first in the *Séances des écoles normales recueillies par les sténographes et revues par les professeurs* distributed to the students to accompany the class exercises and published i *an* IV (1795–1796). These lectures, which are very interesting from several points of view, included Lagrange's interpolation: If y takes the values P, Q, R, S, when $x = p, q, r, s$, then $y = AP + BQ + CR + DS$, with

(This interpolation had already been outlined by Waring in 1779.) The text of the "Leçons" as given in the *Oeuvers* is a much enlarged reissue that appeared in the *Journal de l'école polytechnique* in 1812.

Traité de la résolution des équations numériques de tous les degrés[70] was published in 1798. It is a reissue of memoirs originally published on the same subject in 1769 and 1770, preceded by a fine historical introduction and followed by numerous notes. Several of the latter consider points discussed in other memoirs whether in summary or in a developed form. In this work, whihc was republished in 1808, Lagrange paid tribute to the works of Vandermonde and Gauss.

Théoire des fonctions analytiques contenant les principes de calcul différentiel, dégagés de toute considération d'infiniment petits, d'évanouissants, de limites et de fluxions et réduits à l'analyse algébrique des quantités finies[71] indicates by its title the author's ratjher utopian program. First published in 1813), it returned to themes already considered in 1772. In it Lagrange intended to show lthat power series expansions are sufficient to provide differential calculus with a solid foundation. Today mathematicians are partially returning to this conception in treating the formal calculus of series. As early as 1812, however, J. M. H. Wronski objected to Lagrange's claims. The subsequent opposition of Cauchy was more effective. Nevertheless, Lagrange's point of view could not be totally neglected. Completed by convergence considerations, it dominated the study of the functions of a complex variable throughout the nineteenth century.

Many passages of the *Théorie*, as of the "Leçons" (discussed below), were wholly incorporated into the later didactic works. This is true, for example, of the study of the tangents to curves and surfaces and of "Lagrange's remainder" in the expansion of functions by the Taylor seies.

The "Leçons sur le calcul des fonctions," [72] designed to be both a commentary on and a supplement to the preceding work, appeared in 1801 in the *Journal de l'école polytechnique* as the twelfth part of the "Le…ons de l'école normale." A separate edition of 1806 contained two complementary lectures on the calculus of variations, and the *Théorie des fonctions* also devoted a chapter to this subject. In dealing with it and with all other subjects in these two works, Lagrange abandoned the differential notation and introduced a new vovabulary and a new symbolism: first derivative function *f'*; second derivative function, *f*"; and so on. To a certain extent this symbolism and vocabulary have prevailed.

Without having enumerated all of Lagrange's writings, this study has sought to make known their different aspects ad to place them in approximately chronological order. This attempt will, it is hoped, be of assistance in comprehending the evolution of his thought.

Lagrange was always well informed about his contemporaries and predecessors and often enriched his thinking by a critical reading of their works. His close friendship with d'Alembert should not obscure the frequently striking divergence in their ideas. D'Alembert's mathematical production was characterized by a realism that links him with Newton and Cauchy. Lagrange, on the contrary, displayed in his youth, and sometimes in his later years, a poetic sense that recalls the creative audacity of Leibniz.

Although Lagrange was always very reserved toward Euler, whom he never met, it was the latter, among the older mathematicians, who most influenced him. That is why any study of hid work must be preceded or accompanied by an examination of the work of Euler. Yet even inthe face of this great model he preserved an originality that allowed him to criticize but above all to generalize, to systematize, and to deepen the ideas of his predecessors.

At his death Lagrange left examples to follow, new problems to solve, and techniques to develop in all branches of mathematics. His analytic mind was very different from the more intuitive one of his friend Monge. The two mathematicians in fact complemented each other very well, and together they were the masters of the following generations of French mathematicians, of whom many were trained at the école Polytechnique, where Lagrange and Monge were the two most famous teachers.

NOTES

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2. III, 157–186.

3. I, 3–20.

4. I, 23–36.

5. XIV, 66.

6. I, 39–148.

7. I, 151–316.

8. I, 319–332.

9. I, 334–362.

10. I, 365–468.

11. I, 471–668.

12. XIII, 30.

13. VI, 5–61.

14. V, 5–123.

15. VI, 67–225.

16. VI, 229–324.

17. VI, 324.

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19. VI, 635–709.

20. IV, 111–148.

21. VII, 517–532.

22. VI, 403–503.

23. V, 125–207, pt. 1, published in 1783; V, 211–344. pt. 2, published in 1784.

24. V, 382–414.

25. V, 348–377.

26. V, 418–488.

27. II, 335–374.

28. IV, 439–532.

29. II, 67–121.

30. III, 581–616.

31. III, 661–692.

32. III, 619–649.

33. IV, 695–748.

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35. V, 592–609.

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37. I, 671–731.

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61. V, 544–562.

62. II, 173–234.

63. IV, 151–251.

64. VII, 573–579.

65. VI, 771–804

66. VI, 809–816.

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69. VII, 183–287.

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72. X, 1–451.

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Vol. VI (1873) consists of articles extracted from publications of the Paris Academy of Sciences and of the Class of Mathematical and Physical Sciences of the Institute.

Vol. VII (1877) contains various works that did not appear in the academic publications—in particular, the lectures given at the école Normale.

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Vol. XI (1888) is *Mécanique analytique*, vol. I. This ed. is based on that of 1811, with notes by J. Bertrand and G. Darboux.

Vol. XII (1889) is vol. II of *Mécanique analytique*. Based on the ed. of 1816, it too has notes by Bertrand and Darboux. These two vols. have been reprinted (Paris, 1965).

Vol. XIII (1882) contains correspondence with d'Alembert, annotated by Ludovic Lalanne.

Vol. XV, in preparation, will include some MSS that had been set aside by thecommision of the Institute entrusted with publication of the collected works. This Vol. will also present correspondence discovered since 1892 that has been published in various places or was until now unpublished–particularly the correspondence with Fagnano. It will also provide indexes and chronological tables to facilitate the study of Lagrange's works.

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