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(*b*. Moscow, Russia, 3 September 1884; *d*. Princeton, New Jersey, 5 October 1972)

mathematics.

Lefschetz was the son of Alexander Lefschetz, an importer, and his wife, Vera, who were Turkish citizens. Shortly after his birth the family moved to Paris, where he grew up with five brothers and one sister. French was his native tongue, but he learned Russian and other languages in later years. From 1902 to 1905 Lefschetz studied at the École Centrale, Paris, graduating as *ingenieur des arts et manufactures*. In November 1905 he emigrated to the United States and found a job at the Baldwin Locomotive Works near Philadelphia. In early 1907 he joined the engineering staff of the Westinghouse Electric and Manufacturing Company in Pittsburgh. In November of that year he lost his hands and forearms in a tragic accident.

Lefschetz soon realized that his true bent was mathematics, not engineering. Among his professors at the École Centrale had been Émile Picard and Paul Appell, authors of famous treatises on analysis and analytic mechanics that he now read. In 1910, while teaching apprentices at Westinghouse, Lefschetz determined to make his career in mathematics. He enrolled as a graduate student at Clark University, Worcester, Massachusetts, and obtained the Ph.D. in just one year with a dissertation on a problem in algebraic geometry proposed by W. E. Story. On 17 June 1912 Lefschetz became an American citizen, and on 3 July 1913 he married Alice Berg Hayes, a fellow student at Clark who had received a master's degree in mathematics. She helped him to overcome his handicap, encouraging him in his work and moderating his combative ebullience. They had no children.

From 1911 to 1913 Lefschetz was an instructor at the University of Nebraska, Lincoln, where he taught a heavy load of beginning courses but found ample time to pursue his own work in algebraic geometry. In 1913 he moved to a slightly better position at the University of Kansas in Lawrence. As his work became known in America and Europe, he rose through the ranks to become full professor in 1923. In 1919 he was awarded the Prix Bordin by the Académie des Sciences of Paris and in 1923 the Bôcher Memorial Prize of the American Mathematical Society.

In 1924 Lefschetz accepted a post at Princeton University, where he spent the rest of his life. He had prized the opportunity for solitary research at Nebraska and Kansas, but he welcomed the new world that opened up to him at Princeton. He acquired distinguished geometers as colleagues—James W. Alexander, Luther P. Eisenhart, Oswald Veblen—and met stimulating visitors from abroad, such as Pavel Aleksandrov, Heinz Hopf, M. H. A. Newman, and Hermann Weyl. His first (1926) of some thirty doctoral students was the topologist-to-be Paul A. Smith, who had followed him to Princeton from Kansas.

From his Ph.D. to his appointment to the faculty of Princeton, Lefschetz worked mainly in algebraic geometry, his most important results being presented in his 1921 paper "On Certain Numerical Invariants of Algebraic Varieties with Application to Abelian Varieties" and in his 1924 monograph *L'analysis situs et la géométrie algébrique*. The study of the properties of families of algebraic curves and surfaces began in the nineteenth century as part of the theory of algebraic functions of complex variables. For Lefschetz, too, curves and surfaces—and, more generally, algebraic varieties—were significant representations of the corresponding functions. He was able to solve some of the problems encountered by his predecessors and to enlarge the scope of the subject by the use of new methods. As he put it, "It was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry."

In the 1850's G. F. B. Riemann founded the modern theory of complex algebraic curves by considering, for each curve, an associated surface now called the Riemann surface. The theory was further developed by Guido Castelnuovo, Federigo Enriques, Francesco Severi, and especially Émile Picard. (Lefschetz, while at the École Central, had taken Picard's demanding course.) Riemann and these later mathematicians recognized that it is the topological properties of the Riemann surface (the connectedness properties of the surface as a whole rather than its metrical and local properties) that are significant, yet at the time there was no theory of such properties. In the 1890's Henri Poincaré established such a theory (under the name "analysis situs"), and Lefschetz used Poincaré's results to extend the work of Riemann and his successors.

Riemann had used a series of cuts to turn the Riemann surface into an open 2-cell (and the correspondence between the function and the 2-cell then gave the desired results); Lefschetz used a series of cuts to turn a nonsingular algebraic variety of complex dimension *d* into an open 2*d*-cell. This allowed him to answer many questions (for example, he showed that not all orientable manifolds of even dimension are the carrier manifolds of algebraic varieties) and to extend the theory of integrals of the second kind to double and triple integrals on an algebraic variety of any dimension.

Lefschetz took up Poincaré's study of curves on a surface, which he generalized to the study of subvarieties of an algebraic variety. He found necessary and sufficient conditions for an integral (2*d*-2)-dimensional homology class of variety *V* of complex dimension *d* to contain the cycle of a divisor on *V*. This result and others allowed Lefschetz to make important contributions to the theory of correspondences between curves and to the theory of Abelian varieties. (A much more detailed review by W. V. D. Hodge of Lefschetz's work and influence in algebraic geometry appears in the volume *Algebraic Geometry and Topology)*

According to Hodge, "Our greatest debt to Lefschetz lies in the fact that he showed us that a study of topology was essential for all algebraic geometers." Lefschetz' work in algebraic geometry also gave great impetus to the study of topology, since its value to other areas of mathematics had been demonstrated. In 1923 Lefschetz turned to the development of Poincaré's topology, calling it algebraic topology to distinguish it from the abstract topology of sets of points.

Almost all of Lefschetz' topology resulted from his desire to prove certain fixed-point theorems. Around 1910 L. E. J. Brouwer proved a basic fixed-point theorem: Every continuous transformation of an *n*-simplex into itself has at least one fixed point. In a series of papers Lefschetz obtained a much more general result: For any continuous transformation *f* of a topological space *X* into itself, there is a number *L(f)*, often called the Lefschetz number, such that if $L(f) \neq 0$, then the transformation *f* has a fixed point. *L(f)* is defined as follows: *f* induces a transformation f_p of the *p*th homology group H_p as a vector space over the rational numbers and let $\mathrm{Tr}(f_p)$ be the trace of f_p: then $L(f) = \Sigma\,(-1)^p\,\mathrm{Tr}(f_p)$. For *L(f)* to be well defined, certain restrictions must be placed on *X*; Lefschetz succeeded in progressively weakening these restrictions.

Lefschetz used the following simple example to explain his fixed-point theorem. Let *f* be a continuous transformation of the interval $0 \leq x \leq 1$ into itself. The curve consisting of the points $(x,f(x))$ represents *f*. (See Figure 1.) The diagonal $0 \leq x = y \leq 1$ represents the identity transformation *i*, that is, the transformation that sends each point of the interval to itself. The points of intersection (called the co-incidences) of *f* and *i* are the fixed points of *f*. We want a number that is the same for all continuous transformations of the interval $0 \leq x \leq 1$. The number of coincidences is not constant; *f* and *g* for example, differ in this respect. But if, for a particular transformation, we count the number of crossings from *above* to below (marked *a* in the figure) and the number of crossings from *below* to above (marked *b* in the figure), and if we subtract the latter from the former, we get a number (here, 1) that is the same for all continuous transformations of an interval into itself. That is, for this space (the interval $0 \leq x \leq 1$), the Lefschetz number *L (f)* is 1. Since *L (f)* is not zero, any continuous transformation of $0 \leq x \leq 1$ into itself has a fixed point. (It is intuitively clear that any continuous curve passing from the left side of the square to the right side must intersect the diagonal.)

In 1923 Lefschetz proved this fixed-point theorem for compact orientable manifolds. Since an *n*-cell is not a manifold, this result did not include the Brouwer fixed-point theorem. By introducing the concept of relative homology groups, Lefschetz in 1927 extended his theorem to manifolds with boundary; his theorem then included Brouwer's. He continued to seek generalizations of the theorem; in 1927 he proved it for any finite complex, and in 1936 for any locally connected space. Lefschetz studied fixed points as part of a more general study of coincidences. If *f* and *g* are transformations of space *X* into space *Y*, the points *x* of *X* such that $f(x) = g(x)$ are called the coincidences of *f* and *g* One can prove that under certain conditions two transformations must have coincidences—for example, in Figure 1, if *f* and *g* are continuous and *f* is above *g* at 0 and below *g* at 1, then the number of times *f* crosses *g* from above to below (marked α) minus the number of times *f* crosses *g* from below to above (marked β) is necessarily 1.

In the course of this work Lefschetz invented many of the basic tools of algebraic topology. He made extensive use of product spaces; he developed intersection theory, including the theory of the intersection ring of a manifold; and he made essential contributions to various kinds of homology theory, notably relative homology, singular homology, and cohomology.

A by-product of Lefschetz' work on fixed points

was his duality theorem, which provided a bridge between the classical duality theorems of Poincare and of Alexander. The Lefschetz duality theorem states that the *p*-dimensional Betti number of an orientable *n*-dimensional manifold *M* with regular boundary *L* equals the *(n-p)*-dimensional Betti number of *M* modulo *L* (that is, without *L*). Figure 2 shows an oriented 2–manifold *M* with regular boundary *L* in three parts, one exterior and two interior. The absolute 1–cycles c_1 and c_2 generate the 1–dimensional Betti group of *M* with boundary *L* and the relative 1–cycles d_1 and d_2 generate the relative 1–dimensional Betti group of *M* modulo *L* Thus the 1–dimensional Betti numbers of *M* and *M* modulo *L* are both 2. Cuts along d_1 and d_2 turn the 2–manifold into a 2–cell. (A full exposition of Lefschetz' fixed-point theorem and his duality theorem is in his *Introduction to Topology*, 1949.)

During his years as professor at Princeton (1924– 1953), Lefschetz was the center of an active group of topologists. His *Topology* (1930) and his Algebraic Topology (1942) presented comprehensive accounts of the field and were extremely influential. Indeed, these books firmly established the use of the terms "topology" (rather than "analysis situs") and "algebraic topology" (rather than "combinatorial topology"). (A thorough review by Norman Steenrod of Lefschetz' work and influence in algebraic topology appears in *Algebraic Geometry and Topology*, 1957.)

Lefschetz was an editor of *Annals of Mathematics* from 1928 to 1958, and it was primarily his efforts— insisting on the highest standards, soliciting manuscripts, and securing rapid publication of the most important papers—that made the *Annals* one of the world's foremost mathematical journals. As Steenrod put it, "The importance to American mathematicians of a first-class journal is that it sets high standards for them to aim at. In this somewhat indirect manner, Lefschetz profoundly affected the development of mathematics in the United States."

There was another way in which Lefschetz contributed to the beginning of the publication of advanced mathematics in the United States. As late as the 1930's the American Mathematical Society, whose Colloquium Publications included books by

Lefschetz in 1930 and 1942, was almost the only U.S. publisher of advanced mathematics books. However, two important series of advanced mathematics monographs and textbooks began in 1938 and 1940: the Princeton Mathematical Series and the Annals of Mathematics Studies, both initiated by A. W. Tucker, student and colleague of Lefschetz. Lefschetz wrote two important books for the former series (1949, 1953) and wrote or edited six books for the latter.

In 1943 Lefschetz was asked to consult for the U.S. Navy at the David Taylor Model Basin near Washington, D. C. Working with Nicholas Minorsky, he studied guidance systems and the stability of ships, and became acquainted with the work of Soviet mathematicians on nonlinear mechanics and control theory. Lefschetz recognized that the geometric theory of differential equations, which had begun with the work of Poincare and A. M. Liapunov, could be fruitfully applied, and his background in algebraic geometry and topology proved useful. From 1943 to the end of his life, Lefschetz gave most of his attention to differential equations, doing research and encouraging others.

Lefschetz was almost sixty years old when he turned to differential equations, yet he did important original work. He studied the solutions of analytic differential equations near singular points and gave a complete characterization, for a two-dimensional system, of the solution curves passing through an isolated critical point in the neighborhood of the critical point. Much of his work focused on nonlinear differential equations and on dissipative (as distinct from conservative) dynamic systems. This work contributed to the theory of nonlinear controls and to the study of structural stability of systems. The Russian topologist L. S. Pontriagin, who was a good friend of Lefschetz' both before and after the war, also turned to control theory as a result of his wartime work. (Lawrence Markus' "Solomon Lefschetz: An Appreciation in Memoriam" contains a more detailed account of Lefschetz' work and influence on differential equations.)

In 1946 the newly established Office of Naval Research provided the funding for a differential equations project, directed by Lefschetz, at Princeton. This soon became a leading center for the study of ordinary differential equations, and the project continued at Princeton for five years after Lefschetz' retirement in 1953. In 1957 he established a mathematics center under the auspices of the Research Institute for Advanced Study (RIAS), a branch of the Glen L. Martin Company of Baltimore (now Martin-Marietta). In 1964 Lefschetz and many of the other mathematicians in his group at RIAS moved to Brown University to form the Center for Dynamical Systems (later named the Lefschetz Center for Dynamical Systems). J. P. LaSalle, who had spent the year 1946–1947 with the differential equations project at Princeton and who was Lefschetz' second in command at RIAS, became director at the Brown center. Lefschetz helped to found the *Journal of Differential Equations* and served as an editor for some fifteen years. He continued his work at Brown until 1970.

Lefschetz translated two Russian books on differential equations into English, and he edited several volumes on nonlinear oscillations. He gave constant encouragement to his younger colleagues, in some cases cajoling them into proving important theorems. His work in differential equations showed the usefulness of geometric and topological methods and helped to raise the intellectual stature of applied mathematics.

Throughout his life Lefschetz loved to travel. In the 1920's, and 1930's he made many trips to Europe, especially to France, Italy, and the Soviet Union. During World War II, European travel was impractical, so Lefschetz was visiting professor at the National University of Mexico (1944). Although he did not know Spanish when he arrived there, several weeks later he was giving his lectures in that language. He returned for several months in the academic year 1945–1946, and in the following two decades made many trips to Mexico City, spending most winters there from 1953 to 1966. He helped to build a lively school of mathematics at the National University of Mexico, and in recognition of his efforts the Mexican government in 1964 awarded him the Order of the Aztec Eagle, rarely presented to a foreigner.

Lefschetz was Henry Burchard Fine (research) professor of mathematics at Princeton (1933–1953), succeeding Oswald Veblen, the first holder of the chair (1926–1932). He was chairman of the department of mathematics at Princeton from 1945 until his retirement in 1953. Lefschetz served as president of the American Mathematical Society (1935–1936). The Accademia Nazionale dei Lincei of Rome awarded him the Antonio Feltrinelli International Prize, one of the world's highest mathematical honors, in 1956. In 1964 he was awarded the National Medal of Science by President Johnson "for indomitable leadership in developing mathematics and training mathematicians, for fundamental publications in algebraic geometry and topology, and for stimulating needed research in nonlinear control processes." He was granted honorary degrees by Paris, Prague, Mexico, Clark, Brown, and Princeton. He was a member of the American Philosophical Society and of the National Academy of Sciences, and a foreign member of the Royal Society of London, of the Académie des Sciences of Paris, of the Academia Real de Ciencias of Madrid, and of the Reale Instituto Lombardo of Milan.

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Selected Papers (New York, 1971) brings together "A Page of Mathematical Autobiography," in *Bulletin of the American Mathematical Society*, **74** (1968), 854–879, awarded the society's 1970 Leroy P. Steele Prize: "On Certain Numerical Invariants of Algebraic Varieties with Application to Abelian Varieties," in *Transactions of the American Mathematical Society*, **22** (1921), 327–482, awarded the society's 1924 Bócher Memorial Prize and, in its original French version, the 1919 Prix Bordin of the Paris Academy of Sciences; the 1924 monograph cited above: 16 other principal papers; and a bibliography (to 1970). The paper "The Early Development of Algebraic Topology," in *Boletim da Sociedade brasileira de matematica*, **1** (1971), 1–48, summarizes Lefschetz' view of the development of algebraic topology.

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