Legendre, Adrien-Marie | Encyclopedia.com

(*b*. Paris, France, 18 September 1752;*d*. Paris, 9 January 1833),

mathematics.

Legendre, who came from a well-to-dofamily, studied in Paris at the collége Mazarin (also called Collége des Quarte-Nations). He received an education in science, especially mathematics, that was unusually advanced for Paris schools in the eighteenth century. His mathematics teacher was the Abbé Joseph-Francois Marie, a mathematician of some renown and well-regarded at court. In 1770, at the age of eighteen, Legendre defended his theses in mathematics and physics at the Collége Mazarin. In 1774 Marie utilized several of his essays in a treatise on mechanics.

Legendre's modest fortune was sufficient to allow him to devote himself entirely to research. Nevertheless he taught mathematics at the École Militaire in Paris from 1775 to 1780.

In 1782 Legendre won the prize of the Berlin Academy. The subject of its competition that year concerned exterior ballistics; "Determine the curve described by cannonballs and bombs, taking into consideration the resistance of the air; give rules for obtaining the ranges corresponding to different initial velocities and to different angles of projection." His essay, which was published in Berlin, attracted the attention of Lagrange, who asked Laplace for information about the young author. A few years later the Abbé Marie and Legendre arranged for Lagrange's *Mécanique analytique* (Paris, 1788) to be published and saw it through the press.

Meanwhile, Legendre sought to make himself better known in French scientific circles, particularly at the Académie des Sciences. He conducted research on the mutual attractions of planetary spheroids and on their equilibrium forms. In January 1783 he read a memoir on this problem before the Academy; it was published in the *Recueil des savants étrangers* (1785). He also submitted to Laplace essays on second-degree indeterminate equations, on the properties of continued fractions, on probabilities, and on the rotation of bodies subject to no accelerating force. As a result, on 30 March 1783 he was elected to the Academy as an *adjoint mécanicien,* replacing Laplace, who had been promoted to *associé*.

Legendre's scientific output continued to grow. In July 1784 he read before the Academy his "Recherches sur la figure des planétes" Upon the publication of this memoir, he recalled that Laplace had utilized his works in a study published in the *Mémories de l'Académie des sciences* for 1782 (published in 1784) but written after his own, which allowed "M. de Laplace to go more deeply into this matter and to present a complete theory of it." the famous "Legendre polynomials" first appeared inthese 1784 "Recherches,"

The "Recherches d'analyse indéterminée" (1785) contains, among other things, the demonstration of a theorem that allows decision on the possibility or impossibility of solution of every second-degree indeterminate equation; an account of the laws of reciprocity of quadratic residues and of its many applications; the sketch of a theory of the decomposition of numbers into three squares; and the statement of a theorem that later became famous: "Every arithmetical progression whose first term and ratio ae relatively prime contains an infinite number of prime numbers."

In 1786 Legendre presented a study on the manner of distinguishing maxima from minima in the calculus of variations. The "Legendre conditions" set forth in this paper later gave rise to an extensive literature. Legendre next published, in the *Mémoires de l'académie* for 1786, two important works on integrations by elliptic arcs and on the comparison of these arcs; here can be found the rudiments of his theory of elliptic functions.

In the works cited above, Legendre had marked off his favorite areas of research: celestial mechanics, number theory, and the theory of elliptic functions. Although he did take up other problems in the course of his life, he always returned to these subjects.

Legendre's career at the Academy proceeded without any setbacks. On the reorganization of the mechanics section he was promoted to *associé* (1785). In 1787, along with Cassini **IV** and Méchain, he was assigned by the Academy to the geodetic operations undertaken jointly by the Paris and Greenwich observatories. On this occasion he became a fellow of the Royal Society. His work on this project found expression in the "Mámore sur les opérations trigonométriques dont les résultats dépendent de la figure de la terre." Here are found "Legendre's theorem201C; on spherical triangles:

When the sides of a triangle are very small in relation to the radius of the sphere, it differs very little from a rectilinear triangle. If one subtracts from each of its angles a third of the excess of the sum of the three angles over[the sum of] two right angles, the angles, diminished in this manner, may be considered those of a rectilinear triangle whose sides are equal in length to those of the given triangle.

This memoir also contains his method of indeterminate corrections:

In these calculations there exist some elements that are susceptible to a slight uncertainty. In order to make the calculation only once, and in order to determine the influence of the errors at a glance, I have supposed the value of each principal element to be augmented by an indeterminate quantity that designates the required correction. These literal quantities, which are considered to be very small, do not prevent one from carrying out the calculation by logarithms in the usual manner.

In the *Mémoires de l'Académie* for 1787 Legendre published an important theoretical work stimulated by Monge's studies of minimal surfaces. Entitled "L'intégration de quelques équations aux différences Partielles," it contains the Legendre transformation. Given the partial differential equation

Set $\partial z / \partial x = p$ and $\partial z / \partial y = q$. suppose*R,S,T,* are functions of *p* and *q* only. Legendre is concerned with the plane tangent to the surfacebeing sought: $z = f(x,y)$. The equation of this plane being $pX + qy - Z - v = 0$, he shows that v satisfies the equation

Later, in volume II of his *Traité des fonctions elliptiques* (1826), Legendre employed an analogous procedure to demonstrate "the manner of expressing every integral as an arc length of a curve." If $\int p d\omega$ is the integral in question (*p* being a function of ω), he shows that the arc length of the envelope of the family of straight lines $x \cos \omega + y \sin \omega - p = 0$ is equalto $\int p d\omega + p'$ where p' is the derivative of *p* with respect to ω.

In 1789 and 1790 Legendre presented his "Mémoire sur les intégrales doubles," in which he completed his analysis of the attraction of spheroids ; a study of the case of heterogeneous spheroids; and some investigations of the particular integrals of differential equations.

In April 1792 Legendre read before the Academy an important study on elliptic transcendentals, a more systematic account of material presented in his first works on the question, dating from 1786. The academies were suppressed in August 1793; consequently he published this study himself, toward the end of the same year, in a quarto volume of more than a hundred pages.

The times were difficult for everyone and for Legendre in particular. He may even have been obliged to hide for a time. In any case his "small fortune" disappeared, and he was obliged to find work, especially since he had married a girl of nineteen, Marguerite Couhin. He later wrote to Jacobi:

I married following a bloody revolution that had destroyed my small fortune; we had great problems and some very difficult moments, but my wife staunchly helped me to put my affairs in order little by little and gave me the tranquillity necessary for my customary work and for writing new works that have steadily increased my reputation.

On 13 April 1791 Legendre had been named one of the Academy's three commissioners for the astronomical operations and triangulations necessary for determining the standard meter. His colleagues were Méchain and Cassini **IV,** who four years earlier had participated with him in the geodetic linking of the Paris and Greenwich meridians. On 17 March 1792, however, Legendre had requested to be relieved of this assignment. In ventÔse an II (February-March 1794) the Commission of Public Instruction of the department of Paris appointed him professor of pure mathematics at the short-lived Institut de Marat, formerly the Collége d'Harcourt. During 1794 he was head of the first office of the National Executive Commission of Public Instruction (the second section, Sciences and Letters). He had eight employees under him and was expected to concern himself with weights and measures, inventions and discoveries, and the encouragement of science.

A tireless worker, during this same period Legendre published his *Éléments de géométrie*. This textbook was to dominate elementary instruction in the subject for almost a century. On 29 vendémiaire an II (20 October 1793) the Committee of Public Instruction, of which Legendre soon became senior clerk, commissioned him and Lagrange to write a book entitled *Éléments de calcul et de géométric*. Actually his work must already have been nearly finished. He had probably been working on it for several years, encouraged perhaps by Condorcet, who stated in 1791, in his *Mémrmires sur l'instruction publique:* "I have often spoken of elementary books written for children and for adults, of works designed to serve as guides for teachers …. Perhaps it is of some use to mention here that I had conceived the project for these works and prepared the means necessary to execute them…"

In the meantime the decimal system had been adopted for the measurement of angles; and the survey offices, under the direction of Prony, undertook to calculate the sines of angles in ten-thousandths of a right angle, correct to twenty-two decimal places; the logarithms of sines in hundred-thousandths of a right angle, correct to twelve decimal places; and the logarithms of numbers from 1 to 200,000, also to twelve decimal places. In 1802 Legendre wrote: "These three tables, constructed by means of new techniques based principally on the calculus of differences, are one of the most beautiful monuments ever erected to science." Prony had them drawn up rapidly through a division of labor that permitted him to employ people with low qualifications. The work was prepared by a section of analysts headed by Legendre, who devised new formulas for determining the successive differences of the sines. The other sections had only to perform additions. This collective work resulted in two completely independent copies of the tables, which were mutually verified by their identity. These manuscript tables were deposited at the Bureau des Longitudes. An explanatory article appeared in the *Mémoires de l'Institut* (1801).

Legendre figured neither among the professors of the *écoles normales* of *an* III nor among those of the École Polytechnique. He did, however, succeed Laplace, in 1799, as examiner in mathematics of the graduating students assigned to the artillery. He held this position until 1815, when he voluntarily resigned and was replaced by Prony. He was granted a pension of 3,000 francs, equal to half his salary. He lost it in 1824 following his refusal to vote for the official candidate in an election for a seat in the Institute.

Legendre was not one of the forty-eight scholars selected in August 1795 to form the nucleus of the Institut National, but on 13 December he was elected a resident member in the mathematics section. In 1808, upon the creation of the University, he was named a *conseiller titulaire*. A member of the Legion of Honor, he also obtained the title of Chevalier de l'Empire—a minor honor compared with the title of count bestowed on his colleagues Lagrange, Laplace, and Monge. When Lagrange died in 1813, Legendre replaced him at the Bureau des Longitudes, where he remained for the rest of his life.

We now return to Legendre's scientific publications. The first edition of his *Essai sur la théorie des nombres* appeared in 1798. In it he took up in a more systematic and more thorough fashion the topics covered in his "Recherches" of 1785.

His *Nouvelles méthodes pour la détermination des orbites des comètes* (1806) contains, in a supplement, the first statement and the first published application of the method of least squares. Gauss declared in his *Theoria motus corporum coelestium* (1809) that he himself had been using this method since 1795. This assertion, which was true, infuriated Legendre, who returned to the subject in 1811 and 1820.

Legendre had been dismayed once before by Gauss: in 1801 the latter attributed to himself the law of reciprocity of quadratic residues, which Legendre had stated in 1785. Later, in 1827, Legendre wrote to Jacobi: "How has M. Gauss dared to tell you that the majority of your theorems [on elliptic functions] were known to him and that he had discovered them as early as 1808? This excessive impudence is unbelievable in a man who has sufficient personal merit not to have need of appropriating the discoveries of others."

Both Legendre in his indignation and Gauss in his priority claims were acting in good faith. Gauss considered that a theorem was his if he gave the first rigorous demonstration of it. Legendre, twenty-five years his senior, had a much broader and often a hazier sense of rigor. For Legendre, a belated disciple of Euler, an argumentation that was merely plausible often took the place of a proof. Consequently all discussion of priority between the two resembled a dialogue of the deaf.

In "Analyse des triangles tracés sur la surface d'un sphéroïde" (read before the Institute in March 1806), in which he considered the triangles formed by the geodesics of an ellipsoid of revolution, Legendre generalized his theorem concerning spherical triangles. Gauss provided a much broader generalization of this theorem in 1827, in his "Disquisitiones generales circa superficies curvas."

Legendre brought out the second edition of *Essai sur la théorie des nombres* in 1808. He later wrote two supplements to it (1816, 1825).

The "Recherches sur diverses sortes d'intégrales définies" (1809) continued an early study of Eulerian integrals—the term is Legendre's—in particular of the "gamma function." The earlier work appeared in 1793, in *Mémoire sur les transcendantes elliptiques*. The investigations of 1809 were, in turn, revised completed, and enlarged in later works devoted to elliptic functions (1811, 1816, 1817, 1826).

Volume I of the *Exercices de calcul intégral* (1811) contains a majority of the results that Legendre obtained in the study of elliptic integrals. Volume III was published in 1816. Volume II, which includes the important numerical tables of the elliptic integrals, appeared in 1817.

In 1823 Legendre published "Recherches sur quelques objects d'analyse indéterminée et particuliérement sur le théoréme de Fermat" in *Mémories de l'Académie*. It contains a beautiful demonstration of the impossibility of an integral solution of the equation $x^5+y^5=z^5$, followed by an examination of more complicated cases of the theorem. This memoir was reproduced as the second supplement to the *Essai sur la théorie des nombres* (1825).

In 1825 and 1826, in *Traité des fonctions elliptiques*, Legendre took up once more and developed the essential aspects of his 1811 work, including applications to geometry and mechanics. In 1827, however, Jacobi informed Legendre of his own discoveries in this area. The latter, inspired by the contributions his correspondent and by those of Abel, published three supplements to his *Théorie des fonctions elliptiques* (the title of volume I of the Traité) in rapid succession (1828, 1829, 1832).

In May 1830 the third edition of Legendre's *Théorie des nombres* appeared. In this two-volume work he developed the material of the *Essai* of 1808, adding new thoughts inspired, to a large extent, by Gauss. Jacobi drew his attention to a weak point in his reasoning, and on 11 October 1830 Legendre presented a corrected memoir that appeared in 1832 in the Academy's *Mémoires*. Shortly before his death he referred to this memoir as a necessary complement and conclusion to his *Traité* and expressed the need for printing it at the end of that work. His with was not fulfilled.

The "Réflexions sur différentes maniéres de démontrer la théorie des paralléles ou le théoréme sur la somme des trois angles du triangle" appeared in the *Mémories* in 1833. Legendre had already sent a separately printed copy of it to Jacobi at the end of June 1832.

Legendre died on 9 January 1833, following a painful illness. His health had been failing for several years. His wife, who died in December 1856, made a cult of his memory and until her death displayed a naïve, religious respect for everything that had belonged to him. She left to the village of Auteuil (now part of Paris) the last country house in which they had lived. They had no children.

We shall now return to the three main fields treated in Legendre's works: number theory, elliptic functions, and elementary geometry—more particularly, the theory of parallel lines.

Number Theory. In the introduction to the second edition of his *Théorie des nombres* (1808) Legendre exhibited an admirable concern for rigor. For example, he demonstrated the commutativity of the products of integers by a technique related to Fermat's method of infinite descent. A direct disciple of Euler and Lagrange, he, like them, made frequent use of the algorithm of continued fractions, as much to solve first-degree indeterminate equations as to show that Fermat's equation $x^2 - Ay^2 = 1$ always admits an integral solution.

Moreover, Legendre followed Lagrange step by step in the study of quadratic forms, a study that he completed in some respects. He showed, for example, that every odd number not of the form 8k+7 is the sum of three squares. (He had shown this imperfectly as early as 1785 and in a nearly satisfactory manner in 1798.) On the basis of this result, Cauchy, in 1812, demonstrated Fermat's theorem for the case of polygonal numbers.

Legendre's principal contribution was the law of reciprocity of quadratic residues. He stated it as early as 1785, when he produced a very long and imperfect demonstration of it. In 1801 Gauss subjected it to a thorough criticism and was able to declare that he was the first to have demonstrated the proposition rigorously. In 1808, while preserving his first exposition, improved in 1798, Legendre adopted the proof given by his young critic. In 1830 he added to it that of Jacobi, which he found superior.

We have already mentioned Legendre's contributions to the study of Fermat's great theorem concerning the impossibility of finding an integral solution of the equation $x^n+y^n+z^n=0$. In this connection he had met Dirichlet, a young mathematician of great promise.

A very skillful calculator, Legendre furnished valuable tables listing the quadratic and linear divisors of quadratic forms and the least solutions of Fermat's equation $x^2-Ay^2=\pm1$. For the latter table, published in 1798 and reproduced in a much abridged form in 1808, he later (1830) utilized the corrections made by the Danish mathematician C. F. Degen.

Legendre should be considered a precursor of analytic number theory. His law of the distribution of prime numbers, outlined in 1798 and made more precise in 1808, took the following form: If y is the number of prime numbers less than x, the .He found it, he stated, by induction. In 1830 he pointed out again that it had been found through induction and than "it remains to demonstrate it a priori" and he developed some observations on this subject in a manner similar to Euler's. About 1793 Gauss intuited the law of the asymptotic distribution of prime numbers, which could have occurred to any attentive reader of Euler. But it was Legendre who first drew attention to this remarkable law, which was not truly demonstrated until 1896 (by Charles de la Vallée Poussin and Jacques Hadamard).

On the other hand, Legendre thought he had demonstrated, as early as 1785, that in every arithmetic progression $ax + b$ where a and b are relative primes, there is an infinite number of primes. He even specified that in giving to b the $\phi(a)$ values prime to a and less than this number ($\phi(a)$ being Euler's indicatrix), the prime numbers are distributed almost equally among the $\phi(a)$ distinct progressions. These propositions were first rigorously demonstrated by Dirichlet in 1837.

Giving a rather broad scope to number theory, Legendre sought in 1830 to present Abel's conceptions concerning the algebraic solution of equations. Legendre thought it had been convincingly proved that such a solution is in general impossible for degrees higher than the fourth. He was also interested in the numerical solution of equations. In particular he studied the separation of roots and their expansion as continued fraction. In 1808 he presented a demonstration of the fundamental theorem of algebra that was quite analogous to that given by J. R. Argand in 1806. These essentially correct analytical demonstrations required only a few restatements to be made rigorous.

It should be noted that in 1806 Legendre's attitude toward Argand was very understanding. He did not publicly adopt the latter's ideas on the geometric representation of complex numbers; but thanks to the letter that he wrote to François Argand concernig his brother's discovery, that discovery was not lost, and in 1813 it reached a large audience through Gergonne's *Annales*.

In note 4 of his *Éléments de géométrie*—a note included in the first editions of the work—Legendre, by employing the algorithm of continued fractions, established Lambert's theorem (1761): the ratio of the circumference of a circle to its diameter is an irrational number. He improved this result by showing that the square of this ratio is also irrational, and added: "It is probable that the number π is not even included in the algebraic irrationals, but it appears to be very difficult to demonstrate this proposition rigorously."

Much attracted throughout his life by number theory, Legendre was well aware of the difficulties it presents and in his last years experienced a sort of disenchantment with it. For example, in 1828 he wrote to Jacobi; "I would advise you not to give too much time to investigations of this nature: they are very difficult and are often fruitless."

Number theory, of course, does not constitute the most significant portion of his *oeuvre*. Instead, he should be considered the founder of the theory of elliptic functions. MacLaurin and d'Alembert had studied integrals expressible by arcs of an ellipse or a hyperbola. Fagnano had shown that to any given ellipse or hyperbola two arcs the difference of which equals an algebraic quantity can be assigned in an infinite number of ways (1716). He had also demonstrated that the arcs of Bernoulli's lemniscate $(x^2+y^2)^2=a^2(x^2-y^2)$ can be multiplied and divided algebraically like the arcs of a circle. This was the first demonstration of the use of the simplest of the elliptic integrals, the one later designated by Legendre as *F(x)*and considered by him to govern all the others.

In 1761 Euler had found the complete algebraic integral of a differential equation composed of two separate but similar terms, each of which is integrable only by arcs of conics: , *R* being the square root of a fourth-degree polynomial. This discovery, made almost by chance, allowed Euler to compare—in a more general manner than had ever been done previously—not only arcs of the same conic section or lemniscate but also, in general, all the transcendentals , where *P* is a rational function of *x* and *R* is the square root of a fourthdegree polynomial.

In 1768 Lagrange undertook to incorporate Euler's discovery into the ordinary procedures of analysis and, in 1784 and 1785, he presented a general method for finding the integrals by approximation. Meanwhile, John Landen had demonstrated in 1775 that every arc of a hyperbola can be measured by two arcs of an ellipse. As a result of this memorable discovery the term "Landen's theorem" was applied not only to this result but also to the first known transformation of elliptic integrals.

Such was the state of research in the theory of elliptic transcendentals in 1786, when Legendre published his works on integration by elliptic arcs. The first portion of these, written before he had become aware of Landen's discoveries, contained new ideas concerning the use of elliptic arcs, notably a method of avoiding the use of hyperbolic arcs by replacing them with a suitably constructed table of elliptic arcs. He then gave a new demonstration of Landen's theorem and proved by the same method that every given ellipse is part of an infinite sequence of ellipses, related in such a way that by the rectification of two arbitrarily chosen ellipses the rectification of all the others is obtained. With this theorem it was possible to reduce the rectification of a given ellipse to that of two others differing arbitrarily little from a circle.

But this topic, and the theory of elliptic transcendentals in general, required a more systematic treatment. This is what Legendre, who was virtually alone in his interest in the problem, attempted to provide in his "Mémoire sur les transcendantes elliptiques" (1793). He proposed to compare all functions of this type, classify them into different species, reduce each one to the simplest possible form, evaluate them by the easiest and most rapid approximations, and create a sort of algorithm from the theory as a whole.

Later research having enabled him to perfect this theory in several respects, Legendre returned to it in the *Exercices* of 1811. On this occasion he gave his theory a trigonometric appearance. Setting , with $0 \leq c \leq 1$, he calls *c* the modulus of the function. The integral being taken form 0 to ϕ,ϕ is called its amplitude; is the complement of the modulus. The simplest of the elliptic transcendentals is the integral of the first kind, . The integral of the second kind, which is representable by an elliptic arc of major axis 1 and eccentricity *c,* takes the form $E(\phi) = \int \Delta d\phi$. The integral of the third kind is

with parameter *n*. Every elliptic integral can be expressed as a combination of these three types of transcendentals.

Let ϕ and ψ be two variables linked by the differential equation

The integral of the equation is $F\phi + F\psi = \mathrm{F}\mu$,$\mu$ being an arbitrary constant. Euler's theorem gives

Thus it allows μ to be found algebraically, such that $F\phi + \mathrm{F}\psi = \mathrm{F}\mu$, and then $\mathrm{F}\phi$ to be multiplied by an arbitrary number, whole or rational. From these observations Legendre deduced many consequences for each of the three kinds of integrals.

Furthermore, by designating the integral of the first kind with modules *c* and amplitude ϕ as $\mathrm{F}(c,\phi)$, Legendre was able, with the aid of Landen's theorem, to establish the transformation later called quadratic. Thus, if $\sin(2\phi' - \phi) = c \sin\phi$ and if then . Through repeated use of this transformation Legendre constructed and published (1817) his tables of elliptic functions. In 1825 he wrote:

To render the theory wholly useful it remained to construct a series of tables by means of which one could find, in every given case, the numerical value of the functions. These tables have finally been constructed, after a multitude of investigations undertaken with a view toward discovering the methods and formulas most suitable to diminishing the length and difficulty of the calculations.

The work Legendre published in 1826, volume II of *Traitédes fonctions elliptiques,* contains nine of these tables. The last one is “the general table of the functions *F* and *E* for each [sexagesimal] degree of the amplitude ϕ and of the angle of the modulus θ ($\sin \theta=c$) to ten decimal places for θ less than 45° and nine for θ between 45° and 90°” He wrote to Jacobi regarding these enormous computations that he carried out unassisted: “My goal has always been to introduce into calculation new elements that one can work with in arbitrary numbers, and I devoted myself to an exceedingly long and tedious task in order to construct the tables, a task I do not hesitate to believe is as considerable as that of Briggs’s great tables.”

Volume II of the *Traité* (1826) includes material on the construction of elliptic functions that is of the greatest interest from the point of view of numerical analysis. In particular Legendre presents a symbolic calculus, inspired by Lagrange, linking the expansion of a function by Taylor’s formula with the calculation of its finite differences of various orders. Other investigations of this topic may be found in the analogous studies by Arbogast and Kramp, and by the students of Hindenburg. The same volume of the *Traité* lists—to twelve decimal places—the logarithms of the values of the gamma function $\Gamma(x)$ when x varies in thousandths of an integer and ranges from 1 to 2 inclusively.

In the meantime, around 1825, chance led Legendre to examine two functions *F* linked by relationships between their moduli, on the one hand, and their amplitudes, on the other—relationship that do not arise in the context of Landen’s transformation. In generalizing these relationship Legendre discovered a new transformation closely related to the trisection of the function *F*. This trisection necessitated the solution of a ninth-degree algebraic equation. The new transformation reduced this solution to that of two third-degree equations.

In 1827, however, Jacobi (whose correspondence with Legendre, always of the greatest scientific and human interest, continued until 1832) communicated to Legendre his own discoveries and also informed him of Abel’s Legendre’s attitude in the face of the discoveries of his young rivals was remarkable for its enthusiasm and forthrightness. In the foreword to volume III of his *Traité* he announced:

A young geometer, M. Jacobi of Koenigsberg, who was not aware of the *Traité* [but who, it should be added, was familiar, like Abel, with the *Exercices* of 1811, has succeeded, through his own investigations, in discovering not only the second transformation, which is related to the number 3,but a third related to the number 5, and he has already become certain that there must exist a similar one for every given odd number.

Legendre also drew attention to Abel’s memoirs and analyzed their content. Legendre’s first two supplements are devoted primarily to Jacobi’s works but also to those of Abel, which contained the first appearance of modern elliptic functions, the inverse functions of the Legendre integrals. Legendre discussed their extension to the complex domain and their double periodicity in his usual somewhat ponderous style. The third supplement deals mainly with Abel and his great theorem. Legendre concluded his work on 4 March 1832: “We have only touched the surface of this subject, but it may be supposed that it will be steadily enriched by the works of mathematicians and that eventually it will constitute one of the most beautiful parts of the analysis of transcendental functions.”

In 1869 Charles Hermite made the following judgment concerning Legendre’s writings: “Legendre, who for so many reasons is considered the founder of the theory of elliptic functions, greatly smoothed the way for his successors; it is the fact of the double periodicity of the inverse function, immediately discovered by Abel and Jacobi, that is missing and that gave such a restrained analytical character to his *Traité des fonctions elliptiques*.”

Legendre’s *Éléments de géométrie* long dominated elementary instruction in the subject through its numerous editions and translations. Quite dogmatic in its presentation, this work marked a partial return to Euclid in France. The notes that accompany and enrich the text still have a certain interest. The text itself, virtually unchanged since the first edition, does not take into account the various contributions made by Monge’s disciples. The *Éléments* is above all a typical example of the difficulties encountered by the advocates of non-Euclidean geometries in their struggle to gain acceptance for their conceptions. The first published work of János Bolyai dates from 1832 and is thus contemporary with “Réflexions… sur la théorie des parallèles…,” in which Legendre recalled the efforts (all unsuccessful although he was not convinced of this) that he made between 1794 and 1823 to demonstrate Euclid’s postulate. Let us first mention two very positive achievements that date from 1800. First: “The sum of the three angles of a rectilinear triangle cannot be greater than two right angles,” a proposition he arrived at, of course, by accepting all the axioms, theorems, and postulates preceding the parallel postulate in Euclid’s *Elements*. Second: “If there exists a single triangle in which the sum of the angles is equal to two right angles, then in any triangle the sum of the angles will likewise be equal to two right angles.”

Yet, aside from these beautiful theorems, demonstrated in an impeccable manner, hardly anything but paralogisms are to be found. Like all the disciple of Newton, Legendre believed in absolute space and in the “absolute magnitude” of the sides of a rectilinear triangle. Taking up again a favorite idea of Lagrange’s, outlined by the latter in volume 2 of the *Mémoires de Turin* (1761), he utilized (1794) the “law of homogeneous magnitudes” to establish the theorem of the sum of the angles of a triangle. Suppose a triangle is given with a side a and the adjacent angles B and C. The triangle is then well defined. The third angle A is therefore a function of the given quantities: $A = \phi(B,C,a)$. But A, B, and C are pure numbers and a is a length. Now, by solving the equation $A =\phi(B,C,a,)$ with a as the unknown, the equation $a = f(A,B,C,)$ is obtained “from which it would result that the side *a* is equal to a pure number without dimension, which is absurd." The law of homogeneous magnitudes therefore requires that this length disappear from the formula at the start and thus that $A= \phi(B,C,)$. By considering a right triangle and its altitude it is easily found that $A + B + C = \pi$.

Until the end of his life Legendre remained convinced of the value of this reasoning, and his other attempts at demonstration held only a purely pedagogical interest for him. They all failed because he always relied, in the last analysis, on propositions that were "evident" from the Euclidean point of view. Among these are the following: two convex contours of opposed concavities intersect at a finite distance; from a point within an angle a straight line cutting the two sides can always be drawn; and through three nonaligned points a circumference of a circle can always be passed.

At the end of his *Réflexions,* Legendre even adopted the pseudodemonstration of Louis Bertrand involving infinite spaces of various orders; and he thought he had improved it by what was actually an even more obscure argument. His virtuosity in spherical geometry and spherical trigonometry did not free him from a blind belief in absolute Euclidean space. "It is nevertheless certain," he wrote in 1832, "that the theorem on the sum of the three angles of the triangle should be considered one of those fundamental truths that are impossible to contest and that are an enduring example of mathematical certitude."

Let us conclude this study by emphasizing the transitional character of Legendre's works, which, in time as well as in spirit, are neither completely of the eighteenth century nor of the nineteenth. His scientific activity, extending from about 1770 to the end of 1832, was divided equally between the two centuries. He was a first-rate disciple of Lagrange and above all of Euler. His writings, like theirs, treat both abstract mathematics and the application of mathematics to the system of the world. Yet his boundless confidence in the powers of abstract science bespeaks a certain inïveté. In 1808 he wrote: "It is remarkable that from integral calculus an essential proposition concerning prime numbers can be deduced." The law in question was that of the distribution of the primes, and his remark is very pertinent. He added: "All the truths of mathematics are linked to each other, and all means of discovering them are equally admissible." One can easily agree with him on this point. But he went further, and here he makes one smile: "Consequently we were led to consider functions in order to demonstrate various basic theorems of geometry and mechanics."

Number theory was a sound and difficult school of logic for Legendre, as for all mathematicians who have worked in that field. Yet on several occasions, as in his studies of Eulerian integrals, he employed disconcerting arguments. For example, having established for positive integral values a certain relationship in which the gamma function occurs, he declares that the relationship is true for every value of this variable because the two members of the relationship are continuous function.elsewhere he elaborates on his reasoning: the two members are reduced to the same expression, which he works out; it is an extremely divergent (in the modern sense of the term) series. Still another time he employs an "infinite constant."

Consequently Legendre's writings rapidly became obsolete. Nevertheless he remains a marvelous calculator, a skillful analyst, and, in sum, a good mathematician. In both the theory of elliptic functions and number theory he raised questions that were fruitful subjects of investigation for mathematicians of the nineteenth century.

BIBLIOGRAPHY

I. Original Works. Legendre's writings are *Theses mathematicae ex analysi, geometria, mecanica exerptae, ex collegio Mazarinaeo* (Paris, 1770); Abbé, *Traité de mécanique*(Paris, 1774), which includes several passages by Legendre; *Recherches sur la trajectoire des projectiles dans les milieux résistants* (Berlin, 1782); "Recherches sur la figure des planéts," in *Mémoires de l'Académie des sciences* for 1784, pp. 370ff.; "Recherches d'analyse indéterminée," *ibid.* for 1785, pp. 465-560; "Mémoire sur la manière de distinguer les maxima des minima dans le calcul des variations," *ibid.* for 1786, pp. 7-37, and 1787, 348-351; German trans. in Ostwalds Klassiker der exacten Wissenschaften, no. 47; "Mémoires sur les intégrations par arcs d'ellipse et sur la comparaison de ces arcs," in *Mé moires de l'Académie des sciences* for 1786, pp. 616, 644-673; "Mémoire sur les i'ntégration de quelques équations aux différences partielles," *ibid.* for 1787, pp. 309-351; "Mémoire sur les opérations trigonométriques dont les résultats dépendent de la figure de la terre," *ibid.* for 1787, pp. 352 ff.; "Mémoire sur les intégrales doubles," *ibid.* for 1788, pp. 454-486; "Recherches sur les sphéroïdes homogènes," *ibid.* for 1790; and "Mémoire sur les intégrales particulières des équations différentielles," *ibid.* for 1790.

Works published after the establishment of the republic are *Mémoire sur les transcendates elliptiques* (Paris, 1793); *Éléments de géométrie, avec des notes* (Paris, 1794; 12th ed., 1823), new ed. with additions by A. Blanchet (Paris, 1845; 21st ed., 1876)—these eds. often depart from Legendre's text; English trans. by John Farrar (Cambridge, Mass., 1819) and other English eds. until 1890; German trans. by A. L. Crelle (Berlin, 1822); Romanian trans. by P. Poenaru (Bucharest, 1837); *Essai sur la théorie des nombres* (Paris, 1798; 2nd ed., 1808; with supps., 1816, 1825; 3rd ed., **2** vols., Paris, 1830) and German trans. by Maser as *Zahlentheorie* (Leipzig, 1893); *Méthodes analytiques pour la détermination d'um arc de méridien, par Delambre et Legendre* (Paris, 1799); "Nouvelle formule pour réduire en distances vraies les distances apparentes de la lune au soleil ou à une étoile," in *Mémoires de I'Institut national des sciences et arts,***6** (1806), 30-54; and*Nouvelles méthodes pour la détermination des orbites des comètes…* (Paris, 1806).

Subsequent works are "Analyse des triangles tracés sur la surface d'un sphéroïde," in *Mémoires de l'Institut national des sciences et arts*, **7** , pt. 1 (1806), 130-161; "Recherches sur diverses sortes d'intégrales définites," *ibid.* (1809), pp. 416-509; "Méthode des moindres carrés pour trouver le milieu le plus probable entre les résultats de différentes observations," *ibid.*, pt.

2 (1810), pp. 155-183 with supp. (Paris, 1820); "Mémoire sur l'attractions des ellipsoïdes homogènes," *ibid*., pt. ,**2** (1810), pp.155-183 read 5 Oct. 1812; *Exercices de calcul intégral*, **3** vols. (Paris, 1811-1817); "Sur une méthode d'interpolation employée par Briggs dans la construction de ses grandes tables trigonométriques," in *Connaissance des temps ou des mouvements célestes pour 1817*, **10** (Paris, 1815), 219-222; "Méthodes diverses pour faciliter l'interpolation des grandes tables trigonométriques," *ibid*., pp.302-331; and "Recherches sur quelques objects d'analyse indéterminée et particulièrement sur le théorème de Fermat," in *Mé moires de l'Académie des sciences*, n.s. **6** (1823), 1-60.

Traité des fonctions elliptiques et des intégrales eulériennes, avec des tables pour en faciliter le calcul numérique was published in 3 vols. (Paris, 1825-1828); vol. I contains the theory of elliptic functions and its application to various problems of geometry and mechanics; vol. **II** contains the methods of constructing the elliptical tables, a collection of these tables, the treatise on Eulerian integrals, and an appendix; and vol. III includes various supplements to the theory of Eulerian functions. The three supplements are dated 12 Aug. 1828, 15 Mar. 1829, and 4 Mar. 1832.

Legendre's last works are "Note sur les nouvelles propriétés des fonctions elliptiques découvertes par M. Jacobi," in *Astronomische Nachrichten*, **6** (1828), cols. 205-208; "Mémoire sur la détermination des fonctions Y et Z qui satisfont à l'équation: $4(x^n-1)=(x-1)(Y^2 \pm nZ^2)$," in *Mémoires de l'Académie des sciences,* **11** (1832), 81-99, read 11 Oct. 1830; and "Réflexions sur différentes manières de démontrer la théorie des parallèles ou le théorème sur la somme des trois angles du triangle," *ibid*., **12** (1833), 367-410.

For Legendre's mathematical correspondence with Jacobi, see C. G. J. Jacobi, *Gesammelte Werke*, I (Berlin, 1881), 386-461. For his correspondence with Abel, see N. H. Abel,*Mémorial publié à l'occasion du centenaire de sa naissance,* pt. 2 (Oslo, 1902): pp. 77-79 (Legendre's letter to Abel, 25 Oct. 1828), pp. 82-90 (Abel's letter to Legendre, 25 Nov. 1828), and pp. 91-93 (Legendre's letter to Abel, 16 Jan. 1829).

II. Secondary Literature. On Legendre and his work, see Élie de Beaumont, "éloge historique d'Adrien-Marie Legendre, lu le 25 mars 1861," in *Mémoires de l'Académie des sciences*, **32** (1864), xxxvii–xciv; J. B. J. Delambre, *Rapport historique sur les progrès des sciences mathématiques depuis 1789, et sur leur état actuel* (Paris, 1810), pp. 7-10, 34, 46-96, 135-137, in which S. F. Lacroix was responsible for everything concerning pure mathematics; A. Birembaut, "Les deux déterminations de l'unité de masse du systéme mètrique," in *Revue d'histoire des sciences et de leurs applications*, **12** , no.**1** (1958), 24-54; C. D. Hellman, "Legendre and the French Reform of Weights and Measures," in *Osiris*, **1** (1936), 314-340; Jacob, "A. M. Legendre," in Hoefer, ed., *Nouvelle biographie générale,* XXX (Paris, 1862), cols. 385-388; Gino Loria, *Storia delle matematiche*, 2nd ed. (Milan, 1950), pp. 768-770; Maximilien Marie, *Histoire des mathématiques*, **X** (Paris, 1887). 110-148; L. Maurice, "Mémoire sur les travaux et les écrits de M. Legendre," in *Bibliothèque universelle des sciences, belles-lettres et arts. Sciences et arts* (Geneva), **52** (1833), 45-82; and E. H. Neville, "A Biographical Note," in *Mathematical Gazette*, **17** (1933), 200-201.

See also N. Nielsen, *Géomètres français sous la Révolution* (Copenhagen, 1929), pp. 166-174; Maurice d'Ocagne, *Histoire abrégée des mathématiques* (Paris, 1955), 182-187; Parisot, "Legendre," in Michaud, ed., *Biographie universelle ancienne et moderne*, XXIII, 610-615; Poggendorff, **I,** cols. 1406-1407; Denis Poisson, "Discours prononcé aux funérailles de M. Legendre…," in *Moniteur universel* (20 Jan. 1833), p. 162; J. M. Querard, "Legendre, Adrien-Marie," in *France littéraire*, **5** (1833), 94-95; A. Rabbe, *Biographie universelle et portative des contemporains*, III (Paris, 1834), 234-235; L. G. Simons, "The Influence of French Mathematicians at the End of the Eighteenth Century Upon the Teaching of Mathematics in American Colleges," in *Isis*, **15** , no. 45 (Feb. 1931), 104-123; D. E. Smith, "Legendre on Least Squares," in *A Source Book in Mathematics* (New York, 1929), pp. 576-579; I. Todhunter, *A History of the Progress of the Calculus of Variations* (Cambridge, 1861), ch. 9, pp. 229-253; A. Aubry, "Sur les travaux arithmétiques de Lagrange, de Legendre et de Gauss," in *Enseignement mathématique*, **11** (1909), 430-450; and Ivor Grattan-Guinness, *The Development of the Foundations of Mathematical Analysis from Euler to Riemann* (Cambridge, Mass., 1970), pp. 29, 36-41.

Jean Itard