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(*b*. Ta-hsing [now Peking], China, 1192; *d*. Hopeh province, China, 1279),

mathematics.

Li Chih (literary name, Jen-ch'ing; appellation, Ching-chai) has been described by George Sarton as one of the greatest mathematicians of his time and of his race. His father, Li Yü, served as an attaché under a Jurchen officer called by the Chinese name Hu Sha-hu. Li Yü later sent his family back to his home in Luan-ch'eng, Hopeh province. Li Chih went alone to the Yüan-shih district in the same province for his education.

In 1230 Li Chih went to Loyang to take the civil service examination; after he passed, he was appointed registrar in the district of Kao-ling, Shensi province. Before he reported for duty, however, he was made governor of Chün-chou (now Yü-hsien), Honan province. In 1232 the Mongols captured the city of Chün-chou, and Li Chih was forced to seek refuge in Shansi province, The kingdom of the Jurchen fell into the hands of the Mongols in 1234. From that time on, Li Chih devoted himself to serious study, frequently living in poverty. It was during this period that he wrote his most important mathematical work, the *Ts'e-yüan hai-ching* ("Sea Mirror of the Circle Measurements").

About 1251 Li Chih, finding himself in an improved financial position, returned to the Yüan-shih district of Hopeh province and settled near Feng-lung, a mountain in that district. Although he continued to lead the life of a scholarly recluse, he counted Chang Te-hui and Yuan Yü among his friends; the three of them became popularly known as "the Three Friends of [Feng-]Lung Mountain." In 1257 Kublai Khan sent for Li Chih and asked him about the government of the state, the selection and deployment of scholars for civil service, and the reasons for earthquakes. Li Chih complected another mathematical text, the *I-ku yen-tuan* ("New Steps in Computation") in 1259. Kublai Khan ascended the throne in 1260 and the following year offered Li Chih a government post, which was politely declined with the plea of ill health and old age. In 1264 the Mongolian emperor set up the Han-lin Academy for the purpose of writing the official histories of the kingdoms of Liao and Jurchen, and the following year Li Chih was obliged to join it. After a few months he submitted his resignation, again pleading infirmity and old age. He returned to his home near Feng-lung, and many pupils came to study under him.

Li Chih changed his name to Li Yen at some point in his life because he wished to avoid having the same name as the third Tang emperor, whose dynastic title was Tang Kao-tsung (650-683). This circumstance has given rise to some confusion as to whether Li Yeh was a misprint for Li Chih.

Besides the *Ts'e-yüan hai-ching* and the *I-ku yentuan* Li Chih wrote several other works, including the *Fan shuo*, the *Ching-chai ku-chin chu*, the *Wen chi* and the *Pi-shu ts'ung-hsiao*. Before his death Lt Chih told his son, Li K'e-hsiu, to burn all his books except the *Ts'e-yüan hai-ching*, because he felt that it alone would be of use to future generations. We do not know to what extent his wishes were carried out; but the *I-ku yen-tuan* survived the fire, and the *Ching-chai ku-chin chu* has also come down to us. His other works are now lost, although some passages from the *Fan shuo* are quoted in the *Ching-chaiku-chin chu*. Only the *Ts'e-yüan hai-ching* and the *I-ku yen-tuan* will be further described here, since the other extant work has neither mathematical nor scientific interest.

Originally called the *Ts'e-yüan hai-ching hsi-ts'ao* and completed in 1248, the *Ts'e-yüan hai-ching* was not published until some thirty years later, at about the same time as the *I-ku yen-tuan*. From a preface written by Wang Te-yüan, it appears that there was a second edition in 1287. In the late eighteenth century the *Ts'e-yüan hai-ching* was included in the imperial encyclopedia, the *Ssu-k'u ch'üan-shu*. It came from a copy preserved in the private library of Li Huang (*d*. 1811). A handwritten copy of the book was made by Juan Yuan (1764-1849) from the version in the *Ssu-k'u ch'üan-shu*. Later Ting Chieh presented a handwritten fourteenth-century copy of the *Ts'e-yüan hai-ching hsi-ts'ao* with the seal of Sung Lien (1310-1381) to Juan Yuan. This is probably the copy that is now preserved in the Peking Library. At the request of Juan Yuan, the Ch'ing mathematician Li Jui (1768-1817) collated the two versions in 1797. This has become the most widely circulated edition of the *Ts'e-yüan hai-ching* that exists today. In 1798 Li Jui's version was incorporated in the *Chih-pu-tsu-chai ts'ung-shu* collection, and in 1875 in the *Pai-fu-t'ang suan-hsüeh ts'ung-shu* collection. The modern reproduction in the *Ts'ung-shu chi-ch'eng* series is based on the version in the *Chih-pu-tsu-chai ts'ung-shu* collection.

The *Ts'e-yüan hai-ching* was studied by many eighteenth- and nineteenth-century Chinese mathematicians, such as K'ung Kuang-shen (1752-1786) and Li Shan-lan (1811-1882). A detailed analysis of the work was made by Li Yen (1892-1963), but it has not yet been translated.

The *I-ku yen-tuan* was completed in 1259 and was published in 1282. It has been regarded as a later version of a previous mathematical text, the I-ku-chi, which is no longer extant. The *I-ku yen-tuan* is incorporated in both the *Chih-pu-tsu-chai ts'ung-shu* and the *Pai-fu-t'ang suan-hsüeh ts'ung-shu* collections. The modern reproduction of the *I-ku yen-tuan* in the *Ts'ung-shu chi-ch'eng* series is based on the version in the *Chih-pu-tsu-chai ts'ung-shu* collection. It has been translated into French by L. van Hée.

Li Chih introduced an algebraic process called the *t'ien yüan shu* ("method of the celestial elements" or "coefficient array method") for setting up equations to any degree. The *t'ien yüan shu* occupied a very important position in the history of mathematics in both China and Japan. From the early fourteenth century until algebra was brought to China from the West by the Jesuits, no one in China seemed to understand this method. Algebra enabled Chinese mathematicians of the eighteenth century, especially Mei Ku-ch'eng, to recognize the algebra of the *t'ien yüan shu* and the *ssu yüan shu* of Chu Shihchieh despite their unfamiliar notation. Knowing that algebra originally entered Europe from the East, some enthusiastic Chinese scholars of that time went so far as to claim that the *t'ien yüan shu* had gone from China to the West and there became known as algebra. The *t'ien yüan shu* also exerted a profound influence in Japan, where it became known as the *tengenjutsu*. The seventeenth-century Japanese mathematician Seki Takakazu (also known as Seki Kōwa), for example, developed from the algebra of Li Chih and Chu Shih-chieh a formula for infinite expansion, which is now arrived at by means of the infinitesimal calculus.

Li Chih did not claim to be the originator of the *t'ien yüan shu*. From his *Ching-chai ku-chin chu* it appears that he had copied the method from a certain mathematician in Taiyuan (in modern Shansi) named P'eng Che (literary name, Yen-ts'ai), of whom we know little, Chu Shih-chieh wrote in the early fourteenth century that one of Li Chih's friends, the

famous poet Yuan Hao-wen, was also versed in the method of the celestial element. In the *t'ien yüan shu* method the absolute term is denoted by the character *t*'ai and the unknown by *yüan*, or element. An equation is arranged in a vertical column in which the term containing the unknown is set above the absolute term, the square of the unknown above the unknown, then the cube, and so on in increasing powers. Reciprocals or negative powers can also be placed in descending order after the absolute term. Thus the equation

$$-x^2+8640+652320x^{-1}+4665600x^{-2}=0$$

is represented on the countingboard as in Figure 1*a* with its equivalent in Arabic numerals shown in Figure l*b*. It is sufficient to indicate one position of the unknown and one of the absolute term by writing the words *yüan* and *t' ai*. It is curious that Li Chih reversed the process of expressing algebraic equations in his *I-ku yen-tuan* by writing the unknown below the absolute, the square below the unknown, and so on. For example, the equation $1700- 80x - 0.25x^2 =0$ is shown in Figure 1*c* and in Arabic numerals in Figure 1*d*. Li Chih's method was followed by later Chinese mathematicians.

Li Chih indicated negative quantities by drawing an oblique line over the final digit of the number concerned. He also used the zero symbol as his contemporary Ch'in Chiu-shao did, although there is no evidence that the two ever met or even heard of each other. It is likely that the zero symbol was used earlier in China, and it has even been suggested by Yen Tun-chieh that although the dot (*bindu*) was introduced from India in the eighth century and the circle for zero appeared in a magic square brought from the area of Islamic culture in the thirteenth century, the circle was nevertheless evolved independently in China from the square denoting zero sometime during the twelfth century.

Li Chih made use of numerical equations up to the sixth degree, He did not describe the procedure of solving such equations, which omission indicates that the method must have been well known in China during his time. This method must be similar to that rediscovered independently by Ruffini and Horner in the early nineteenth century, as described by Ch'in Chiu-shao. Li Chih stabilized the terminology used in connection with equations of higher degrees in the form

$$+ax^6+bx^5+cx^4+dx^3+ex^2+fx+g=0.$$

The absolute term g is called by the general term *shih*, or by the more specific terms *p' ing shih, fang shih, erh ch' eng fang shih, san ch'eng fang shih, shih ch' eng fang shih,* and *wu ch' eng fang shih* for linear equations, quadratic equations, cubic equations, quartic equations, and equations of the fifth and sixth degrees, respectively. The coefficient of the highest power of x (in this case, a) in the equation is called *yü, yü fa*, or *ch'ang fa*. The coefficient of the lowest power of x (in this case, f) is called *ts'ung* or *ts'ung fang*. All coefficients between the lowest and the highest powers are described by the word *lien*. For a cubic equation the coefficient of x^2 is known by the term *lien*. For a quartic equation the coefficient of x^2 is called *ti i lien*, that is, the first *lien*, and the coefficient of x^3 is *ti erh lien*, that is, the second *lien*. Hence, in the sixth-degree equation above we have *e, ti i lien* (first *lien*); *d, ti erh lien* (second *lien*); *c, ti san lien* (third *lien*); and *b, ti shih lien* (fourth *lien*).

All the above, except in the case of the absolute term, apply only to positive numbers. To denote negative numbers, Li Chih added either the word *i* or the word *hsü* before the terms applying to coefficients of the highest and lowest powers of x and before the word *lien*. He did not use different terms to distinguish between positive and negative absolute terms and, unlike Ch'in Chiu-shao, he did not make it a rule that the absolute term must be negative.

It is interesting to see how Li Chih handled the remainder in extracting a square root. An example is encountered in the equation

$$-22.5x^2-648x+23002=0,$$

which occurred in the fortieth problem of his I-*ku yen-tuan*. He put $y = nx$, where $n = 22.5$, and transformed the equation into

$$-y^2-648y+517545=0.$$

From the above, $y = 465$, and hence $x = 202\ 2/3$, The same method was also used by Ch'in Chiu-shao.

The *Ts'e-yüan hai-ching* includes 170 problems dealing with various situations based on a circle inscribed in or circumscribing a right triangle. The same question is asked in all these problems and the same answer obtained. The book begins with a diagram showing a circle inscribed in a right triangle, *ABC* (Fig. 2). The square *CDEF* circumscribing the

circle lies along the base and height of ΔABC and intersects the hypotenuse *AB* at *G* and *H,* from which perpendiculars *GJ* and *HK* are dropped on *BC* and *AC,* respectively. *GJ* and *HK* intersect at *L*. Through *O,* the center of the circle, *MNPOQ* is drawn parallel to *AC,* meeting *AB* at *M* and *BC* at *Q* and Cutting *ED* at *N* and *HK* at *P*. Also through *O* is drawn *RSTOU,* parallel to *BC* meeting *AB* at *R* and *AC* at *U* and cutting *EF* at *S* and *GJ* at *T*. Finally, *MV* is drawn parallel to *BC,* meeting *AC* at *V,* and *RW* is drawn parallel to *AC,* meeting *BC* at *W*.

Special terms are then given for the three sides of each of the fifteen triangles in the diagram. These are followed by a list of relationships between the sides of some of these triangles and the circle. For example: the sides of $\Delta\ ABC$ and the diameter of the inscribed circle have the relationship $D = 2ab/(a + b + c)$; the sides of $\Delta\ ARU$ and the diameter of a circle with its center at one of the sides and touching the other two sides have the relationship $D = 2ar/(r + u)$; and the sides of $\Delta\ AGD$ and the diameter of an escribed circle touching the side *ED* and the sides *AG* and *AD* produced have the relationship $D = 2ag/(g + d - a)$. Similarly, for $\Delta\ MBQ, D=2mb/(m + q)$; for $\Delta HBF, D= shb/\{h +f - b)$; for $\Delta MRO; D =2mr/o;$ for $\Delta GHE, D = 2hg/(h +g-e);$ for $\Delta MGN, D = 2mg/(n - m);$ and for $\Delta HRS, D = 2hr/(s - r)$.

All the above are given in chapter 1 of the book. In the subsequent chapters Li Chih showed how these results can be applied to various cases. For example, the second problem in chapter 2 says:

Two persons, *A* and *B,* start from the western gate [of a circular city wall], *B* [first] walks a distance of 256 *pu* eastward, Then *A* walks a distance of 480 *pu* south before he can see *B,* Find [the diameter of the wall] as before.

The equation for ΔMBQ was then applied directly to give the diameter of the circular city wall.

Li Chih showed how to solve a similar problem by the use of a cubic equation. The fourth problem in chapter 3 says:

A leaves the western gate [of a circular city wall] and walks south for 480 *pu*. *B* leaves the eastern gate and walks straight ahead a distance of 16 *pu,* when he just begins to see *A*. Find [the diameter of the city wall] as before.

Here Li Chih found *x,* the diameter of the city wall, by solving the cubic equation

$$x^3+cx^2-4cb^2=0,$$

where $c = 16$ *pu* and $b = 480$ *pu*. This obviously came from the quartic equation

$$x^4+2cx^3+c^2x^2-4cb^2x-4c^2b^2=0,$$

which can be derived directly from the equation for ΔMBQ. In doing this Li Chih had discarded the factor $(x + c)$, knowing that the answer $x = -c$ was inadmissible. It is interesting to compare this with the tenth-degree equation used by Ch'in Chiu-shao for the same purpose.

All the 170 problems in the *T'e-yüan hai-ching* have been studied by Li Yen. To illustrate Li Chih's method of solving these problems, we shall follow step by step the working of problem 18 in chapter 11, which says:

135 *pu* directly out of the southern gate [of a circular city wall] is a tree. If one walks 15 *pu* out of the northern city gate and then turns east for a distance of 208 *pu,* the tree becomes visible. Find [the diameter of the city wall] as before. [The answer says 240 *pu* as before.]

Using modern conventions but following the traditional Chinese method of indicating the cardinalpoints in such a way that south is at the top, north at the bottom, east to the left, and west to the right, the problem is as illustrated in Figure 3.

First, the method is given:

Take the product of the distance to the east and that to the south [that is, $c(b+b')$], square it, and make it the *shih* [that is, $c^2(b+b')^2$ is taken as the absolute term]. Square the distance to the east, multiply it by the distance to the south, and double it to form the t*ts'ung* [that is, the coefficient of x, the radius, is $2c(b+b')^2$]. Put aside the square of the distance to the east [that is, $(b + b')^2$]; add together the distance to the south and north, subtract the sum from the distance to the east, square the result, and subtract this from the distance to the east [that is, $(b+b')^2 - \{(b+b') - (c + c')\}^2$]; put aside this value. Again add together the distances to the south and north, multiply the sum first by the distance to the east and then by 2 [that is, $2(b + b')(c + c')$]; subtracting from this the amount just set aside gives the *tai i i lien* [that is, the coefficient of x^2 is $-2\{(b + b')(c + c')\} - [(b + b')^2 - \{(b+b')-(c +c')\}^2]$]. Multiply the distance to the east by 4 and put this aside [that is, $4(b + b')$]; add the distances to the south and north, subtract the sum from the eastward distance and multiply by 4 [that is, $4\{(b + b') + (c + c')\}$]; subtracting this result from the amount put aside gives the *tai erh i lien* [that is, the coefficient of x^3 is $-4(b + b') - 4\{(b + b') + (c + c')\}$). Take 4 times the *hsü yü* [that is, $-4x^4$]. Solving the quartic equation gives the radius.

The procedure is as follows;

Set up one celestial element to represent the radius. (This is the *kao kou.*) Adding to this the distance to the south gives Figure 4*a* (below), which represents the

kao hsien (that is, the vertical side *MO* in *MRO* in Figure 3 is x+135). Put down the value for the *ta kou* (base of *MB'Q'*) of 208 and multiply it by the *Kao hsien* (x+135); the result is 208x+28080 (Figure 4*b*).

Dividing this by the *kao kou* gives 208 + 28080x^{-1} (FIGURE 4*c*).

This is the *ta hsien* (the hypotenuse *MB'* of $\Delta MB'Q'$, and squaring this yields 43264 + 11681280x^{-1} + 788486400x^{-2}(Figure 4*d*).

The above is temporarily set aside. Take 2 as the celestial element and add this to the sum of the distances to the south and north, giving 2x + 150 (Figure 4*e*),

or the *ta ku* (the vertical side *MQ'* of $\Delta MB'Q'$). Subtracting from this the value of the *ta kou*, 208, 2x—58 is obtained (Figure 4*f*).

This is known as the *chiao*. Squaring it gives the *chiao mi,* $4x^2 - 232x + 3364$, shown in Figure 4*g,*

Subtracting the *chiao mi* from the quantity set aside (Figure 4d) yields—$4x^2+232x + 39900 + 11681280x^{-1} + 788486400x^{-2}$ (Figure 4*h*).

This value, known as *erh-chih-chi*, is set aside at the lefthand side of the countingboard. Next multiply the taku *(MQ')* by the *ta kou (B'Q'),* which yields 416x + 31200 (Figure 4*i*).

This is the *chih chi*, which, when doubled, gives 832x + 62400 (Figure 4*j*).

This value is the same as that set aside at the left (that is, the value represented by Figure 4h). Equating the two values, one obtains $-4x^4 - 600x^3 - 22500x^2 + 11681280x + 788486400 = 0$ (Figure 4*k*).

Note that the position of the celestial element is no longer indicated here.

Solving this as a quartic equation, one obtains 120 *pu,* the radius of the circular city wall, which corresponds with the required answer.

Although written much later than the *Ts'e-yüanhai-ching,* the *I-ku yen-tuan* is considerably simpler in its contents. It is thought that Li Chih took this opportunity to explain the *t'ien yüan shu* method in a less complicated manner after finding his first book too difficult for people to understand. This second mathematical treatise has also been regarded as a later version of another work, *I-ku chi,* published between 1078 and 1224 and no longer extant. According to a preface in Chu Shih-chieh's *Ssu-yüan yu-chien,* the *I-ku chi* was written by a certain Chiang Chou of P'ing-yang. Out of a total of sixty-four problems in the *I-ku yen-tuan,* twenty-one are referred to as the "old method" *(chiu shu),* which presumably means the *I-ku chi*. Sixteen of these twenty-one problems deal with the quadratic equation

$ax^2 + bx - c = 0,$

where $a > 0$ or $a < 0$, $b > 0$ and $c > 0$. When $c > 0$ and $b > 0$, they are called by the terms *shih* and *ts'ung,* respectively. When $a > 0$, it is known as *lien* instead of the more general term *yü,* and when $a < 0$, it is also known as *lien,* but is followed by the words *chien ts'ung*.

Divided into three chapters, the *I-ku yen-tuan* deals with the combination of a circle and a square or, in a few cases, a circle and a rectangle. A full translation of the first problem in chapter 1 is given below:

A square farm with a circular pool of water in the center has an area 13 *mou* and 7 1/2 tenths of a *mou* [that is, 13.75 *mou*]. The pool is 20 *pu* from the edge [1 *mou* = 240 square *pu*]. Find the side of the square and the diameter of the pool.

Answer: Side of square = 60 *pu,* diameter of pool = 20 *pu*.

Method: Put down one [counting rod] as the celestial element to represent the diameter of the pool. By adding twice the distance from the edge of the pool to the side of the farm, the side of the square farm is given by $x + 40$ [Figure 5*a*].

The square of the side gives the area of the farm and the circular pool. That is, the total area is given by $x^2 + 80x + 1600$ [Figure 5*b*].

Again, put down one [counting rod] as the celestial element to denote the diameter of the pool, Squaring the diameter, and multiplying the result by 3 [the ancient approximate value of π], then dividing the

result by 4, yields the area of the pool: $0.75x^2$ [Figure *5c*].

Subtracting the area of the pool from the total area gives $0.25x^2+80x+1600$ [Figure 5*d*].

The given area is 3,300 [square] *pu*. Equating this with the above yields$-0.25x^2-80x+1700=0$ [Figure 5*e].*

Applying the method of solving quadratic equations shows the diameter of the pool to be 20 *pu*. If the distance from the side of the square to the edge of the pool is doubled and added to the diameter of the pool, the side of the farm is found to be 60 *pu*.

The first ten problems in the *I-ku yen-tuan* deal with a circle in the center of a square, each with different given parameters; the next ten problems are concerned with a square inside a circle. Problems 21 and 22, the last two problems in chapter I, are concerned only with squares. In chapter 2, problems 23-29, we have the combination of a square with a circumscribed circle, while problem 30 gives the combination of two circles. Problem 31 concerns a rectangle with a circle in the center, and problems 32-37 give a circle with a rectangle at its center. In problem 38 two rectangles are given. Problems 39-42 treat various cases of a circle inside a rectangle. Problem 43, the first in chapter 3, deals with the three different values of π: the ancient value π =3, the “close” value π = 22/7, and Liu Hui’s value π =3.14. Problem 44 is concerned with a trapezium, problem 45 with a square inside another square, problem 46 with a circle set outside but along the extended diagonal of a square, problem 47 with a square within a rectangle, and problem 48 with a rectangle within a square. In problems 49-52 a square is placed in the center of a larger square so that the diagonal of one is perpendicular to two sides of the other. In problems 53 and 54 the central square is replaced by a rectangle. Problems 55 and 56 are concerned with the annulus, and problems 57 and 58 with a rectangle inside a circle. In problem 59 a square encloses a circle, which in turn encloses another square at the center; and in problem 60 a circle encloses a square, which in turn encloses a circle at the center. Problem 62 concerns a square placed diagonally at a corner of another square. Problem 63 concerns a circle and two squares with another circle enclosed in one of them. The last problem, 64, has an annulus enclosed by a larger square.

Li Chih and Ch’in Chiu-shao were contemporaries, but they never mentioned each other in their writings. Li Chih lived in the north and Ch’in Chiu-shao in the south during the time when China was ruled in the south by the Sung dynasty and in the north first by the Jurchen Tartars and later by the Mongols. It is very likely that the two never even heard of each other. The terminology they used for equations of higher degree is similar but not identical. They also employed the so-called celestial element in different ways. Li Chih used it to denote the unknown quantity; but to Ch’in Chiu-shao the celestial element was a known number, and he never used the term in connection with his numerical equations. Ch’in Chiu-shao went into great detail in explaining the process of root extraction of numerical equations, but he did not describe how such equations were constructed by algebraic considerations from the given data in the problems. On the contrary, Li Chih concentrated on the method of setting out such equations algebraically without explaining the process of solving them. Thus Li Chih was indeed, as George Sarton says, essentially an algebraist.

BIBLIOGRAPHY

Works that may be consulted for further information on Li Chih and his writings are Ch’ien Pao-tsung, *Chung-kuo suan hsüeh-shih* [“History of Chinese Mathematics”] (Peking, 1932), pp. 116-124; Ch’ien Pao-tsung *et al., Sung Yuan shu-hsüeh-shih lun-wen-chi* [“Collection of Essays on Chinese Mathematics in the Periods of Sung and Yuan”] (Peking, 1966), pp. 104-148; L. van Heé, “Li Yeh, mathématicien chinois du XIII[e] siècle,” in T’oung Pao, **14** (1913), pp. 537-568; Juan Yuan, Ch’ou-jen chuan [“Biographies of Mathematicians and Astronomers”] (Shanghai, 1935); Li Yen, *Chung-kuo shu-hsüeh ta-kang* [“Outline of Chinese Mathematics”], I (Shanghai, 1931), pp. 141-156; *Chung-kuo suan-hsüeh-shih* [“History of Chinese Mathematics”] (Shanghai, 1937), pp. 99-100; and “Chung-suan-shih lun-ts’ung,” in *Gesammelte Abhandlungen über die Geschichte der chinesischen Mathematik,***4,** no. 1 (1947), 15-251; Li Yen and Tu Shih-jan, *Chung-kuo ku-tai shu-hsüeh chien-shih* [“Concise History of Ancient Chinese Mathematics”], II (Peking, 1964), pp. 145, 147; Yoshio Mikami, *The Development of Mathematics*

in China and Japan (Leipzig, 1913), pp. 79-84; Joseph Needham, *Science and Civilisation in China,* III (Cambridge, 1959), esp. pp. 40-41; George Sarton, *Introduction to the History of Science,* 3 vols. (Baltimore, 1927-1947), esp. vol. 2, pp. 627-628; Sung Lien *et al., Yuan shih* ["Official History of the Yuan Dynasty"] (ca. 1370), ch. 160; and Alexander Wylie, *Notes on Chinese Literature* (Shanghai, 1902), p. 116.

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