Kodaira, Kunihiko | Encyclopedia.com

Complete Dictionary of Scientific Biography COPYRIGHT 2008 Charles Scribner's Sons
15-19 minutes

(*b*. Tokyo, Japan, 16 March 1915; *d*. Kofu, Japan, 26 July 1997)

mathematics, complex manifolds, complex structures.

Kodaira was one of the leading figures in complex algebraic geometry and function theory in the second half of the twentieth century. He was one of the first to apply modern topological methods to the classification of algebraic surfaces and then did pioneering work with Donald Spencer on the deformation of complex structures on a manifold. He was awarded a Fields Medal in 1954.

Early Career . Kodaira graduated from the Mathematics Department of Tokyo Imperial University in 1938 and proceeded to take a degree in the Department of Physics there in 1941. He became a professor of physics there in 1944, a position he retained until 1951, by which time he had become internationally recognized as a mathematician. He had by then obtained a PhD in mathematics, and a rewritten version of his thesis, titled "Harmonic Fields in Riemannian Manifolds (Generalized Potential Theory)" and published in the prestigious *Annals of Mathematics* (1949), came to the attention of Hermann Weyl. Weyl was then at the Institute for Advanced Study in Princeton, New Jersey, and he saw that Kodaira had made a significant new contribution to the study of harmonic integrals, one of the central topics in mathematics and one, indeed, that Weyl had himself worked on. Therefore, he invited Kodaira to the institute. Kodaira left for the United States in 1949, the start of his eighteen-year-long stay there. At Princeton he divided his time between the institute and Princeton University. In the 1960s he had positions at Harvard University in Cambridge, Massachusetts; Johns Hopkins University in Baltimore, Maryland; and Stanford University in Stanford, California, before returning to Tokyo in 1967. Early on in his American career he achieved high recognition with the award of a Fields Medal in 1954. In presenting the award to him, Weyl praised him for having the courage to attack the primary concrete problems in all their complexity and for having, as a result, found the right general concepts to resolve the difficulties and ease further progress. By the phrase "primary concrete problems," Weyl meant the fundamental questions concerning the existence of harmonic forms with prescribed singularities and periods.

Topology and the Riemann-Roch Theorem . In the early 1950s, Princeton was the setting for one of the most important mathematical developments of the twentieth century, and Kodaira played a leading role. These developments revolved around the introduction of structural methods to questions in the area of complex function theory (especially in several variables) and geometry. The new theories accomplished a number of things. They helped generalize results known in the one variable case to the much harder case of several variables, which in geometric terms amounts to passing from the case of curves to manifolds of higher dimension. They also gave quite precise answers to difficult existence questions. It had long been known, for example, that a Riemann surface generally admits a variety of different complex structures. The new methods permitted one to answer a similar question for higher dimensional complex manifolds, and much of Kodaira's later work was concerned with the theory of the deformation of complex structures on a complex manifold.

Kodaira continued to work with his American and European friends and colleagues on his return to Japan in 1967—he conducted a lifelong collaboration with Donald Spencer at Princeton—but back in Tokyo he also produced a considerable number of gifted students who did much to sustain Japan's reputation as a major mathematical center despite the isolation of the war years.

George Riemann's work in the 1850s on complex function theory, then in its infancy, can be thought of as exploiting the close ties that exist between a complex function of a single complex variable and a real-valued function of two real variables. Unfortunately, there is no such close identification of complex functions and harmonic functions in higher dimensions, and after a considerable amount of work by Émile Picard and Solomon Lefschetz, it was the great achievement of William V. D. Hodge in the 1930s to forge the first full general theory. One of the central results of his book *The Theory and Applications of Harmonic Integrals* (1941) was that harmonic forms exist in profusion on a complex manifold, indeed there is a unique one of any rank with preassigned periods. However, this proof of this result was found to contain a serious gap, and it was filled, independently in 1942–1943, by Weyl and Kodaira. It was Kodaira's generalization of this result to harmonic forms with singularities that caught Weyl's attention in 1949.

In order to develop the consequences of Hodge's theory, Kodaira turned to the study of the Riemann-Roch theorem. This result, in the one variable case discussed by Riemann in 1857 and his student, Gustav Roch, in 1862, gives real insight into the existence of meromorphic functions on a Riemann surface having prescribed singularities. It had long been known that there should be similar results in higher dimensions, and the Italian school of algebraic geometers around 1900 had found the right generalization for surfaces. Later, results were obtained for complex manifolds of dimension three, but at each stage the result was harder to understand. In the one variable case, the Riemann-Roch theorem expresses the dimension of the space of meromorphic functions in terms of various other numbers that are easy to compute in any given case.

In particular, the answer is given in this form: the dimension is $d + 1 - g + h$, where d and h are numbers determined by the singularities and g is a number determined by the Riemann surface. In most cases, d is the number of singular points and h is necessarily positive, so one sees even without computing h that there will be nontrivial functions with these singularities whenever d is greater than g. However, when two variables are involved, the relevant formula is of the form $d + 1 - g + h - k$, so no simple argument is available and it is necessary to find h and k explicitly. This was often impossible, because no good geometric meaning attached to them. Furthermore, whereas the set of singularities of a function (like its zero set) is a set of points when the function is a function of a single variable, the set of singularities (its divisor) forms a curve on a surface (and a surface in a three-dimensional variety) so the geometry of the situation becomes more and more complicated with each increase in dimension.

Kodaira began by rederiving the Riemann-Roch theorem in dimension two, but in a much more perspicuous form. He then did the same for three-dimensional manifolds, each time applying and deepening the theory of harmonic forms. In the course of this work Kodaira, sometimes in joint papers with Spencer, showed that three arithmetic genera introduced by Francesco Severi in the 1920s and 1930s as fundamental quantities for describing a divisor on a complex surface were in fact the same. Kodaira's method was to solve the problem first for algebraic varieties, then for Kähler varieties, and then for arbitrary compact complex manifolds; he also had significant things to say about when a complex manifold is in fact algebraic. On the basis of the successful treatment of the low dimensional cases, the path to the general n-dimensional setting was finally open, as Weyl was to comment. What became visible was the extent to which purely topological considerations intervened in an analytic and geometric setting.

Algebraic Topological Methods . The methods of algebraic topology had been described by Samuel Eilenberg and Norman Steenrod in their influential book, *Foundations of Algebraic Topology* (1952). At about the same time, the Cartan seminar in Paris was developing the methods of sheaf theory and showing how they could be applied to the study of functions on complex manifolds. The geometry of such problems was also clarified by the introduction of the concept of fiber bundles, especially the simplest case of line and vector bundles. All these overlapping branches of mathematics share two common features: they had their origins in investigations begun in the 1930s, and further research on them was interrupted by World War II. As a result, it was often a new generation of young mathematicians that took up their study after the war. All of these concepts (cohomology, sheaves, and bundles) progressively appear in Kodaira's work in the 1950s, and they rapidly came to be associated with its success.

In 1954 Kodaira was able to characterize those complex manifolds that are algebraic: they are the ones that carry a Hodge metric. A topic that Kodaira studied in great detail, with the full rigor of modern methods, was the classification of complex surfaces. This had first been accomplished by Guido Castelnuovo and Federigo Enriques in the years before the World War I for algebraic surfaces. They correctly distinguished several major types, but their classification left the properties of some of these types of surface little understood. Kodaira reworked their accounts with a view to illuminating both the function theory of these surfaces and their geometry, and he extended it to complex two-dimensional manifolds that are not algebraic surfaces. A result of Wei-Liang Chow and Carl Ludwig Siegel showed that the dimension of the field of meromorphic functions on a complex surface is at most two. Kodaira, in work with Chow, showed that a complex surface is a nonsingular algebraic surface embedded in projective space if and only if this dimension is two.

This leaves the case where this dimension is either one or zero. Kodaira next showed that when the dimension is one, the complex manifold fibers as an elliptic bundle over a curve (the surface is then called an elliptic surface), and he then began the exploration of the dimension zero case. The final analysis of all complex surfaces depended on whether the surface carries an exceptional curve or not and upon its geometric genus. A curve on a surface is exceptional if there is a birational map of the surface that maps the curve to a point. The geometric genus measures the dimension of the space of holomorphic functions on the surface. If a surface does not have an exceptional curve, it is either a surface with geometric genus zero or a surface with geometric genus greater than zero and satisfying an inequality involving the first Chern class. Surfaces of this kind with geometric genus zero break into two types, according as the first Betti number is or is not one. If the surface carries an exceptional curve, it may be a K_3 surface, a complex torus, or an elliptic surface. Kodaira went on to make a detailed investigation of the difficult cases where the dimension of the field of meromorphic functions is less than two. Kodaira also showed that every K_3 surface is a deformation of a nonsingular quartic surface in projective three-space, and an elliptic surface is a deformation of an algebraic surface if and only if its first Betti number is even. All this work depended heavily on the Riemann-Roch theorem, which was the focus of much work by others, notably Michael Atiyah, Friedrich Hirzebruch, and I. M. Singer in the complex and differentiable settings and by Alexandre Grothendieck, Armand Borel, and Jean-Pierre Serre in the setting of the new algebraic geometry.

Deformations of Complex Structures . Starting in the late 1950s, Kodaira and Spencer switched their attention to the topic of varying or deforming the complex structure on a compact complex manifold, a vast generalization of the work done by Riemann a century before for Riemann surfaces, which are one-dimensional compact complex manifolds. In his book *Complex Manifolds and Deformation of Complex Structures* (1986, p. vii), Kodaira said: “The process of the development was the most interesting experience in my whole mathematical life. It was similar to an experimental science developed by the interaction between experiments (examination of examples) and theory.” Their research, which was partly inspired by related work of Alfred Frölicher and Albert Nijenhuis, focused on an unexpected and initially inexplicable numerical coincidence. It was possible to argue that an infinitesimal deformation of the complex structure on a compact complex manifold *M* should be represented by an element of a certain cohomology group, while there seemed to be no reason to believe that every element of this group represented a deformation of the structure. However, as it happened, in many cases the dimension of this group turned out to equal the number of effective parameters involved in defining a structure on *M*, which suggested strongly that every element of the group did indeed represent a deformation. They then began working on proving the conjecture that this equality always holds.

Kodaira and Spencer asked themselves what the consequences would be of an element of the cohomology group not representing a deformation. Consequences of this kind are called obstructions in the language of cohomology theories, and they showed that the obstructions were elements of a second cohomology group. So if this group vanished there were no obstructions, and every element of the first cohomology group represented a deformation. They rederived Riemann's result in their setting. (Riemann had not presented his profound claims in a rigorous way.) They checked the correctness of the result they were seeking for complex analytic hypersurfaces and found a single counterexample to the claim. This led to the introduction of the concept of completeness for a family. All confirmations of the conjecture concerned complete families; the single counterexample was not complete. Finally, by 1960 Kodaira and Spencer were led to obtain sufficient conditions for the conjecture to be true. These were the vanishing of the second cohomology group mentioned above and also of another cohomology group. As Kodaira observed in his *Complex Manifolds*, these sufficient conditions were unduly restrictive, and the conjecture held in many cases when the conditions were not satisfied. In 1961 Masatake Kuranishi contributed an essential theorem on the existence of deformations that further strengthened belief in the conjecture, and Kodaira and Spencer now confirmed it for a wide class of complex, two-dimensional onal manifolds (complex surfaces). In 1962, David Mumford devised a new counterexample that involved a carefully constructed three-dimensional compact complex manifold. Then, in 1967, Arnold Kas found a counterexample among the family of elliptic surfaces, and writing in *Complex Manifolds*, Kodaira said: "In this way the conjecture … ultimately turned out to be false," and he and Spencer ultimately failed to find a "simple and useful criterion" for it to be true (1986, p. 319). A very modest way of concluding a program of work that had opened a whole new field in the study of complex manifolds.

BIBLIOGRAPHY

WORKS BY KODAIRA

"Harmonic Fields in Riemannian Manifolds (Generalized Potential Theory)." *Annals of Mathematics* 50 (1949): 587–665.

"The Theorem of Riemann-Roch on Compact Analytic Surfaces." *American Journal of Mathematics* 73 (1951): 813–875.

With D. C. Spencer. "On Deformations of Complex Analytic Structures, I-II." *Annals of Mathematics* 67 (1958): 328–466. "On Compact Analytic Surfaces, II–III." *Annals of Mathematics* 77 (1963): 563–626.

Collected Works. 3 vols. Tokyo: Iwanami Shoten; Princeton, NJ: Princeton University Press, 1975. This contains a complete bibliography of Kodaira's works.

Complex Manifolds and Deformation of Complex Structures. Translated by Kazuo Akao. New York and Tokyo: Springer-Verlag, 1986.

OTHER SOURCES

Eilenberg, Samuel, and Norman Steenrod. *Foundations of Algebraic Topology*. Princeton, NJ: Princeton University Press, 1952.

Hirzebruch, Friedrich. "Kunihiko Kodaira: Mathematician, Friend, and Teacher." *Notices of the American Mathematical Society* 45, no. 11 (1998): 1456–1462.

Hodge, W. V. D. *The Theory and Applications of Harmonic Integrals*. Cambridge, U.K.: Cambridge University Press, 1941.

Jeremy Gray