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(*b*. Breslau, Germany [now Wroclaw, Poland], 8 November 1843; *d*. Bad Homburg, Germany, 20 September 1930)

mathematics.

Pasch studied chemistry at Breslau but changed to mathematics at the suggestion of Heinrich Schröter, to whom, along with Kambly, his teacher at the Elisabeth Gymnasium, he dedicated his dissertation (1865). Later, at Berlin, he was influenced by Weierstrass and Kronecker. He maintained his mathematical activity with scarcely a break for sixty-five years, for the first seventeen years in algebraic geometry and later in foundations, the work on which his fame rests. His first two papers were written in collaboration with his lifelong friend J. Rosanes. Except for rapid promotion, Pasch's career at the University of Giessen was not unusual: in 1870, *Dozent*; in 1873, extraordinary professor; in 1875, after an offer of an extraordinary professorship from the University of Breslau, ordinary professor. In 1888 he obtained the chair left vacant by the death of Heinrich Baltzer. He was also active in administration, becoming dean in 1883 and rector in 1893–1894. In order to dedicate himself more fully to his scientific work, he retired in 1911. In celebration of his eightieth birthday Pasch reveived honorary Ph.D.'s from the universities of Frankfurt and Freiburg. He was a member of the Deutsche Mathematiker-Vereinigung. His name is perpetuated in Pasch's axiom, which states that in a plane, if a line meets one side of a triagle, then it meets another. His outward life was simple, although saddened by the early death of his wife and one of two daughters. He died while on a vacation trip away from Giessen.

The axiomatic method as it is understood today was initiated by Pasch in his *Vorlesungen über neuere Geometrie* (Leipzig, 1882; 2nd ed., Berlin, 1926). It consists in isolating from a given study certain notions that are left undefined and are expressly declared to be such (the *Kernbegriffe*, in Pasch's terminology of 1916), and certain theorems that are accepted without proof (the *Kernsätze*, or axioms). From this initial fund of notions and theorems, the other notions are to be defined and the theorems proved using only logical arguments, without appeal to experience or intuition. The resulting theory takes the form of purely logical relations between undefined concepts.

To be sure, there are preliminary explanations, and a definite philosophy is disclosed for choosing the axioms. According to Pasch the initial notions and theorems should be founded on observations. Thus the notion of point is allowed but not that of line, since no one has ever observed a complete (straight) line; rather, the notion of segment is taken as primitive. Similarly, a planar surface, but not a plane, is primitive.

Pasch's analysis relating to the order of points on a line and in the plane is both striking and pertinent to its understanding. Every student can draw diagrams and see that if a point B is between point A and point C, then C is not between A and B, or that every line divides a plane into two parts. But no one before Pasch had laid a basis for dealing logically with such observations. These matters may have been considered too obvious; but the result of such neglect is the need to refer constantly to intuition, so that the logical status of what is being done cannot become clear. According to Pasch, the appeal to intuition formally ceases once the *Kernbegriffe* and *Kernsätze* are stated.

The higher geometry of Pasch's day was projective geometry that used real numbers as coordinates. Pasch therefore considered that the foundation would be laid once the coordinates had been introduced. In doing this he presented notions of congruence, which were nonprojective. This is somewhat disappointing in view of Staudt's 1847 program for founding projective geometry solely on projective terms (though we may emphasize that the congruence axioms are original with Pasch). But F. Klein had uncovered some nonrigorous thinking in Staudt's proof that a one-to-one mapping between two lines that sends harmonic quadruples into harmonic quadruples is uniquely determined by the images of three points—without, however, obtaining notable success in clarifying this matter. Pasch proved this fundamental theorem on the basis of the Archimedean character of the ordering on the line, and not on its completeness, as Klein proposed to do. The congruence notions were introduced, at least in part, in order to state Archimedes' axiom. Once the fundamental theorem was proved, the introduction of coordinates could be easily accomplished—as M. Dehn remarks in a historical appendix to Pasch's *Vorlesungen*—on the basis of the Eudoxian theory of book V of Euclid's *Elements*. But for Pasch this procedure was complicated by his empiricist point of view.

It would be easy to overlook the significance of Pasch's foundational achievements for several reasons. First, it is now a commonplace to present theories in an axiomatic way, so that even logic itself is presented axiomatically. Thus Pasch's innovation achieves the status of being a trifle.

There are also widespread misconceptions as to what is in book I of the *Elements*: it is thought that Euclid had an axiomatic way of presenting geometry. This view is further confounded by a lack of clarity as to what the axiomatic method is and what geometry is. Anyone who looks at book I of the *Elements* with modern hindsight sees that something is wrong, but it would take delicate historical considerations to place the source of the faults in a correct light.

The Greeks of Euclid's time had the axiomatic method; Aristotle's description of it can be considered a close approximation to the modern one. Or, better yet, one may consider Eudoxus' theory of magnitude as presented in book V of the *Elements*. Except for style (which, however, may indicate a difference in point of view), the procedure presented there coincides with Pasch's. It is known, however, that the *Elements* is a compilation of uneven quality, so that even with the definitions, postulates, and common notions of book I, it is unwarranted to assume that book I is written from the same point of view as book V.

In some versions of book I, as it has come down to us, there are five "common notions" and five postulates. T. L. Heath considers it probable that common notions 4 and 5 were interpolations; and P. Tannery maintains that they were none of them authentic. The first three postulates are the "postulates of construction," the fourth states that all right angles are equal, and the fifth is the parallel postulate. It has been argued that the first three postulates were meant to help meet the injunction to limit the means of construction to "straightedge and compass "; that there was no intention to say anything about space. One could eliminate these three, as well as the fourth, without changing the rigor of the book or the points of view disclosed relative to geometry or to the axiomatic method. The fifth does not appear until proposition 29, so that the first twenty-eight propositions (minus the constructions) are, from a modern axiomatic point of view, based on nothing.

Although deduction is a prominent feature of book I of the *Elements*, the contents of the book and the history of the parallel postulate show that geometry was conceived as the study of a definite object, "external space." With the invention of non-Euclidean geometry around 1800, it began to dawn on mathematicians that their concern is with deduction, and not with a supposed external reality. Applications, if any, may be left to the physicist. With G. Fano's miniature projective plane of just seven points and seven lines (1892), the revolution may be considered to have been completed. Hilbert, through his work in geometry and logic, consolidated it.

Pasch initiated the axiomatic method, although the foundational developments of his time were against this point of view. Thus Cantor's striking discoveries were based, from an axiomatic point of view, on nothing. Dedekind rightly contrasted his own treatment of magnitude with Eudoxus'; Dedekind's was constructive, whereas Eudoxus' was axiomatic. As late as 1903 Frege poked fun at the axiomatic method as presented in Hilbert's *Grundlagen der Geometrie*.

The Italian geometers, particularly Peano, continued Pasch's work. In 1889 Peano published both his exposition of geometry, following Pasch, and his treatment of number. It is tempting to see in the former work a source of the latter, although there were many other sources.

Pasch played a crucial role in the innovation of the axiomatic method. This method, with contributions from logic and algebra, is a central feature of twentieth-century mathematics.

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