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(*b*. Nancy, France, 29 April 1854; *d*. Paris. France, 17 July 1912)

mathematics, mathematical physics, celestial mechanics.

The development of mathematics in the nineteenth century began under the shadow of a giant, Carl Friedrich Gauss; it ended with the domination by a genius of similar magnitude, Henri Poincaré. Both were universal mathematicians in the supreme sense. and both made important contributions to astronomy and mathematical physics. If Poincaré's discoveries in number theory do not equal those of Gauss, his achievements in the theory of functions are at least on the same level—even when one takes into account the theory of elliptic and modular functions, which must be credited to Gauss and which represents in that field his most important discovery, although it was not published during his lifetime. If Gauss was the initiator in the theory of differentiable manifolds, Poincaré played the same role in algebraic topology. Finally, Poincaré remains the most important figure in the theory of differential equations and the mathematician who after Newton did the most remarkable work in celestial mechanics. Both Gauss and Poincaré had very few students and liked to work alone; but the similarity ends there. Where Gauss was very reluctant to publish his discoveries, Poincaré's list of papers approaches five hundred, which does not include the many books and lecture notes he published as a result of his teaching at the Sorbonne.

Poincaré's parents both belonged to the upper middle class, and both their families had lived in Lorraine for several generations. His paternal grandfather had two sons: Léon, Henri's father, was a physician and a professor of medicine al the University of Nancy; Antoine had studied at the École Polytechnique and rose to high rank in the engineering corps. One of Antoine's sons, Raymond, was several times prime minister and was president of the French Republic during World War I; the other son, Lueien, occupied high administrative functions in the university. Poincaré's mathematical ability became apparent while he was still a student in the *lycee*, He won first prizes in the *concours général* (a competition between students from all French *lycées*) and in 1873 entered the École Polytechnique at the top of his class; his professor al Nancy is said to have referred to him as a "monster of mathematics." After graduation he followed courses in engineering at the EcoIe des Mines and worked briefly as an engineer while writing his thesis for the doctorate in mathematics which he obtained in 1879. Shortly afterward he started teaching at the University of Caen, and in 1881 he became a professor at the University of Paris, where he taught until his untimely death in 1912. At the early age of thirty-three he was elected to the Académie des Sciences and in 1908 to the Académie Française. He was also the recipient of innumerable prizes and honors both in France and abroad.

Function Theory. Before he was thirty years of age, Poincaré became world famous with his epoch-making discovery of the "automorphic functions" of one complex variable (or, as he called them, the "fuchsian" and "kleinean" functions). The study of the modular function and of the solutions of the hypergcomctric equation had given examples of analytic functions defined in an open connected subset *D* of the complex plane, and "invariant" under a group *G* of transformations of *D* onto itself, of the form

G being "properly discontinuous," that is, such that no point z of *D* is the limit of an infinite sequence of transforms (distinct from z) of a point $z' \varepsilon D$ by a sequence of elements. $T_n \varepsilon G$. For instance, the modular group consists of transformations (1), where a, b, c, d are integers and $ad - bc = 1$; *D* is the upper half plane $\mathcal{I}z > 0$, and it can be covered, without overlapping, by all transforms of the fundamental domain defined by. $|z| \geq 1$, $|\mathcal{B}z| \leq 1/2$. Using non-Euclidean geometry in a very ingenious way, Poincard was able to show that for any properly discontinuous group *G* of transformations of type (1), there exists similarly a fundamental domain, bounded by portions of straight lines or circles, and whose transforms by the elements of *G* cover *D* without overlapping. Conversely, given any such "circular polygon" satisfying some explicit conditions concerning its angles and its sides, it is the fundamental domain of a properly discontinuous group of transformations of type (1). The open set *D* may be the half plane $\mathcal{I}z > 0$, or the interior or the exterior of a circle; when it is not of this type, its boundary may be a perfect non-dense set, or a curve that has either no tangent at any point or no curvature at any point.

Poincaré next showed—by analogy with the Weierstrass series in the theory of elliptic functions—that for a given group G, and a rational function *H* having no poles on the boundary of *D*, the series

where the transformations

are an enumeration of the transformations of *G*, and *m* is a large enough integer, converges except at the transforms of the poles of *H* by *G*; the meromorphic function Θ thus defined in *D*, obviously satisfies the relation for any transformation (1) of the group G. The quotient of two such functions, which Poincaré called thetafuchsian, corresponding to the same integer m, gives an automorphic function (meromorphic in D). It is easy to show that any two automorphic functions X, Y (meromorphic in D and corresponding to the same group G) satisfy an "algebraic" relation $P(X, Y) = 0$, where the genus of the curve $P(x, y) = 0$ is equal to the topological genus of the homogeneous space D/G and can be explicitly computed (as Poinearé showed) from the fundamental domain of G. Furthermore, if $v_1 = (dX/dz)^{1/2}$, $v_2 = zu_1$, v_1, and v_2 are solutions of a linear differential equation of order 2:

$$d^2v/dX^2 = \phi(X, Y)v,$$

where ϕ is rational in X and Y, so that the automorphic function X is obtained by "inverting" the relation $z = v_1(X)/v_2(X)$. This property was the starting point of Poincaré's researches, following a paper by I. L. Fuchs investigating second-order equations. $y'' + P(x)y' + Q(x)y = 0$, with rational coefficients P, Q, in which the inversion of the quotient of two solutions would give a meromorphic function; hence the name be chose for his automorphic functions.

But Poincaré did not stop there. Observing that his construction of fuchsian functions introduced many parameters susceptible of continuous variation, he conceived that by a suitable choice of these parameters, one could obtain for an "arbitrary" algebraic curve $P(x, y) = 0$, a parametric representation by fuchsian functions, and also that for an arbitrary homogeneous linear differential equation of any order

$$y_{(n)} + P_1(x)y_{(n-1)} + \dots + P_n(x)y = 0,$$

where the P_j, are algebraic functions of x, one could express the solutions of that equation by "zetafuchsian" functions (such a function $\mathbf{F}$ takes its value in a space $\mathbf{C}^p$; in other words, it is a system of p scalar meromorphic functions and is such that, for any transformation (1) of the fuchsian group G to which it corresponds, one has $\mathbf{F}(T \cdot z) = \varrho(T) \cdot \mathbf{F}(z)$, where ϱ is a linear representation of G into $\mathbf{C}^p$). The "continuity method" by which he sought to prove these results could not at that time be made rigorous, due to the tact, of proper topological concepts and results in the early 1880's; but after Brouwer's fundamental theorems in topology, correct proofs could be given using somewhat different methods.

Much has been written on the "competition" between C. F. Klein and Poincaré in the discovery of automorphic functions. Actually there never was any real competition, and Klein was miles behind from the start. In 1879 Klein certainly knew everything that had been written on special automorphic functions, a theory to which he had contributed by several beautiful papers on the transformation of elliptic functions. He could not have failed in particular to notice the connection between the fundamental domains of these functions, and non Euclidean geometry, since it was he who, after Cayley and Beltrami, had clarified the concept of Euclidean "models" for the various non Euclidean geometries, of which the "Poincaré half plane" was a special example.

On the other hand, Poincaré's ignorance of the mathematical literature, when he started his researches, is almost unbelievable. He hardly knew anything on the subject beyond Hermite's work on the modular functions; he certainly had never read Riemann, and by his own account had not even heard of the "Dirichlet principle," which he was to use in such imaginative fashion a few years later. Nevertheless, Poincaré's idea of associating a fundamental domain to any fuchsian group does not seem to have occurred to Klein, nor did the idea of "sing" non Euclidean geometry, which is never mentioned in his papers on modular functions up to 1880. One of the questions Klein asked Poincaré in his letters was how he had proved the convergence of the "theta" series. It is only after realizing that poincaré was looking for a theorem that would give a parametric representation by meromorphic functions of all algebraic curves that Klein set out to prove this by himself and succeeded in sketching a proof independently of Poincaré. He used similar methods (suffering from the same lack of rigor).

The general theory of automorphic functions of one complex variable is one of the few branches of mathematics where Poincaré left little for his successors to do. There is no "natural" generalization of automorphic functions to several complex variables. Present knowledge suggests that the general theory should be linked to the theory of symmetric spaces G/K of E. Cartan (G semisimple real Lie group, K maximal compact subgroup of G), and to the discrete subgroups Γ of G operating on G/K and such that G/Γ has finite measure (C. L. Siegel), But from that point of view, the group $G = \mathbf{SL}(2, \mathbf{R})$, which is at the basis of Poincaré's theory, appears as very exceptional, being the only simple Lie group where the conjugacy classes of discrete subgroups Γ depend on continuous parameters (A. Weil's rigidity theorem). The "continuity" methods dear to Poincaré are therefore ruled out; in fact the known discrete groups I Ì G for which G/Γ has finite measure are defined by arithmetical considerations, and the automorphic functions of several variables are thus much closer to number theory than for one variable (where Poincaré very early had noticed the particular "fuchsian groups" deriving from the arithmetic theory of ternary quadratic forms, and the special properties of the corresponding automorphic functions).

The theory of automorphic functions is only one of the many contributions of Poincaré to the theory of analytic functions, each of which was the starting point of extensive Théories. In a short paper of 1883 he was the first to investigate the links between the genus of an entire function (defined by properties of its Weierstrass decomposition in primary factors) and the coefficients of its Taylor development or the rate of growth of the absolute value of the function; together with the Picard theorem, this was to lead, through the results of Hadamard and E. Borel, to the vast theory of entire and meromorphic functions that is not yet exhausted after eighty years.

Automorphic functions had provided the first examples of analytic functions having singular points that formed a perfect non dense set, as well as functions having curves of singular points. Poincaré gave another general method to form functions of this type by means of series of rational functions, leading to the theory of monogenic functions later developed by E. Bore] and A. Denjoy.

It was also a result from the theory of automorphic functions, namely the parametrization theorem of algebraic cones, that in 1883 led Poincaré to the general "uniforinization theorem," which is equivalent to the existence of a conformal mapping of en arbitrary simply connected noncompact Riemann surface on the plane or on an open disc. This time he saw that the problem was a generalization of Dirichlet's problem, and Poincaré was the first to introduce the idea of "exhausting" the Riemann surface by an increasing sequence of compact regions and of obtaining the conformal mapping by a limiting process. Here again it was difficult at that time to build a completely satisfactory proof, and Poincaré himself and Koebe had to return to the question in 1907 before it could be considered as settled.

Poincaré was even more an initiator in the theory of analytic functions of several complex variables—which was practically nonexistent before him. His first result was the theorem that a meromorphic function F of two complex variables is a quotient of two entire functions, which in 1883 he proved by a very ingenious use of the Dirichlet principle applied to the function log $|F|$; in a later paper (1898) he deepened the study of such "pluriharmonic" functions for any number of complex variables and used it in the theory of Abelian functions. Still later (1907) after the publication of F. M. Hartogs' theorems, he pointed out the completely new problems to which led the extension of the concept of "conformal mapping" for functions of two complex variables. These were the germs of the imposing "analytic geometry" (or theory of analytic manifolds and analytic spaces) which we know today, following the pioneering works of Cousin, Hartogs, and E. E. Levi before 1914; H. Carton, K. Oka, H. Behnke, and P. Lelong in the 1930's; and the tremendous impulse given to the theory by cohomological ideas after 1945.

Finally, Poincaré was the first to give a satisfactory generalization of the concept of "residue" for multiple integrals of functions of several complex variables, after earlier attempts by other mathematicians had brought to light serious difficulties in this problem. Only quite recently have his ideas come to full fruition in the work of J. Leray, again using the resources of algebraic topology.

Abelian Functions and Algebraic Geometry. As soon as he came info contact with the work of Riemann and Weierstrass on Abelian functions and algebraic geometry, Poincaré was very much attracted by those fields. His papers on these subjects occupy in his complete works as much space as those on automorphic functions, their dates ranging from 1881 to 1911. One of the main ideas in these papers is that of "reduction" of Abelian functions. Generalizing particular cases studied by Jacobi, Weierstrass, and Picard, Poincaré proved the general "complete reducibility" theorem, which is now expressed by saying that if A is an Abelian variety and B an Abelian subvariety of A, then there exists an Abelian subvariety C of A such that $A = B + C$ and B Ç C is a finite group. Abelian varieties can thus be decomposed in sums of "simple" Abelian varieties having finite intersection. Poincaré noted further that Abelian functions corresponding to reducible varieties (and varieties products of elliptic curves, that is, Abelian varieties of dimension 1) are "dense" among all Abelian functions—a result that enabled him to extend and generalize many of Riemann's results on theta functions, and to investigate the special properties of the theta functions corresponding to the Jacobian varieties of algebraic curves.

The most remarkable contribution of Poincaré to algebraic geometry is in his papers of 1910–1911 on algebraic curves contained in an algebraic surface $F(x, y, z) = 0$. Following the general method of Picard, Poincaré considers the sections of the surface by planes y = coast.; the genus p of such a curve C_y is constant except for isolated values of y.

It is possible to define p Abelian integrals of the first kind on C_y, $\nu_1, \ldots, \nu_p$, which are analytic functions on the surface (or rather, on its universal covering). Now, to each algebraic curve, I' on the surface, meeting a generic C_y in m points, Poinecaré associates p functions $\nu_1 \ldots, \nu_p$ of y, $u_j(y)$ being the sum of the values of the integral ν_j at the m points of intersection of $\mathbf{C}_y$ and **I** '; furthermore, he is able to characterize these "normal functions" by properties where the curve □ does not appear anymore, and thus he obtains a kind of analytical "subatitue" for the algebraic curve. This remarkable method enabled him to obtain simple proofs of deep results of Picard and Severi, as well as the first correct proof of a famous theorem stated by Castelnuovo, Enriques, and Severi, showing that the irregularity $q = p_g - p_a$ of the surface (p_g and p_a being the geometric and the arithmetic genus) is exactly the maximum dimension of the "continuous nonlinear systems"of curves on the surface. The method of proof suggested by the Italian geometers was later found to be defective, and no proof other than Poincaré 's was obtained until 1965. His method has also shown its value in other recent questions (Igusa, Griffiths), and it is very likely that its effectiveness is far front exhausted.

Number Theory. Poincaré was a student of Hermite, and some of his early work deals with Hermite's method of "continuous reduction" in the arithmetic theory of forms, and in particular the finiteness theorem for the classes of such forms (with nonvanishing discriminant) that had just been proved by C. Jordan. These papers bring sonic complements and precisions to the results of Hermite and Jordan, without introducing any new idea. In connection with them Poincaré gave the first general definition of the genus of a form with integral coefficients, generalizing those of Gauss and Eisenstein; Minkowski had arrived independently at that definition at the same time.

Poincaré's last paper on number theory (1901) was most influential and was the first paper on what we now call "algebraic geometry over the field of rationale" (or a field of algebraic numbers). The subject matter of the paper is the Diophantine problem of finding the points with rational coordinates on a curve $f(x, y) = 0$, where the coefficients of, f are rational numbers.

Poincaré observed immediately that the problem is invariant under birational transformations, provided the latter have rational coefficients. Thus he is naturally led to consider the genus of the curve $f(x, y) = 0$, and his main concern is with the case of genus 1; using the parametric representation of the curve by elliptic functions (or, as we now say, the Jacobian of the curve), he observes that the rational points correspond on tile Jacobian to a subgroup, and he defines the "rank" of the curve as the rank of that subgroup. It is likely that Poincaré conjectured that the rank is always finite; this fundamental fact was proved by L. J. Mordell in 1922 and generalized to curves of arbitrary genus by A. Weil in 1929. These authors used a method of "auoinfinite descent" based upon the bisection or elliptic (or Abelian) functions; Poincaré had developed in his paper similar computations related to the trisection of elliptic functions, and it is likely that these ideas wore at the origin of Mordell's proof. The MordellWeil theorem has become fundamental in the theory of Diophantine equations, but many questions regarding the concept of rant, introduced by Poincaré remain unanswered, and it is possible that a deeper study of his paper may lead to new results.

Algebra. It is not certain that Poincaré knew Kronecker's dictum that algebra is only the handmaiden of mathematics, and has no right to independent existence. At any rate Poincaré never studied algebra for its own sake, but only when he needed algebraic results in problems of arithmetic or analysis. For instance, his work on the arithmetic theory of forms led him to the study of forms of degree ≥ 3, which admit continuous groups of automorphisms. It seems that it is in connection with this problem that his attention was drawn to the relation between hypercomplex systems (over **R** or **C**) and the continuous group defined by multiplication of invertible elements of the system; the short note he published on the subject in 1884 inspired later work of Study and E. Cartan oil hypercomplex systems. A little-known fact is that Poincaré returned to noncommutative algebra in a 1903 paper on algebraic integrals of linear differential equations. His method led him to introduce the group algebra of the group of the equation (which then is finite), and to split it (according to H. Maschke's theorem, which apparently he did not know but proved by referring to a theorem of Frobenius) into simple algebras over C (that is, matrix algebras). He then introduced for the first time the concepts of left and right ideals in an algebra, and proved that any left ideal in a matrix algebra is a direct sum of minimal left ideals (a result usually credited to Wedderburn or Artin).

Poincaré was one of the few mathematicians of his time who understood and admired the work of Lie and his continuators on "continuous groups." and in particular the only mathematician who in the early 1900's realized the depth and scope of E. Cartan's papers. In 1899 Poincaré became interested in a new way to prove Lie's third fundamental theorem and in what is now called the Campbell-Hausdorff formula; in his work Poincaré substantially defined for the first time what we now call the "enveloping algebra" of a Lie algebra (over the complex field) and gave a description of a "natural" basis of that algebra deduced from a given basis of the Lie algebra; this theorem (rediscovered much later by G. Birkhoff and E. Witt, and now called the "Poincaré Birkhoff-Witt theorem") has become fundamental in the modern theory of Lie algebras.

Differential Equations and Celestial Mechanics . The theory of differential equations and its applications to dynamics was clearly at the center of Poincaré's mathematical thought; from his first (1878) to his last (1912) paper, he attacked the theory from all possible angles and very seldom let a year pass without publishing a paper on the subject. We have seen already that the whole theory of automorphic functions was from the start guided by the idea of integrating linear differential equations with algebraic coefficients. Poincaré simultaneously investigated the local problem of a linear differential equation in the neighborhood of an "irregular" singular point, showing for the first time how asymptotic developments could be obtained for the integrals. A little later (1884) he took up the question, also started by **I. L** . Fuchs, of the determination of all differential equations of the first order (in the complex domain) algebraic in y and y' and having fixed singular points; his researches were to be extended by Picard for equations of the second order, and to lead to the spectacular results of Painlevé and his school at the beginning of the twentieth century.

The most extraordinary production of Poincaré, also dating from his prodigious period of creativity (1880–1883) (reminding us of Gauss's *Tagebuch* of 1797–1801), is the qualitative theory of differential equations. It is one of the few examples of a mathematical theory that sprang apparently from nowhere and that almost immediately reached perfection in the hands of its creator. Everything was new in the first two of the four big papers that Poincaré published on the subject between 1880 and 1886.

The Problems . Until 1880, outside of the elementary types of differential equations (integrable by "quadratures") and the local "existence theorems," global general studies had been confined to linear equations, and (with the exception of the Sturm-Liouville theory) chiefly in the complex domain. Poincaré started with general equations $dx/X = dy/Y$, where X and Y, are "arbitrary" polynomials in x, y, everything being real, and did not hesitate to consider the most general problem possible, namely a qualitative description of all solutions of the equation. In order to handle the infinite branches of the integral curves, he had the happy idea to project the (x, y) plane on a sphere from the center of the sphere (the center not lying in the phane), thus dealing for the first trine with the integral curves of a vector field on a compact manifold.

The Methods . The starting point was the consideration of the "critical points" of the equation, satisfying $X = Y = 0$. Poincaré used the classification of these points due to Cauchy and Briot-Bouquet (modified to take care of the restriction to real coordinates) in the well-known categories of "nodes," "saddles," "spiral points," and "centers." In order to investigate the shape of an integral curve, Poincaré introduced the fundamental notion of "transversal" arcs, which are not tangent to the vector field at any of their points. Functions $F(x, y)$ such that $F(x, y) = C$ is a transversal for certain values of C also play an important part (their introduction is a forerunner of the method later used by Liapunov for stability problems).

The Results . The example of the “classical” differential equations had led one to believe that “general” integral curves would be given by an equation $\Phi(x, y) = C$, where $\Phi$is analytic, and the constant C takes arbitrary values. Poincaré showed that on the contrary this kind of situation prevails only in “exceptional” cases, when there are no nodes nor spiral points among the critical points. In general, there are no centers—only a finite number of nodes, saddles, or spiral points; there is a finite number of closed integral curves, and the other curves either join two critical points or are “asymptotic” to these closed curves. Finally, he showed how his methods could be applied in explicit cases to determine a subdivision of the sphere into regions containing no closed integral or exactly one such curve.

In the third paper of that series Poincaré attacked the more general case of equations of the first order $F(x, y, y') = 0$, where F is a polynomial. By the consideration of the surface $F(x, y, z) = 0$, he showed that the problem is a special case of the determination of the integral curves of a vector field on a compact algebraic surface S. This immediately led him to introduce the genus p of S as the fundamental invariant of the problem, and to discover the relation

$N + F — C = 2 — 2\mathrm{p}$ (3)

where N, F, and C are the numbers of nodes, spiral points, and saddles. He then proceeded to show how his previous results for the sphere partly extend to the general case, and then made a detailed and beautiful study of the case when S is a torus ($p = 1$), so that there may be no critical point; in that case, he is confronted with a new situation—the appearance of the “ergodic hypothesis” for the integral curves. He was not able to prove that the hypothesis holds in general (under the smoothness conditions imposed on the vector field), but later work of Denjoy showed that this is in fact the case.

In the fourth paper Poincaré finally inaugurated the qualitative theory for equations of higher order, or equivalently, the study of integral curves on manifolds of dimension ≥3. The number of types of critical points increases with the dimension, but Poincaré saw how his relation (3) for dimension 2 can be generalized, by introducing the “Kronecker index” of a critical point, and showing that the sum of the indices of the critical points contained in a bounded domain limited by it transversal hypersurface Σ depends only on the Betti numbers of Σ. It seems hopeless to obtain in general a description of all integral curves as precise as the one obtained for dimension 2. Probably inspired by his first results on the three-body problem (dating from 1883), Poincaré limited himself to the integral curves that are “near” a closed integral curve C_0. He considered a point M on C_0. and a small portion Σ of the hypersurface normal to C_0 at M. If point P of Σ is close enough to M, the integral curve passing through P will cut Σ again for the first time at a point $T(P)$, and one thus defines a transformation T of Σ into itself, leaving M invariant, which can be proved to be continuously differentiable (and even analytic if one starts with analytic data). Poincaré then showed how the behavior of integral curves “near C_0” depends on the eigenvalues of the linear transformation tangent to T at M, and the classification of the various types is therefore closely similar to the classification of critical points.

After 1885 most of Poincaré’s papers on differential equations were concerned with celestial mechanics, and more particularly the three-body problem. It seems that his interest in the subject was first aroused by his teaching at the Sorbonne; then, in 1885, King Oscar II of Sweden set up a competition among mathematicians of all countries on the n-body problem. Poincaré contributed a long paper, which was awarded first prize, and which ranks with his papers on the qualitative theory of differential equations as one of his masterpieces. Its central theme is the study of the periodic Solutions of the three body problem when the masses of two of the bodies are very small in relation to the mass of the third (which is what happens in the solar system). In 1878 G. W. Hill had given an example of such solutions; in 1883 Poincaré proved—by a beautiful application of the Kronecker index—the existence of a whole continuum of such solutions. Then in his prize memoir he gave another proof for the “restricted” three body problem, when one of the small masses is neglected, and the other μ is introduced as a parameter in the Hamiltonian of the system. Starting from the trivial existence of periodic solutions for $\mu = 0$, Poincaré proved the existence of “neighboring” periodic solutions for small enough μ by an application of Cauchy’s method of majorants. He then showed that there exist solutions that are asymptotic to a periodic solution for values of the time tending to $+ \infty$ or ∞, or even for both (“doubly asymptotic” solutions). It should be stressed that in order to arrive at these results, Poincaré first had to invent the necessary general tools: the “variational equation” giving the derivative of a vector solution **f** of a system of differential equations, with respect to a parameter, as a solution of a linear differential equation; the “characteristic exponents” corresponding to the case in which **f** is periodic; and the “integral invariants” of a vector field, generalizing the particular case of an invariant volume used by Lionville and Boltzmann.

Celestial Mechanics . The works or Poincaré on celestial mechanics contrasted sharply with those of his predecessors. Since Lagrange, the mathematical and numerical study of the solar system had been carried out by developing the coordinates of the planets in series of powers of the masses of the planets or satellites (very small compared with that of the sun); the coefficients of these series would then be computed, as functions of the time t, by various processes of approximation, from the equations obtained by identifying in the equations of motion the coefficients of the powers of the masses. At first the functions of t defined in this manner contained not only trigonometric functions such as $\sin(at + b)$ (a, b constants) but also terms such as $t \cdot \cos(at + b)$, and so forth, which for large t were likely to contradict the observed movements, and showed that the approximations made were unsatisfactory. Later in the nineteenth century these earlier approximations were replaced by more sophisticated ones, which were series containing only trigonometric functions of variables of type $a_n t + b_n$; but nobody had ever proved that these series were convergent, although most astronomers believed they were. One of Poincaré’s results was that these series cannot be uniformly convergent, but may be used to provide asymptotic developments of the coordinates.

Thus Poincaré inaugurated the rigorous treatment of celestial mechanics, in opposition to the semiempirical computations that had been prevalent before him. However, he was also keenly interested in the “classical” computations and published close to a hundred papers concerning various aspects of the theory of the solar system, in which he suggested innumerable improvements and new techniques. Most of his results were developed in his famous three-volume *Les méthods nouvelles de la mécanique céleste* and later in his *Leçons de mécanique céleste*. From the theoretical point of view, one should mention his proof that in the "restricted" three-body problem, where the Hamiltonian depends on four variables (x_1, x_2, y_1, Y_2) and the parameter μ, and where it is analytic in these five variables and periodic of period 2π in y_1, and y_2, then there is no "first integral" of the equations of motion, except the Hamiltonian, which has similar properties. Poincaré also started the study of "stability" of dynamical systems, although not in the various more precise senses that have been given to this notion by later writers (starting with Liapunov). The most remarkable result that he proved is now known as "Poincaré's recurrence theorem" : for "almost all" orbits (for a dynamical system admitting a "positive" integral invariant), the orbit intersects an arbitrary nonempty open set for a sequence of values of the time tending to + ∞. What is particularly interesting in that theorem is the introduction, probably for the first time, of null sets in a question of analysis (Poincaré, of course, did not speak of measure, but of "probability").

Another famous paper of Poincaré in celestial mechanics is the one he wrote in 1885 on the shape of a rotating fluid mass submitted only to the forces of gravitation. Maclaurin had found as possible shapes some ellipsoids of revolution to which Jacobi had added other types of ellipsoids with unequal axes, and P. G. Tait and W. Thomson some annular shapes. By a penetrating analysis of the problem, Poincaré showed that still other "pyriform" shapes existed. One of the features of his interesting argument is that, apparently for the first time, he was confronted with the problem of minimizing a quadratic form in "infinitely" many variables.

Finally, in one of his later papers (1905), Poincaré attacked for the first time the difficult problem of the existence of closed geodesics on a convex smooth surface (which he supposed analytic). The method by which he tried to prove the existence of such geodesics is derived from his ideas on periodic orbits in the three-body problem. Later work showed that this method is not conclusive, but it has inspired the numerous workers who finally succeeded in obtaining a complete proof of the theorem and extensive generalizations.

Partial Differential Equations and Mathematical Physics . For more than twenty years Poincaré lectured at the Sorbonne on mathematical physics; he gave himself to that task with his characteristic thoroughness and energy, with the result that he became an expert in practically all parts of theoretical physics, and published more than seventy papers and books on the most varied subjects, with a predilection for the Théories of light and of electromagnetic waves. On two occasions he played an important part in the development of the new ideas and discoveries that revolutionized physics at the end of the nineteenth century. His remark on the possible connection between X rays and the phenomena of phosphorescence was the starting point of H. Becquerel's experiments which led him to the discovery of radioactivity. On the other hand, Poincaré was active in the discussions concerning Lorentz' theory of the electron from 1899 on; Poincaré was the first to observe that the Lorentz transformations form a group, isomorphic to the group leaving invariant the quadratic form $x^2 + y^2 + z^2 - t^2$; and many physicists consider that Poincaré shares with Lorentz and Einstein the credit for the invention of the special theory of relativity.

This persistent interest in physical problems was bound to lead Poincaré into the mathematical problems raised by the partial differential equations of mathematical physics, most of which were still in a very rudimentary state around 1880. It is typical that in all the papers he wrote on this subject, he never lost sight of the possible physical meanings (often drawn from very different physical Théories) of the methods he used and the results he obtained. This is particularly apparent in the first big paper (1890) that he wrote on the Dirichlet problem. At that time the existence of a solution inside a bounded domain D limited by a surface S was established (for an arbitrary given continuous function on S) only under rather restrictive conditions on S, by two methods due to C. Neunmann and H. A. Schwarz. Poincaré invented a third method, the "sweeping out process": the problem is classically equivalent to the existence of positive masses on S whose potential V is equal to 1 in D and continuous in the whole space. Poincaré started with masses on a large sphere Σ containing D and giving potential 1 inside Σ. He then observed that the classical Poisson formula allows one to replace masses inside a sphere C by masses on the surface of the sphere in such a way that the potential is the same outside C and has decreased inside C. By covering the exterior of D by a sequence (C_n) of spheres and applying repeatedly to each C_n (in a suitable order) the preceding remark, he showed that the limit of the potentials thus obtained is the solution V of the problem, the masses initially on Σ having been ultimately "swept out" on S. Of course he had to prove the continuity of V at the points of S, which he did under the only assumption that at each of these points there is a half-cone (with opening $2\alpha > 0$) having the point as vertex and such that the intersection of that half-cone and of a neighborhood of the vertex does not meet D (later examples of Lebesgue showed that such a restriction cannot be eliminated). This very original method was later to play an important part in the renewal of potential theory that took place in the 1920's and 1930's, before the advent of modern Hilbert space methods.

In the same 1890 paper Poincaré began the long, and only partly successful, struggle with what we now call the problem of the eigenvalues of the Laplacian. In several problems of physics (vibrations of membranes, cooling of a solid, theory of the tides, and so forth), one meets the problem of finding a function u satisfying in a hounded domain D an equation of the form

and oil the boundary S of D the condition

where du/dn is the normal derivative and λ and k are constants. Heuristic variational arguments (generalizing the method of Riemann for the Dirichlet principle) and the analogy with the Sturm Liouville problem (which is the corresponding problem for functions of a single variable) lead to the conjecture that for a given k there exists an increasing sequence of real numbers ("eigenvalues") such that the problem is only solvable when. λ is equal to one of the λ_n, and then has only one solution u_n such that, the "eigenfunctions", u_n forming an orthonormal system. In the case of the vibrating membrane, the corresponds to the experimentally detectable "harmonics." But a rigorous proof of the existence of the λ_n and the had u_n not been found before Poincaré; for the case $k = 0$, Schwarz had proved the existence of λ_1, by the following method: the analogy with the Sturm-Liouovlle problem suggested that for any smooth function *f*, the equation

would have for λ distinct from the λ_n a unique solution $u(\lambda, x)$satisfying(5), and which would be a meromorphic function of λ, having the λ_n as simple poles. Schwarz had shown that, as a function of λ, the solution $u(\lambda, x)$ was equal to a power series with a finite radius of convergence. Picard had been able to prove also the existence of λ_2. In 1894 Poincaré (always in the case $k = 0$) succeeded in proving the above property of $u(\lambda, x)$, by an ingenious adaptation of Schwarz's method, using in addition an inequality of the type

(C constant depending only on D)

valid for all smooth functions *V* such that (the forerunner of numerous similar inequalities that play a fundamental part in the modern theory of partial differential equations). But he could not extend his method for $k \neq 0$ on account of the difficulty of finding a solution of (6) having a normal derivative on *S* (he could only obtain what we now would call a "weak" derivative, or derivative in the sense of distribution theory).

Two years later he met similar difficulties when he tried to extend Neumann's method for the solution of the Dirichlet problem (which was valid only for convex domains *D*). Through a penetrating discussion of that method (based on so called "double layer" potentials), Poincaré linked it to the Schwarz process mentioned above, and was thus led to a new "boundary problem" containing a parameter λ: find a "single layer" potential ϕ defined by masses on *S*, such that $(d\phi/dn)_i = -\lambda(d\phi/dn)_e$, where the suffixes *i* and *e* mean normal derivatives taken toward the interior and toward the exterior of *S*. Here again, heuristic variational arguments convinced Poincaré that there should be a sequence of "eigenvalues" and corresponding "eigenfunctions" for this problem, but for the same reasons lit was not able to prove their existence. A few years later, Fredholm's theory of integral equations enabled him to solve all these problems; it is likely that Poincareé's papers had a decisive influence on the development of Fredholm's method, in particular the idea of introducing a variable complex parameter in the integral equation. It should also be mentioned that Fredholm's determinants were directly inspired by the theory of "infinite determinants" of H. von Koch, which itself was a development of much earlier results of Poincaré in connection with the solution of linear differential equations.

Algebraic Topology . The main leitmotiv of Poincaré's mathematical work is clearly the idea of "continuity": whenever he attacks a problem in analysis, we almost immediately see him investigating what happens when the conditions of the problem are allowed to vary continuously. He was therefore bound to encounter at every turn what we now call topological problems. He himself said in 1901, "Livery problem 1 had attacked led me to *Analysis situs*," particularly the researches on differential equations and on the periods of multiple integrals. Starting in 1894 he inaugurated in a remarkable series of six papers—written during a period of ten years—the modern methods of algebraic topology. Until then the only significant step had been the generalizations of the concept of "order of connection" of a surface, defined independently by Riemann and Betti, and which Poincaré called "Betti numbers" (they are the numbers $1 + h_j$, where the h_j are the present-day "Betti numbers"): but practically nothing had been done beyond this definition. The machinery of what we now call simplicial homology is entirely a creation of Poincaré: concepts of triangulation of a manifold, of a simplicial complex, of barycentric subdivision, and of the dual complex, of the matrix of incidence coefficients of a complex, and the computation of Betti numbers from that matrix. With the help of these tools, Poincaré discovered the generalization of the Euler theorem for polyhedra (now known as the Euler-Poincuré formula) and the famous duality theorem for the homology of a manifold; a little later he introduced the concept of torsion. Furthermore, in his first paper he had defined the fundamental group of a manifold (or first homotopy group) and shown its relations to the first Betti number. In the last paper of the series he was able to give an example of two manifolds having the same homology but different fundamental groups. In the first paper he had also linked the Betti numbers to the periods of integrals of differential forms (with which he was familiar through his work on multiple integrals and on invariant integrals), and stated the theorem which G. de Rham first proved in 1931. It has been rightly said that until the discovery of the higher homotopy groups in 1933, the development of algebraic topology was entirely based on Poincaré's ideas and techniques.

In addition, Poincaré also showed how to apply these new tools to some of the problems for which he had invented them. In two of the papers of the series on analysis situs, he determined the Betti numbers of an algebraic (complex) surface, and the fundamental group of surfaces defined by an equation of type $z^2 = F(x, y)$ (*F* polynomial), thus paving the way for the later generalizations of Lefschetz and Hodge. In his last paper on differential equations (1912). Poincaré reduced the problem of the existence of periodic solutions of the restricted three body problem (but with no restriction oil the parameter μ) to a theorem of the existence of fixed points for a Continuous transformation of the plane subject to certain conditions, which was probably the first example of an existence proof in analysis based on algebraic topology. He did not succeed in proving that fixed point theorem, which was obtained by G. D. Birkhoff a few months after Poincaré's death.

Foundations of Mathematics. With the growth of his international reputation, Poincaré was more and more called upon to speak or write on various topics of mathematics and science for a wider audience, a chore for which he does not seem to have shown great reluctance. (In 1910 he even was asked to comment on the influence of comets on the weather!) His vivid style and clarity of mind enhanced his reputation in his time as the best expositor of mathematics for the layman. His well-known description of the process of mathematical discovery remains unsurpassed and has been on the whole corroborated by many mathematicians, despite the fact that Poincaré's imagination was completely atypical; and the pages he devoted to the axioms of geometry and their relation to experimental science are classical. Whether this is enough to dub him a "philosopher," as has often been asserted, is a question which is best left for professional philosophers to decide, and we may limit ourselves to the influence of his writings on the problem of the foundations of mathematics.

Whereas Poincaré has been accused of being too conservative in physics, he certainly was very openminded regarding new mathematical ideas. The quotations in his papers show that he read extensively, if not systematically, and was aware of all the latest developments in practically every branch of mathematics. He was probably the first mathematician to use Cantor's theory of sets in analysis; he had met concepts such as perfect non-dense sets in his work on automorphic functions or on differential equations in the early 1880's. Up to a certain point, he also looked with favor on the axiomatic trend in mathematics, as it was developing toward the end of the nineteenth Century, and he praised Hilbert's *Grundlagen der Geometrie*. However, Poincaré's position during the polemics of the early 1900's about the "paradoxes" of set theory and the foundations of mathematics has made him a precursor of.the intuitionist School. He never stated his ideas on these questions very clearly and mostly confined himself to criticizing the schools of Russell, Peano, and Hilbert. Although accepting the "arithmetization" of mathematics, Poincaré did not agree to the reduction of arithmetic to the theory of sets nor to the Peano axiomatic definition of natural numbers. For Poincaré (as laser for L. E. J. Brouwer) the natural numbers constituted a fundamental intuitive notion, apparently to be taken for granted without further analysis; he several times explicitly repudiated the concept of an infinite set in favor of the "otential infinite," but he never developed this idea systematically. He obviously had a blind spot regarding the formalization of mathematics, and poked fun repeatedly at the efforts of the disciples of Peano and Russell in that direction; but, somewhat paradoxically, his criticism of the early attempts of Hilbert was probably the starting point of some of the most fruitful of the later developments of matamathematics. Poincaré stressed that Hilbert's point of view of defining objects by a system of axioms was only admissible if one could prove a priori that such a system did not imply contradiction, and it is well known that the proof of noncontradiction was the main goal of the theory which Hilbert founded after 1920. Poincaré seems to have been convinced that such attempts were hopeless, and K. Gödel's theorem proved him right; what Poincaré failed to grasp is that all the work spent on matamathematics would greatly improve our understanding of the nature of mathematical reasoning.

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