Bertrand Arthur William Russell | Encyclopedia.com

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(*b*. Trelleck, Monmouthshire, England, 18 May 1872: *d*. Plas Penrhyn, near Penrhyndeudraeth, Wales, 2 February 1970)

mathematical logic.

The Russell family has played a prominent part in the social, intellectual, and political life of Great Britain since the time of the Tudors; Russells were usually to be found on the Whig side of politics, with a firm belief in civil and religious liberty, as that phrase was interpreted by the Whigs. Lord John Russell (later first earl Russell), the third son of the sixth duke of Bedford, was an important figure in nineteenth-century politics: He was a leader in the struggle to establish the great Reform Act of 1832, held several high offices of state, and was twice prime minister in Whig and Whig-Liberal administrations. His eldest son, known by the courtesy title of Viscount Amberley, married Katherine Stanley, of another famous English family, the Stanleys of Alderley. The young couple were highly intelligent and were in strong sympathy with most of the reforming and progressive movements of their time, a stance that made them far from popular with the conservative section of the aristocracy. Unhappily, neither enjoyed good health; the wife died in 1874 and the husband in 1876. There were two children, Frank and Bertrand, the latter the younger by about seven years.

The Russell family did not approve of the arrangements made by Viscount Amberley for the upbringing of the two children in the event of his death. When this occurred, the boys were made wards in chancery and placed in the care of Earl Russell and his wife, who were then living at Pembroke Lodge in Richmond Park, a house in the gift of the Crown. Bertrand's grandfather died in 1878, but his grandmother lived until 1898 and had a strong influence on his early life.

Like many Victorina children of the upper class, the boy was educated at home by a succession of tutors, so that when he entered Trinity College, Cambridge, as a scholar in 1890, he had no experience of communal life in an educational establishment save for a few months in a "cramming" school in London. At Trinity he was welcomed into a society that for intellectual brilliance could hardly have been bettered anywhere at that time. He obtained a first class in the mathematical tripos and in the moral sciences tripos, although the formality of examinations seems not to have appealed to him. He remarks in his autobiography that the university teachers "contributed little to my enjoyment of Cambridge," and that "I derived no benefits from lectures."

A great stimulus to Russell's development was his election in 1892 to the Apostles. This was a small, informal society, founded about 1820, that regarded itself—not without some justification—as composed of the intellectual cream of the university; its main object was the completely unfettered discussion of any subject whatsoever. One member was A.N. Whitehead, then a mathematical lecturer at Cambridge, who had read Russell's papers in the scholarship examination and had in consequence formed a high opinion of his ability. Through the society Russell acquired a circle of gifted friends: the philosophers G.E. Moore and Ellis McTaggart, the historian G.M. Trevelyan and his poet brother R.C. Trevelyan, the brilliant brothers Crompton and Theodore Llewelyn Davies, and later the economist J.M. Keynes and the essayist Lytton Strachey.

In the latter part of the nineteenth century, progressive opinion at Cambridge had begun to maintain that university dons should regard research as a primary activity, rather than as a secondary pursuit for leisure hours after teaching duties had been performed. This opinion was particularly strong in Trinity, where A.R. Forsyth, W.W. Rouse Ball, and Whitehead encouraged the researches of younger men such as E.W. Barnes, G.H. Hardy, J.H. Jeans, E.T. Whittaker, and Russell; these themselves exercised a great influence on the next generation, the remarkable set of Trinity mathematicians of the period 1900–1914.

One mode of encouragement was the establishment of prize fellowships, awarded for original dissertations; such a fellowship lasted for six years and involved no special duties, the object being to give a young man an unhindered opportunity for intellectual development. Russell was elected in 1895, on the strength of a dissertation on the foundations of geometry, published in 1897. During the later part of his tenure and after it lapsed, he was not in residence; but in 1910 the college appointed him to a special lectureship in logic and the philosophy of mathematics.

During World War I pacifism excited emotions much more bitter than was the case in World War II. Russell's strongly held views made him unpopular in high places; and when in 1916 he published a leaflet protesting against the harsh treatment of a conscientious objector, he was prosecuted on a charge of making statements likely to prejudice recruiting for and discipline in the armed services, and fined £100. The Council, the governing body of Trinity, then dismissed him from his lectureship, and Russell broke all connection with the college by removing his name from the books. In 1918 another article of his was judged seditious, and he was sentenced to imprisonment for six months. The sentence was carried out with sufficient leniency to enable him to write his very useful *Introduction to Mathematical Philosophy* in Brixton Prison.

Many members of Trinity felt that the Council's action in dismissing Russell in 1916 was excessively harsh. After the war the breach was healed: in 1925 the college invited Russell to give the Tarner lectures, later published under the title *The Analysis of Matter;* and from 1944 until his death he was again a fellow of the college.

In the prologue to his *Autobiography* Russell tells us that three strong passions have governed his life: "the longing for love, the search for happiness, and unbearable pity for the suffering of mankind." His writings and his actions testify to the perseverance with which he pursued his aims from youth to extreme old age, undeterred by opposition and regardless of obloquy. Russell's perseverance did not necessarily imply obstinacy, for his mind was never closed; and if his acute sense of logic revealed to him a fallacy in his argument, he would not cling to a logically indefensible view but would rethink his position, on the basis of his three strong principles. But it was also his devotion to logic that led him frequently to reject the compromises so often forced on the practical politician.

Russell—not surprisingly, in view of his ancestry—was always ready to campaign for "progressive" causes. About 1907 he fought hard for women's suffrage, a cause that provoked more opposition and rowdyism in the United Kingdom than any other political question during this century—even more than the pacifism for which Russell was prosecuted in World War I. After the war Russell continued his search for a genuine democracy, in which freedom for the individual should be compatible with the common good; his experiments in education were designed to contribute to this end. Never insular, he would expose what he saw as the faults of his own country or of the English-speaking nations as caustically as he would those of the totalitarian regimes; but that the growth of the latter could be met only by war was a conclusion to which he came very reluctantly. At the close of **World War II** , his vision of humanity inevitably destroying itself through the potency of nuclear weapons caused him to lead a long campaign for nuclear disarmament.

A long list of books bears witness to Russell's endeavor to encourage human beings to think clearly, to understand the new scientific discoveries and to realize some of their implications, and to abhor injustice, violence, and war. *The Impact of Science on Society, History of Western Philosophy, Common Sense and Nuclear Warfare, Marriage and Morals, Freedom and Organisation*, and *Prospects of Industrial Civilisation*, to name only a few, show how earnestly he sought to promote his ideals. All were written in an English that was always clear and precise, and often beautiful. Critics might disagree with his opinions but seldom could misunderstand them. An occasional didactic arrogance might offend, but it could be forgiven in view of the author's manifest sincerity.

A few of Russell's many honors were fellowship of the Royal Society in 1908, the Order of Merit in 1949, and the Nobel Prize for literature in 1950.

Russell has told us in his autobiography how he began the study of geometry, with his elder brother as tutor, at the age of eleven. Like almost every other English boy of his time, he began on Euclid; unlike almost every other English boy, he was entranced, for he had not known that the world contained anything so delicious. His brother told him that the fifth proposition of book, **I** , the notorious *pons asinorum*, was generally considered difficult; but Russell found it no trouble. Having been told, however, that Euclid proved things, he was disappointed at having to begin by assimilating an array of axioms and would not accept this necessity until his brother told him that unless he did so, his study of geometry could not proceed, thus extorting a reluctant acceptance. The anecdote is not irrelevant; Russell's mathematical work, which occupied him until he was over forty, was almost entirely concerned with probing and testing the foundations of mathematics, in order that the superstructure might be firmly established.

Russell's fellowship thesis was revised for publication in 1897 as *An Essay on the Foundations of Geometry*. Its basic theme was an examination of the status assigned to geometry by Kant in his doctrine of synthetic a priori judgments. Analytic propositions are propositions of pure logic; but synthetic propositions, such as "New York is a large city," cannot be obtained by purely logical processes. Thus all propositions that are known through experience are synthetic; but Kant would not accept the converse, that only such propositions are synthetic. An empirical proposition is derived from experience; but an a priori proposition, however derived, is eventually recognized to have a basis other than experience.

Kant's problem was to determine how synthetic a priori judgments or propositions are possible. He held that Euclidean geometry falls into this category, for geometry is concerned with what we perceive and thus is conditioned by our perceptions. This argument becomes dubious when the full implication of the existence of non-Euclidean geometries is appreciated; but although these geometries were discovered about 1830, the philosophical implications had hardly been fully grasped by the end of the nineteenth century. One considerable step was taken by Hilbert when he constructed his formal and abstract system based on his epigram that in geometry it must be possible to replace the words "points," "lines," and "planes" by the words "tables," "chairs," and "beer mugs"; but his *Grundlagen der Geometrie* was not published until 1899.

To Russell, non-Euclidean spaces were possible, in the philosophical sense that they are not condemned by any a priori argument as to the necessity of space for experience. His examination of fundamentals led him to conclude that for metrical geometry three axioms are a priori: (1) the axiom of free mobility, or congruence: shapes do not in any way depend on absolute position in space; (2) the axiom of dimensions: space must have a finite integral number of dimensions; (3) the axiom of distance: every point must have to every other point one and only one relation independent of the rest of space, this relation being the distance between the two points.

For projective geometry the a priori axioms are (1) as in metrical geometry; (2) space is continuous and infinitely divisible, the zero of extension being a point; (3) two points determine a unique figure, the straight line. For metrical geometry an empirical element enters into the concept of distance, but the two sets are otherwise equivalent. In the light of modern views on the nature of a geometry, these investigations must be regarded as meaningless or at least as devoted to the wrong kind of question. What remains of interest in the *Foundations of Geometry* is the surgical skill with which Russell can dissect a corpus of thought, and his command of an easy yet precise English style.

Following the publication of the *Foundations of Geometry*, Russell settled down to the composition of a comprehensive treatise on the principles of mathematics, to expound his belief that pure mathematics deals entirely with concepts that can be discussed on a basis of a small number of fundamental logical concepts, deducing all its propositions by means of a small number of fundamental logical principles. He was not satisfied with his first drafts, but in July 1900 he went to Paris with Whitehead to attend an International Congress of Philosophy. Here his meeting with Peano brought about, in his own phrase, "a turning point in my intellectual life." Until then he had had only a vague acquaintance with Peano's work, but the extraordinary skill and precision of Peano's contributions to discussions convinced Russell that such mastery must be to a large extent due to Peano's knowledge of mathematical logic and its symbolic language.

The work of Boole, Peirce, and Schröder had constructed a symbolic calculus of logic; and their success contributed to Peano's systematic attempt to place the whole of mathematics on a purely formal and abstract basis, for which purpose he utilized a symbolism of his own creation. This enabled him and to analyze the logic, basis, and structure of such mathematical concepts as the positive integers. The apparently trivial symbolism that replaces "The entity x is a member of the class A" with "$x \in A$" leads not only to brevity but, more importantly, to a precision free from the ambiguities lurking in the statement "x is A." One result of Peano's work was to dispose of Kant's synthetic a priori judgments.

Russell rapidly mastered Peano's symbolism and ideas, and then resumed the writing of his book on principles; the whole of the first volume was completed within a few months of his meeting with Peano. Some sections were subjected to a thorough rewriting, however; and volume **I** of *Principles of Mathematics* was not published until 1903. The second edition (1937) is perhaps more valuable for the study of the development of Russell's ideas, for it both reprints the first edition and contains a new introduction in which Russell gives his own opinion on those points on which his views had changed since 1903; but in spite of Hilbert and Brouwer, he is still firm in his belief that mathematics and logic are identical.

The second volume of the *Principles*, to be written in cooperation with Whitehead, never appeared because it was replaced by the later *Principia Mathematica*. It was to have been a completely symbolic account of the assimilation of mathematics to logic, of which a descriptive version appears in volume **I** . The main sections of volume **I** treat indefinables of mathematics (including a description of Peano's symbolic logic), number, quantity, order, infinity and continuity, space, and matter and motion. On all these topics Russell's clarity of thought contributed to the establishment of precision; thus his analysis of the words "some," "any," "every," for instance, is very searching. In two places, at least, he was able to throw light on familiar but vexed topics of mathematical definition and technique.

That the positive integer 2 represents some property possessed by all couples may be intuitively acceptable but does not supply a precise definition, since neither existence nor uniqueness is guaranteed in this way. Russell's definition of a number uses a technique of equivalence that has had many further applications. The definition had already been given by Frege, but his work was not then known to Russell;[1] indeed, it was known to hardly any mathematician of the time, since Frege's style and symbolism are somewhat obscure. Two classes, A and B, are said to be similar $(A \sim B)$ if each element of either class can be uniquely mated to one element of the other class; clearly A is similar to itself. Similarity is a transitive and symmetrical relation; that is, if $A \sim B$ and $B \sim C$, then $A \sim C$, and if $A \sim B$, then $B \sim A$. The (cardinal) number of a class A is, then, the class consisting of all classes similar to A. Thus every class has a cardinal number, and similar classes have the same cardinal number. The null class ϕ is such that $x \in \phi$ is universally false, and its cardinal is denoted by O. A unit class contains some term x and is such that if y is a member of this class, then $x = y$; its cardinal is denoted by 1. The operations of addition and multiplication are then readily constructed. We thus have a workable definition with no difficulties about existence or uniqueness; but—and this is a considerable concession—the concept of "class" must be acceptable.

A similar clarification of the notion of a real number was also given by Russell. Various methods of definition were known, one of the most popular being that of Dedekind. Suppose that p and q are two mutually exclusive properties, such that every rational number possesses one or the other. Further, suppose that every rational possessing property p is less than any rational possessing property q. This process defines a section of the rationals, giving a lower class L and an upper class R. If L has a greatest member, or if R has a least member, the section corresponds to this rational number. But if L has no greatest and R no least member, the section does not determine a rational. (The case in which L has a greatest and R has a least member cannot arise, since the rationals are dense.) Of the two possible cases, to say that the section in one case corresponds to a rational number and in the other it corresponds to or represents an irrational number is—if instinctive—not to define the irrational number, for the language is imprecise. Russell surmounts this difficulty by simply defining a real number to be a lower section L of the rationals. If L has a greatest member or R a least member, then this real number is rational; in the other case, this real number is irrational.

These are two of the outstanding points in the *Principles*. But the concept of "class" is evidently deeply involved, and the "contradiction of the greatest cardinal" caused Russell to probe the consequences of the acceptance of the class concept more profoundly, particularly in view of Cantor's work on infinite numbers. Cantor proved that the number of subclasses that can be formed out of a given class is greater than the number of members of the class, and thus it follows that there is no greatest cardinal number.

Yet if the class of all objects that can be counted is formed, this class must have a cardinal number that is the greatest possible. From this contradiction Russell was led to formulate a notorious antinomy: **A** class may or may not be a member of itself. Thus, the class of men, mankind, is not a man and is not a member of itself; on the other hand, the class consisting of the number 5 is the number 5, that is, it is a member of itself. Now let W denote the class of all classes not members of themselves. If W is not a member of itself, then by definition it belongs to W—that is, it is a member of itself. If W is a member of W, then by definition it is not a member of itself—that is, it is not a member of W. In the main text this contradiction is discussed but not resolved. In an appendix, however, there is a brief anticipation of an attempt to eliminate it by means of what Russell called the "theory of types," dealt with more fully in *Principia Mathematica*.

A commonsense reaction to this contradiction might well be a feeling that a class of objects is in a category different from that of the objects themselves, and so cannot reasonably be regarded as a member of itself. If x is a member of a class A, such that the definition of x depends on A, the definition is said to be impredicative; it has the appearance of circularity, since what is defined is part of its own definition. Poincaré suggested that the various antinomies were generated by accepting impredicative definitions,[2] and Russell enunciated the "vicious circle principle" that no class can contain entities definable only in terms of that class. This recourse, however, while ostracizing the antinomies, would also cast doubt on the validity of certain important processes in mathematical analysis; thus the definition of the exact upper bound is impredicative.

To meet the difficulty, Russell devised his theory of types. Very crudely outlined, it starts with primary individuals; these are of one type, say type 0. Properties of primary individuals are of type 1, properties of properties of individuals are of type 2, and so on. All admitted properties must belong to some type. Within a type, other than type 0, there are orders. In type 1, properties defined without using any totality belong to order 0; properties defined by means of a totality of properties of a given order belong to the next higher order. Then, finally, to exclude troubles arising from impredicative definitions, Russell introduced his axiom of reducibility: for any property of order other than 0, there is a property over precisely the same range that is of order 0; that is, in a given type any impredicative definition is logically equivalent to some predicative definition.

Whitehead and Russell themselves declared that they were not entirely happy about this new axiom. Even if an axiom may well be arbitrary, it should, so one feels, at least be plausible. Among the other axioms this appears as anomalous as, for instance, the notorious axiom of parallels seems to be in Euclidean geometry.

F.P. Ramsey showed that the antinomies could be separated into two kinds: those which are "logical," such as Russell's, and those which are "semantical," such as that involved in the assertion "I am lying." [3] The first class can then be eliminated by the simpler theory of types, in which the further classification into orders is not required. But even so, Ramsey did not regard this reconstruction of the Whitehead-Russell position as altogether satisfactory. Weyl pointed out that one might just as well accept the simpler axiomatic set theory of Zermelo and Fraenkel as a foundation, and remarked that a return to the standpoint of Whitehead and Russell was unthinkable.[4]

Before leaving the *Principles*, some other matters are worthy of note. First, the important calculus of relations, hinted at by De Morgan and explored by C. S. Peirce, is examined in detail.[5] Second, Russell was not concerned merely with foundations; he took the whole mathematical world as his parish. There is thus a long examination of the nature of space and of the characteristics of projective, descriptive, and metrical geometries. Third, there is an analysis of philosophical views on the nature of matter and motion. Finally, he draws, possibly for the first time, a clear distinction between a proposition, which must be true or false, and a propositional function, which becomes a proposition, with a truth value, only when the argument is given a determinate value; the propositional function "x is a prime number" becomes a proposition, which may be true or false, when x is specified.

In the three volumes of *Principia Mathematica* (1910–1913) Whitehead and Russell took up the task, attempted in Russell's uncompleted *Principles*, of constructing the whole body of mathematical doctrine by logical deduction from the basis of a small number of primitive ideas and a small number of primitive principles of logical inference using a symbolism derived from that of Peano but considerably extended and systematized.

Associated with elementary propositions p, q, the primitive concepts are (1) negation, the contradictory of p, not-p, denoted by $\sim p$; (2) disjunction, or logical sum, asserting that at least one of p and q is true, denoted by $p \vee q$; (3) conjunction, or logical product, asserting that both p and q are true, denoted by $p \cdot q$; (4) implication, p implies q, denoted by $p \supset q$; (5) equivalence, p implies q and q implies p, denoted by $p \equiv q$. If a proposition is merely to be considered, it may be denoted simply by p; but if it is to be asserted, this is denoted by $\vdash p$, so that $\vdash$ may be read as "It is true that...." The assertion of a propositional function for some undetermined value of the argument is denoted by $\exists$, which may be read as "There exists a... such that...."

Dots are used systematically in place of brackets; the rule of operation is that the more dots, the stronger their effect. An example will show the way in which the dots are used. The proposition "if either p or q is true, and either p or 'q implies r' is true, then either p or r is true" may be written as

$$\vdash :: p \vee q : p \cdot \vee \cdot q \supset r : \supset \cdot p \vee r.$$

The five concepts listed above are not independent. In the *Principia* negation and disjunction are taken as fundamental, and the other three are then defined in terms of these two. Thus conjunction, $p \cdot q$, is defined as $\sim(\sim p \vee \sim q)$; implication, $p \supset q$, as $\sim p \vee q$; and equivalence, $p \equiv q$, as $(p \supset q) \cdot (q \supset p)$, which can of course now be expressed entirely in terms of the symbols $\sim$ and $\vee$. Another elementary function of two propositions is incompatibility; p is incompatible with q if either or both of p and q are false, that is, if they are not both true; the symbolic notation is p/q. Negation and disjunction are then definable in terms of incompatibility:

$$\sim p = p/p, p \vee q = p/\sim q.$$

Thus the five concepts (1) —(5) are all definable in terms of the single concept of incompatibility; for instance, $p \supset q = p/(q/q)$. This reduction was given by H. M. Sheffer in 1913,[6] although Willard Quine points out that Peirce recognized the possibility about 1880.

The primitive propositions first require two general principles of deduction—anything implied by a true proposition is true, and an analogous statement for propositional functions: when $\phi(x)$ and "$\phi(x)$ implies $\psi(x)$" can be asserted, then $\psi(x)$ can be asserted. There are then five primitive propositions of symbolic logic:

(1) Tautology. If either p is true or p is true, then p is true:

$$\vdash : p \vee p \cdot \supset \cdot p.$$

(2) Addition. If q is true, then "p or q" is true:

$$\vdash : q \cdot \supset \cdot p \vee q.$$

(3) Permutation. "p or q" implies "q or p":

$$\vdash : p \vee q \cdot \supset \cdot q \vee p.$$

(4) Association. If either p is true or "q or r" is true, then either q is true or "p or r" is true:

$$\vdash : p \vee (q \vee r) \cdot \supset \cdot q \vee (p \vee r).$$

Summation. If q implies r, then "p or q" implies "p or r":

$$\vdash :. q \supset r \cdot \supset : p \vee q \cdot \supset \cdot p \vee r.$$

Following up Sheffer's use of incompatibility as the single primitive concept, Nicod showed that the primitive propositions could be replaced by a single primitive proposition of the form

$$p \cdot \supset \cdot q \cdot r : \supset \cdot t \supset t \cdot s/q \supset p/s,$$

which, since $p \supset q = p/(q/q)$, can be expressed entirely in terms of the stroke symbol for incompatibility.[7]

The second edition of the *Principia* (1925–1927) was mainly a reprint of the first, with small errors corrected; but its worth to the student is considerably increased by the addition of a new introduction, of some thirty-four pages, in which the authors give an account of modifications and improvements rendered possible by work on the logical bases of mathematics following the appearance of the first edition—for instance, the researches of Sheffer and Nicod just mentioned. The authors are mildly apologetic about the notorious axiom of reducibility; they are not content with it, but are prepared to accept it until something better turns up. In particular, they refer to the work of Chwistek and of Wittgenstein, without, however, being able to give wholehearted approval.[8] Much of the introduction is devoted to Wittgenstein's theory and its consequences. Here they show that the results of volume I of the *Principia* stand, although proofs have to be revised; but they cannot, on Wittgenstein's theory, reestablish the important Dedekindian doctrine of the real number, nor Cantor's theorem that $2^n > n$, save for the case of n finite. The introduction was also much influenced by the views of Ramsey, whose death in 1930, at the age of twenty-seven, deprived Cambridge of a brilliant philosopher.

The publication of the *Principia* gave a marked impulse to the study of mathematical logic. The deft handling of a complicated but precise symbolism encouraged workers to use this powerful technique and thus avoid the ambiguities lurking in the earlier employment of ordinary language. The awkwardness and inadequacy of the theory of types and the axiom of reducibility led not only to further investigations of the Whitehead-Russell doctrine but also to an increased interest in rival theories, particularly Hilbert's formalism and Brouwer's intuitionism. Perhaps because none of these three competitors can be regarded as finally satisfactory, research on the foundations of mathematics has produced new results and opened up new problems the very existence of which could hardly have been foreseen in the early years of this century. Whitehead and Russell may have failed in their valiant attempt to place mathematics once and for all on an unassailable logical basis, but their failure may have contributed more to the development of mathematical logic than complete success would have done.

The *Introduction to Mathematical Philosophy* (1919), written while Russell was serving a sentence in Brixton Prison, is a genuine introduction but certainly is not "philosophy without tears"; it may perhaps best be described as *une oeuvre de haute vulgarisation*. The aim is to expound work done in this field, particularly by Whitehead and Russell, without using the complex symbolism of *Principia Mathematica*. Russell's mastery of clear and precise English stood him in good stead for such a task, and many young students in the decade 1920–1930 were first drawn to mathematical logic by a study of this efficient and readable volume.

To explain the arrangement of the book, Russell remarks, "The most obvious and easy things in mathematics are not those that come logically at the beginning: they are things that, from the point of view of logical deduction, come somewhere near the middle." Taking such things as a starting point, a close analysis should lead back to general ideas and principles, from which the starting point can then be deduced or defined. This starting point is here taken to be the familiar set of positive integers; the theory of these, as shown by Peano, depends on the three primitive ideas of zero, number, and successor, and on five primitive propositions, one of which is the principle of mathematical induction. The integers themselves are then defined by the Frege-Russell method, using the class of all similar classes, and the relation between finiteness and mathematical induction is established. Order and relations are studied next, to enable rational, real, and complex numbers to be defined, after which the deeper topics of infinite cardinals and infinite ordinals can be broached. It is then possible to look at certain topics in analysis, such as limit processes and continuity.

The definition of multiplication when the number of factors may be infinite presents a subtle difficulty. If a class A has m members and a class B has n members, the product m X n can be defined as the number of ordered couples that can be formed by choosing the first term of the couple from A and the second from B. Here m or n or both may be infinite, and the definition may readily be generalized further to the situation in which there is a finite number of classes, $A, B, \ldots, K$. But if the number of classes is infinite, then in defect of a rule of selection, we are confronted with the impossible task of making an infinite number of arbitrary acts of choice.

To turn this obstacle, recourse must be had to the multiplicative axiom, or axiom of selection: given a class of mutually exclusive classes (none being null), there is at least one class that has exactly one term in common with each of the given classes. This is equivalent to Zermelo's axiom that every class can be well-ordered, that is, can be arranged in a series in which each subclass (not being null) has a first term.[9] This matter is dealt with in *Principia Mathematica*, but here Russell offers a pleasant illustration provided by the arithmetical perplexity of the millionaire who buys a pair of socks whenever he buys a pair of boots, ultimately purchasing an infinity of each. In dealing with the number of boots, the axiom is not required, since we can choose, say, the right boot (or the left) from every pair. But no such distinction is available in counting the socks, however, and here the axiom is needed.

The last six chapters of the *Introduction* are concerned with the theory of deduction and the general logical bases of mathematics, including an analysis of the use and nature of classes and the need, in Russell's theory, for a doctrine of types.

Among the essays collected in *Mysticism and Logic* (1921) are some that deal, in popular style, with Russell's views on mathematics and its logical foundations. One of these, "Mathematics and the Metaphysicians," written in 1901, had appeared in *International Monthly*. The editor had asked Russell to make the article "as romantic as possible," and hence it contains a number of quips, some now famous, in which the air of paradox masks a substantial degree of truth. To say that pure mathematics was discovered by George Boole in 1854 is merely Russell's way of stating that Boole was one of the first to recognize the identity of formal logic with mathematics, a point of view firmly held by Russell. In emphasizing that pure mathematics is made up of logical steps of the form "If p, then q" —that is, if such and such a proposition is true of anything, then such and such another proposition is true of that thing—Russell remarks that it is essential not to discuss whether the first proposition is really true, and not to mention what the anything is, of which it is supposed to be true. He is thus led to his oft-quoted description of mathematics as "the subject in which we never know what we are talking about, nor whether what we are saying is true." He comments that many people may find comfort in agreeing that the description is accurate.

Russell's gifts as a popularizer of knowledge are shown in a number of his other books, such as *The Analysis of Matter* and *The ABC of Relativity*, in which problems arising from contemporary physics are discussed. He never wholly divorced mathematics from its applications; and even his first book, on the foundations of geometry, had its origin in his wish to establish the concept of motion and the laws of dynamics on a secure logical basis. In these later books his critical skill is exercised on the mathematical foundations of physics and occasionally is used to provide alternatives to suggested theories.

For instance, the advent of relativity, bringing in the notion of an event as a point in the space-time continuum, had encouraged Whitehead to deal with the definition of points and events by the application of his principle of extensive abstraction, discussed in detail in his *An Enquiry Concerning the Principles of Natural Knowledge* (1919). This principle has a certain affinity with the Frege-Russell definition of the number of a class as the class of all similar classes. To state the application very crudely, a point is defined as the set of all volumes that enclose that point; Whitehead is of course careful to frame the principle in such a way as to avoid the circularity suggested in this crude statement. This idea is then used to define an event. In *The Analysis of Matter*, Russell argues that while logically flawless, this definition, in the case of an event, does not seem genuinely to correspond to the nature of events as they occur in the physical world, and that it makes the large assumption that there is no minimum and no maximum to the time extent of an event. He develops an alternative theory involving an ingenious application of Hausdorff's axioms for a topological space.[10]

Whatever the final verdict on Russell's work in symbolic logic maybe, his place among the outstanding leaders in this field in the present century must be secure.

NOTES

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2. H. Poincaré, "La logique de l'infini," in Scientia, **12** (1912).

3. F. P. Ramsey, "The Foundations of Mathematics," in *Proceedings of the London Mathematical Society*, 2nd ser., **25** (1926).

4. H. Weyl, "Mathematics and Logic," in *American Mathematical Monthly*, **53** (1946); and "David Hilbert and His Mathematical Work," in Bulletin of the American Mathematical Society, **50** (1944).

5. A. De Morgan, "On the Logic of Relations," in *Transactions of the Cambridge Philosophical Society*, **10** (1864); C. S. Peirce, "On the Algebra of Logic," in *American Journal of Mathematics*, **3** (1880).

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7. Jean Nicod, "A Reduction in the Number of the Primitive Propositions of Logic," in *Proceedings of the Cambridge Philosophical Society. Mathematical and Physical Sciences*, **19** (1919).

8. Leon Chwistek, "The Theory of Constructive Types," in *Annales de la Société mathématique de Pologne* (1924); L. Wittgenstein. Tractatus Logico-Philosophicus (London, 1922).

9. E. Zermelo, "Beweis, dass jede Menge wohlgeordnet werden kann," in *Mathematische Annalen*, **59** (1904); see also *ibid.*, **65** (1908).

10. F. Hausdorff, *Grundzğe der Mengenlehre* (Leipzig, 1914).

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I. original works. Russell's contributions to mathematical logic are best studied through *An Essay on the Foundations of Geometry* (London, 1897); *The Principles of Mathematics*, **1** (London, 1903; 2nd ed. with new intro., 1937); *Principia Mathematica*, 3 vols. (London, 1910–1913; 2nd ed., with new intro., 1925–1927), written with A. N. Whitehead; and *Introduction to Mathematical Philosophy* (London, 1919).

The Autobiography of Bertrand Russell, 3 vols. (London, 1967–1969), contains a nontechnical account of his mathematical work in vol. **I** and early chs. of vol. **II** . *Mysticism and Logic* (London, 1917) contains some essays of a popular nature.

II. Secondary Literature. The primary authority for Russell's life in his *Autobiography*, cited above. G. H. Hardy. *Bertrand Russell and Trinity* (London, 1970), clears away some misconceptions concerning Russell's relations with the college. An able and witty criticism of Russell's logical ideas is presented by P. E. B. Jourdain, given in *The Philosphy of Mr. B*rtr* nd R*ss *;ll* (London, 1918).

For useful surveys of the doctrine of *Principia Mathematica*, see S. K. Langer. *An Introduction to Symbolic Logic* (New York, 1937); S. C. Kleene, *Introduction to Metamathematics* (New York, 1952), which has a valuable selected bibliography; and, by F. P. Ramsey, several items in *The Foundations of Mathematics and Other Logical Essays* (London, 1931). *The Philosophy of Bertrand Russell*, P. A. Schilpp, ed. (Chicago, 1944), contains a section by Gödel on Russell's mathematical logic.

To trace the many publications related directly or indirectly to Russell's work, see A. Church, "A Bibliography of Symbolic Logic," in *Journal of Symbolic Logic*, **1** (1936) and **3** (1938); it is also available as a separate volume. For the literature since 1935, the reader is advised to consult the volumes and index parts of the *Journal of Symbolic Logic*.

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