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(*b*. Grasswil, Bern, Switzerland, 15 January 1814; *d*. Bern, Switzerland, 20 March 1895)

mathematics.

Schläfli, the son of Johann Ludwig schläfli, a citizen of Burgdorf, and Magdalena Aebi, attended primary school in Burgdorf. With the aid of a scholarship he was able to study at the Gymnasium in Bern, where he displayed a gift for mathematics. He enrolled in the theological faculty at Bern but, not wishing to pursue an ecclesiastical career, decided to accept a post as teacher of mathematics and science at the *Burgerschule* in Thun. He taught there for ten years, using his few free hours to study higher mathematics. In the autumn of 1843 Jakob Steiner, who was traveling to Rome with Jacobi, Dirichlet, and Borchardt, proposed that they take Schläfli with them as interpreter. Schläfli thus had an opportunity to learn from the leading mathematicians of his time. Dirichlet instructed him daily in number theory, and Schläfli's later works on quadratic forms bear the mark of this early training. During this period Schläfli translated two works by Steiner and two by Jacobi into Italian.

In 1848 Schläfli became a *Privatdozent* at Bern, where, as he expressed it, he was "confined to a stipend of Fr. 400 and, in the literal sense of the word, had to do without (*darben musste*)." His nomination as extraordinary professor in 1853 did not much improve his situation, and it was not until he became a full professor in 1868 that he was freed from financial concerns. Schläfli's scientific achievements gained recognition only slowly. In 1863 he received an honorary doctorate from the University of Bern; in 1868 he became a corresponding member of the Istituto Lombardo di Scienze e Lettere in Milan and was later accorded the same honor by the Akademie der Wissenschaften in Göttingen (1871) and the Accademia dei Lincei (1883). He won the Jakob Steiner Prize for his geometric works in 1870.

While at Bern, Schläfli was concerned with two major problems, one in elimination theory and the other in n-dimensional in geometry, and he brought his results together in two extensive works . The first problem is discussed in "Ueber die Resultanteeines Systems mehrerer algebraischer Gleichungen. Ein Beitrag zur Theorie der Elimination" (published in *Denkschriften der Akademie der Wissenschaften*, **4**). Schläfli summarized the first part of this work in a letter to Steiner:

For a given system of n equations of higher degree with n unknowns, **I** take a linear equation with literal (undetermined) coefficients $a, b, c, \cdots$ and show how one can thus obtain true resultants without burdening the calculation with extraneous factors. If everything else is given numerically, then the resultant must be decomposable into factors all of which are linear with respect to $a, b, c, \cdots$. In the case of each of these linear polynomials the coefficients of $a, b, c, \cdots$ are then values of the unknowns belonging to a *single* solution.

Drawing on the works of Hesse, Jacobi, and Cayley, Schläfli presented applications to special cases. He then developed the fundamental theorems on class and degree of an algebraic manifold, theorems that attracted the interest of the Italian school of geometers. The work concluded with an examination of the class equation of third-degree curves. Through this publication Schläfli became acquainted with Arthur Cayley, whose paper "Sur un théorème de M. Schläfli" begins: "In § 13 of a very interesting memoir by M. Schläfli one finds a very beautiful theorem on resultants." The acquaintance led to an extensive correspondence and opened the way for Schläfli to publish in English journals. In his obituary of Schläfli, F. Brioschi wrote:

While rereading this important work recently it occurred to me that it displays the outstanding characteristics of Schläfli's work as a whole. These are, first, deep and firsthand knowledge of the writings of other authors; next, a desire and ability to generalize results; and, finally, great penetration in investigating problems from very different points of view.

The second of the two major works, "Theorie der vielfachen Kontinuität," was rejected by the academies of Vienna and Berlin because of its great length and was not published until 1901 (in *Neue Denkschriften der Schweizerischen naturforschenden Gesellschaft*). For many years only sections of it appeared in print–in the journals of Crelle and Liouville and in the *Quarterly Journal of Mathematics*. The core of this work consisted of the detailed theory of regular bodies in Euclidean space R_n of n dimensions and the associated problems of the regular subdivision of the higher dimensional spheres. Schläfli based his investigation of regular polytopes on his discovery that such objects can be characterized by certain symbols now known as Schläfli symbols. His definition was recursive: $\{k_1\}$ is the symbol of the plane regular k_1-gon. $\{k_1, \cdots ,k_{n-1}\}$ is the Schläfli symbol of that regular polytope the boundary polytopes of which have the symbol $k_1, \cdots ,k_{n-2}$ and the vertex polytopes of which have the symbol $\{k_2, \cdots ,k_{n-1}\}$.

Schläfli discovered a way of finding all regular polytopes by calculating the numbers $k_1, \cdots, k_{n-1}$ in the following manner: In the plane, for every k_1 there exists a $\{k_1\}$. In considering n-space he started from Euler's theorem on polyhedrons, which he formulated and proved for R_n: assume a polytope with a_0 vertexes, a_1 edges, a_2 faces and so on in higher dimensions until a_{n-1} boundary polytopes of dimension $n-1$, and $a_n = 1$. Then it is true that

$$a_0 - a_1 + a_2 - \cdots (-)^{n-1} a_{n-1} + (-1)^n a_n = 1.$$

For $n = 3$ the Euler theorem on polyhedrons becomes $a_0 - a_1 + a_2 = 2$. Since, further, for $\{k_1, k_2\}$ it is true that $k_2 \cdot a_0 = 2a_1 = k_1 \cdot a_2$, it follows that

$$a_0 : a_1 : a_2 : 1 = 4k_1 : 2k_1k_2 : 4k_2 : [4-(k_1 - 2)(k_2 - 2)].$$

The nature of the problem requires a positive value for [$4-(k_1-2)(k_2-2)$"]; therefore $(k_1-2)(k_2-2)$ can take only the values 1,2,3. This yields the following possibilities; {3,3} tetrahedron, {3,4} octahedron, {3,5} icosahedron, {4,3} cube, and {5,3} dodecahedron. For $n = 4$ the Euler equation $a_0 - a_1 + a_2 - a_3 = 0$ becomes homogeneous and yields only the ratios of the a_i. Schläfli therefore determined the radius of the circumscribed sphere of a $\{k_1, k_2, k_3\}$ of edge length 1. If this radius is to be real, then it must be true that . This condition yields the six bodies {3,3,3}, {4,3,3}, {3,3,4}, {3,4,3}, {5,3,3}, and {3,3,5}. Schläfli proved further that in every R_n with $n > 4$ there are only three regular solids: {3,3, ⋯ , 3}, regular simplex; {4,3,⋯, 3}, n-dimensional cube; and {3,3,⋯,3,4}, regular n-dimensional octahedron.

Schläfli achieved another beautiful result by considering the unit sphere in R_n and n hyperplanes through the origin $(1) = 0, \cdots, (n) = 0$. Specifically, he found that the inequalities $(1) \geq 0, \cdots, (n) \geq 0$ determine a spherical simplex with surface S_n. Schläfli proved , where S_{n-2} is the surface of a boundary simplex of two dimensions less and λ is a suitable angle between two such simplexes, and the summation extends over all such boundary simplexes. Let O_n be the surface of the sphere; then the Schläfli function f_n is defined by . It can be proved that . where a_k is proportional to the k+1 Bernoulli number B_k. This equation states that the Schläfli function in a space of odd dimension can be reduced to Schläfli functions in spaces of even dimension. Concerning this discovery Schläfli wrote to Steiner: "I believe I am not overestimating the importance of this general theorem if I set it beside the most beautiful results that have been achieved in geometry."

Besides the theory of Schläfli functions the second section of the paper included a detailed treatment of the decomposition of an arbitrary spherical simplex into right-angled simplexes. The section concluded with a theorem on the sum of the squares of the projections of a ray on the vertex rays of a regular polytope, a question that has interested researchers in recent times.

The third section, headed "Verschiedene Anwendungen der Theorie der vielfachen Kontinuität, welche das Gebiet der linearen und sphärischen übersteigen," contains both applications of theorems of Binet, Monge, Chasles, and Dupin to quadratic continua in R_n and Schläfli's own discoveries. After first determining the midpoint, major axes, and conjugate diameters for a quadratic continuum, he demonstrates the law of inertia of the quadratic forms by means of continuity considerations. Among other results presented is a generalization of a theorem of Binet for a system of conjugate radii: the sum of the squares of all m-fold parallelepipeds constructed out of the conjugate radii of a system is equal to the sum obtained when the system is formed from the major axes. Schläfli then divided the quadratic continua into two classes and generalized Monge's theorem on the director circle, or great circle, of a central conic section. He also examined confocal systems and showed that in R_3 their determination depends on a third-order linear differential equation.

After "Theorie der vielfachen Kontinuität" had appeared in its entirety, P. H. Schoute wrote in 1902:

This treatise surpasses in scientific value a good portion of everything that has been published up to the present day in the field of multidimensional geometry. The author experienced the sad misfortune of those who are ahead of their time: the fruits of his most mature studies cannot bring him fame. And in this case the success of the division of the cubic surfaces was only a small compensation; for, in my opinion, this achievement, however valuable it might be, is far from conveying the genius expressed in the theory of manifold continuity.

Steiner communicated to Schläfli Cayley's discovery of the twenty-seven straight lines on the third'degree surface. Schläfli thereupon found the thirty-six "doubles sixes" on this surface and then the division of the cubic surface into twenty-two species according to the nature of the singularities. Schlaäfli also solved problems posed by the Italian school of geometers. He gave a condition under which a manifold has constant curvature: its geodesic lines must appear as straight lines in a suitable coordinate system. He also investigated the space of least dimension in which a manifold can be imbedded; his conjecture on this question was demonstrated by M. Janet and E. Cartan (1926–1927). Schläfli's work on the division of third-order surfaces led him to assert the one-sidedness of the projective plane in a letter to Felix Klein in 1874.

Schläfli wrote a work on the composition theory of quadratic forms in which he sought to provide the proof of the associative law that was lacking in Gauss's treatment of the subject. Schläfli's posthumous papers contain extensive tables for the class number of quadratic forms of both positive and negative determinants.

Schläfli's geometric and arithmetical studies were equaled in significance by his work in function theory. Stimulated by C. G. Neumann's investigations (1867) and following up the representation of the gamma function by a line integral, Schläfli gave the integral representation of the Bessel function $J_n(z)$ for arbitrary n, even where n is not integral. He also wrote an outstanding work on elliptic modular functions (1870) that gave rise to the designation "Schläfli modular equation." An examination of his posthumous manuscripts reveals that in 1867, ten years before Dedekind, Schläfli discovered the domain of discontinuity of the modular group and used it to make a careful analysis of the Hermite modular functions from the analytic, number theoretic, and geometric points of view. As early as 1868, moreover, Schläfli employed means that Weber discovered only twenty years later and termed *f*-functions or class invariants.

Besides his mathematical achievements, Schläfli was an expert on the flora of the canton of Bern and an accomplished student at languages. He possessed a profound knowledge of the *Veda*, and his posthumous manuscripts include ninety notebooks of Sanskrit and commentary on the *Rig'Veda*.

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Johann Jakob Burckhardt