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(*b*. Dorpat, Germany [now Tartu, E.S.S.R.], 13 January 1876; *d*. Berlin, Germany, 6 December 1959)

mathematics.

Schmidt's most significant contributions to mathematics were in integral equations and in the founding of Hilbert space theory. Specifically, he simplified and extended David Hilbert's results in the theory of integral equations; and he formalized Hilbert's distinct ideas on integral equations into the single concept of a Hilbert space, in the process introducing many geometrical terms. In addition he made contributions in the fields of partial differential equations and geometry. The most important of these discoveries were the extensions of the isoperimetric inequality, first to *n*-dimensional Euclidean space and then to multidimensional hyperbolic and spherical spaces. Although his methods were classical rather than abstractionist, nevertheless he must be considered a founder of modern functional analysis.

The son of Alexander Schmidt, a medical biologist, Erhard studied at Dorpat, Berlin, and finally at Göttingen, where he was a doctoral candidate under Hilbert. His degree was awarded in 1905 after the presentation of his thesis, "Entwicklung willkürlicher Funktionen nach Systemen forgeschriebener," After short periods as a teacher in Bonn. Zurich, Erlangen, and Breslau (now Wroclaw), in 1917 he went to the University of Berlin, where he was to remain the rest of his life. In 1946 he became the first director of the Research Institute for Mathematics of the German Academy of Sciences, a post he held until 1958. He was also one of the founders and first editors of *Mathematische Nachrichten* (1948).

The integral equation on which Schmidt's reputation is based has the form

In (1), $K(s, t)$-called the kernel-and $f(s)$ are known functions and ϕ is an unknown function that is to be found. This equation has a long history. Interest in it stemmed from its many applications: for example, if (1) can be solved, then the partial differential equation $\Delta u=\partial^2u/\partial x^2+\partial^2u/\partial y^2=0$ with the prescribed condition $u(x, y)=b(s)$ of arc length s on the boundary of a given region of the plane can also be solved. This differential equation arises in many problems of physics.

From the early nineteenth century on, there were many attempts to solve equation (1), but only partial results were obtained until 1903. In that year Ivar Fredholm was able to present a complete solution to (1), although in his theory the parameter λ plays no significant role. Fredholm showed that for a fixed λ and K either (1) has a unique solution for every function f. or the associated homogeneous equation.

has a finite number of linearly independent solutions: in this case (1) has a solution ϕ only for those f that satisfy certain orthogonality conditions.

In 1904 Hilbert continued the study. He first used a complicated limiting process involving infinite matrices to show that for the fixed but symmetric kernal K ($K(s, t)=K(t, s)$), there would always be values of λ—all real—for which (2) had nontrivial solutions. These λ's he called the *eigenvalues* associated with K, and the solutions he called *eigenfunctions*. He also proved that if f is such that there exists g continuous on [0,1] with

then $f(s)$ can be expanded in a series in eigenfunctions of K, that is,

where $\{\phi_p\}$ is an orthonormal[1] set of eigenfunctions of K.

A year later Hilbert introduced the concept of infinite bilinear forms into both the theory of integral equations and the related topic of infinite matrices. He discovered the concept of complete continuity[2] for such forms and then showed that if $\{a_{ij}: i, j=1, 2, \cdots\}$ are the coefficients of a completely continuous form, then the infinite system of linear equations

either has a unique square summable[3] solution $\{x_i: i = 1, 2,\cdots\}$ for every square summable sequence $\{a_i\}$ or the associated homogeneous system $x_i+\sum a_{ij}x_j=0$ has a finite number of linearly independent solutions. In the latter case, (3) will have solutions only for those sequences $\{a_i\}$ that satisfy certain orthogonality conditions. Hilbert then went on to prove again Fredholm's result converting equation (1) to equation (3) by using Fourier co-efficients.

Schmidt's paper on integral equation (1) appeared in two parts in 1907. He began by reproving Hiklbert's earlier results concerning symmetric kernels. He was able to simplify the proofs and also to show that Hilbert's theorems were valid under less restrictive conditions. Included in this part of the work is the well-known Gram-Schmidt process for the construction of a set of orthonormal functions from a given set of linearly independent functions.

Schmidt then went on to consider the case of (1) in which the kernel $K(s, t)$ is no longer symmetric. He showed that in this case, too, there always will be eigenvalues that are real. The eigenfunctions, however, now occur in adjoint pairs: that is, ϕ and ψ are adjoint eigenfunctions belonging to · if ϕ satisfies

which is called an eigenfunction of the first kind, and ψ satisfies

an eigenfunction of the second kind. Moreover, if $\phi=\phi_1+i\phi_2$, $\psi=\psi_1+i\psi_2$, then ϕ_1 and ψ_1 and ψ_1 are an adjoint pair of eigenfunctions, as are ϕ_2 and ψ_2. Thus, it is only necessary to consider real pairs of eigen-functions.

Other extensions of the symmetric to the unsymmetric case were also developed by Schmidt. As a broadening of Hilbert's result. Schmidt proved (Hilbert-Schmidt theorem) that if *f* is such that there is a function *g* continuous on $[a, b]$ with

then *f* can be represented by an orthonormal series of the eigenfunctions of the first kind of *K*: and if

then *f* has a representation in a series of the second kind of eigenfunctions. He also proved a type of diagonalization theorem: If $x(s)$ and $y(s)$ are continuous on $[a, b]$, then

where $\{\phi_p\}$ and $\{\psi_p\}$ are orthonormal sets of eigenfunctions of the first or second kinds and λ_p are the associated eigenvalues.

The idea behind Schmidt's work is extremely simple. From the kernel $K(s, t)$ of equation (1) he constructed two new kernels:

and

which are both symmetric. Then ϕ and ψ are an adjoint pair eigenfunctions belonging to λ if and only if

and

that is, ϕ is an eigenfunction belonging to λ^2 of $\bar{K}$ and ψ is an eigenfunction belonging to λ^2 of $\bar{K}$. Thus Schmidt could then apply much of the earlier theory of symmetric kernels.

Schmidt's contributions to Hilbert space theory stem from Hilbert himself. Before Hilbert there had been some attempts to develop a general theory of infinite linear equations, but by the turn of the twentieth century only a few partial results had been obtained. Hilbert focused the attention of mathematicians on the connections among infinite linear systems, square summable sequences, and matrices of which the entries define completely continuous bilinear forms. These equations were of importance since their applications were useful not only in integral equations but also in differential equations and continued fractions.

In 1908 Schmidt published his study on the solution of infinitely many linear equations with infinitely many unknowns. Although his paper is in one sense a definitive work on the subject, its chief importance was the explicit development of the concept of a Hilbert space and also the geometry of such space–ideas that were only latent in Hilbert's own work.

A vector or point *z* of Schmidt's space *H* was a square summable sequence of complex numbers, $\{z_n\}$. The inner product of two vectors *z* and *w*– denoted by (z, w)–was given by the formula

and a norm-denoted by $\|z\|$- was defined by . The vectors *z* and *w* were defined to be perpendicular or orthogonal if $(z, w)=0$, and Schmidt showed that any set of mutually orthogonal vectors must be linearly independent. The Gram-Schmidt orthogonalization process was then developed for linearly independent sets, and from this procedure necessary and sufficient conditions for a set to be linearly independent were derived.

Schmidt then considered convergence. If $\{z^n\}$ is a sequence of vector of *H*, then $\{z^n\}$ is defined to converge strongly in *H* to *z* if and $\{z^n\}$ is said to be a strong Cauchy sequence if independently in *p* and *n*. He then showed that every strong Cauchy sequence in *H* converges strongly to some element of *H*. Then the nontrivial concept of a closed subspace *A* of *H* was introduced. Schmidt showed how such subspaces could be constructed and then proved the projection theorem: If *z* is a vector *H* and *A* is a closed subspace of *H*, then *z* has a unique representation $z=a+w$ where *a* is an *A* and *w* is orthogonal to every vector in *A*. Furthermore, $\|w\|=\min \|z-y\|$ where *y* is any element of *A*, and this minimum is actually assumed only for $y=a$. Finally, these results were used to establish necessary and sufficient conditions under which the infinite system of equations

has a square summable solution $\{z_p\}$ where $\{c_n\}$ is a square summable sequence and, for each *n*, $\{a_{np}\}$ is also square summable. He then obtained specific representations for the solutions.

Schmidt's work on Hilbert space represents a long step toward modern mathematics. He was one of the earliest mathematicians to demonstrate that the ordinary experience of Euclidean concepts can be extended meaningfully beyond geometry into the idealized constructions of more complex abstract mathematics.

NOTES

1. The set $\{\phi_p\}$ is orthonormal if

and

2. The form $K(x, x)$ is completely continuous at a if

implies that

where $a=(a_1, a_2,\cdots)$ and $\in^{(b)}=(\in_1^{(b)}, \in_2^{(b)},\cdots)$. In a Hilbert space this is stronger than ordinary continuity (in the norm topology).

3. The sequence of (complex) numbers $\{b_x\}$ is square summable if

BIBLIOGRAPHY

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II. Secondary Literature. On Schmidt and his work, see Ernst Hellinger and Otto Toeplitz, "Integral-gleichungen und Gleichungen mit unendlichvielen Unbekannten," in *Encyklopädie der Mathematischen Wissenschaften*, IIC, 13 (Leipzig, 1923–1927), 1335–1602. This article, also published under separate cover, is an excellent general treatise, and specifically shows the relationship between integral equation theory and the theory of infinite linear systems.

Michael Bernkopf