Jakob Steiner | Encyclopedia.com

Complete Dictionary of Scientific Biography COPYRIGHT 2008 Charles Scribner's Sons
40-50 minutes

(b. Utzensdorf, Bern, Switzerland, 18 March 1796;

d. Bern, Switzerland, 1 April 1863)

mathematics.

Life . Steiner was the youngest of the eight children of Niklaus Steiner (1752–1826), a small farmer and tradesman, and the former Anna Barbara Weber (1757–1832), who were married on 28 January 1780. The fourth child, Anna Barbara (1786–1870), married David Begert: and their daughter Elisabeth (*b*. 1815) married Friedrich Geiser, a butcher, in 1836. To this marriage was born the mathematician Karl Friedrich Geiser (1843–1934), who was thus a grandnephew of Steiner.[1]

Steiner had a poor education and did not learn to write until he was fourteen. As a child he had to help his parents on the farm and in their business: his skill in calculation was of great assistance. Steiner’s desire for learning led him to leave home, against his parents’ will, in the spring of 1814 to attend Johann Heinrich Pestalozzi’s school at Yverdon, where he was both student and teacher. Pestalozzi found a brilliant interpreter of his revolutionary ideas on education in Steiner, who characterized the new approach in an application to the Prussian Ministry of Education (16 December 1826):

The method used in Pestalozzi’s school, treating the truths of mathematics as objects of independent reflection, led me, as a student there, to seek other grounds for the theorems presented in the courses than those provided by my teachers. Where possible I looked for deeper bases, and I succeeded so often that my teachers preferred my proofs to their own. As a result, after I had been there for a year and a half, it was thought that I could give instruction in mathematics[2].

Steiner’s posthumous papers include hundreds of pages of manuscripts containing both courses given by his fellow teachers and his own ideas. These papers include the studies “Einige Gesetze über die Teilung der Ebene und des Raumes,” which later appeared in the first volume of Crelle’s *Journal für die reine und angewandte Mathematik*. Steiner stated that they were inspired by Pestalozzi’s views.

In the application of 1826 Steiner also wrote:

Without my knowing or wishing it, continuous concern with teaching has intensified by striving after scientific unity and coherence. Just as related theorems in a single branch of mathematics grow out of one another in distinct classes, so, I believed, do the branches of mathematics itself. I glimpsed the idea of the organic unity of all the objects of mathematics; and I believed at that time that I could find this unity in some university, if not as an independent subject, at least in the form of specific suggestions.

These two statements provide an excellent characterization of Steiner’s basic attitude toward teaching and research. The first advocates independent reflection by the students, a practice that was the foundation of Steiner’s great success as a teacher. At first, in Berlin, he was in great demand as a private teacher; among his students was the son of Wilhelm von Humboldt. Steiner often gave his courses as colloquiums, posing questions to the students. This direct contact with the students was often continued outside the classroom. The second statement expresses the idea that guided all his work: to discover the organic unity of all the objects of mathematics, an aim realized especially in his fundamental research on synthetic geometry. Steiner left Yverdon in the autumn of 1818 and went to Heidelberg, where he supported himself by giving private instruction. His most important teacher there was Ferdinand Schweins, whose lectures on combinatorial analysis furnished the basis for two of Steiner’s works[3].

At this time Steiner also studied differential and integral calculus and algebra. In addition, lectures at Heidelberg stimulated the careful work contained in manuscripts on mechanics from 1821, 1824, and 1825, upon which Steiner later drew for investigations on the center of gravity[4].

Following a friend’s advice, Steiner left for Berlin at Easter 1821. Not having passed any academic examinations, he was now obliged to do so in order to obtain a teaching license. He was only partially successful in his examinations and therefore received only a restricted license in mathematics, along with an appointment at the Werder Gymnasium. The initially favorable judgment of his teaching was soon followed by criticism that led to his dismissal in the autumn of 1822. From November 1822 to August 1824 he was enrolled as a student at the University of Berlin, at the same time as C. G. J. Jacobi. He again earned his living by giving private instruction until 1825, when he became assistant master (and in 1829 senior master) at the technical school in Berlin. On 8 October 1834 he was appointed extraordinary professor at the University of Berlin, a post he held until his death.

Steiner never married. He left a fortune of about 90,000 Swiss francs, equivalent to 24,000 thaler[5]. He bequeathed a third of it to the Berlin Academy for establishment of the prize named for him,[6] and 60,000 francs to his relatives. In addition he left 750 francs to the school of his native village, the interest on which is still used to pay for prize awarded to students adept at mental computations[7]. Steiner, with a yearly income of between 700 and 800 thaler, amassed this fortune by giving lectures on geometry[8].

Students and contemporaries wrote of the brilliance of Steiner's geometric research and of the fiery temperament he displayed in leading others into the new territory he had discovered. Combined with this were very liberal political views. Moreover, he often behaved crudely and spoke bluntly, thereby alienating a number of people. Thus it is certain that his dismissal from the Werder Gymnasium cannot have been merely a question of his scholarly qualifications. Steiner attributed this action to his refusal to base his course on the textbook written by the school's director, Dr. Zimmermann. The latter, in turn, reproached Steiner for using Pestalozzi's methods, claiming that they were suitable only for elementary instruction and therefore made Steiner's teaching deficient. Steiner also experienced difficulties at the technical school, where he was expected to follow, without question, the orders of the director, K. F. von Klöden, Klöden, however, felt that Steiner did not treat him with proper respect, and made exacting demands of him that were of a magnitude and severity that even a soldier subject to military discipline could hardly be expected to accept.

Steiner's scientific achievements brought him an honorary doctorate from the University of Königsberg (20 April 1833) and membership in the Prussian Academy of Sciences (5 June 1834). He spent the winter of 1854–1855 in Paris and became a corresponding member of the French Academy of Sciences. He had already been made a corresponding member of the Accademia dei Lincei in 1853. A kidney ailment obliged him to take repeated cures in the following years, and he lectured only during the winter terms.

Mathematical Work . Having set himself the task of reforming geometry, Steiner sought to discover simple principles from which many seemingly unrelated theorems in the subject could be deduced in a natural way. He formulated his plan in the preface to *Systematische Entwicklung der Abhängigkeit geometrischer Gestalten voneinander, mit Berücksichtigung der Arbeiten alter und neuer Geometer über Porismen, Projections-Methoden, Geometrie der Lage, Transversalen, Dualität und Reciprocität* (1832), dedicated to Wilhelm von Humboldt:

The present work is an attempt to discover the organism [*Organisums*] through which the most varied spatial phenomena are linked with one another. There exist a limited number of very simple fundamental relationships that together constitute the schema by means of which the remaining theorems can be developed logically and without difficulty. Through the proper adoption of the few basic relations one becomes master of the entire field. Order replaces chaos: and one sees how all the parts mesh naturally, arrange themselves in the most beautiful order, and form well-defined groups. In this manner one obtains, simultaneously, the elements from which nature starts when, with the greatest possible economy and in the simplest way, it endows the figures with infinitely many properties. Here the main thing is neither the synthetic nor the analytic method, but the discovery of the mutual dependence of the figures and of the way in which their properties are carried over from the simpler to the more complex ones. This connection and transition is the real source of all the remaining individual propositions of geometry. Properties of figures the very existence of which one previously had to be convinced through ingenious demonstrations and which, when found, stood as something marvelous are now revealed as necessary consequences of the most common properties of these newly discovered basic elements, and the former are established a priori by the latter.[9]

Also in the preface Steiner asserted that this work would contain "a systematic development of the problems and theorems concerning the intersection and tangency of the circle in the plane and on spherical surfaces and of spheres." The plan was not carried out, and the manuscript of this part was not published until 1931.[10] But many of the observations, theorems, and problems included in it appeared in "Einige geometrische Betrachtungen" (1826), Steiner's first long publication.[11]

The earliest detailed account of some of the sources of Steiner's concepts and theorems can be found in the posthumously published *Allgemeine Theorie über das Berühren und Schneiden der Kreise und der Kugeln, worunter eine grosse Anzahl neuer Untersuchungen und Sätze, in einem systematischen Entwicklungsgange dargestellt*. . . .[12] The headings of the sections describe its contents: "I. Of Centers, Lines, and Planes of Similitude in Circles and Spheres. II. Of the Power and the Locus of Equal Powers With Respect to Circles and Spheres. III. Of the Common Power in Circles and in Spheres. IV. Of Angles at Which Circles and Spheres Intersect."

In the foreword to *Allgemeine Theorie*, F. Gonseth stated in current terminology the basic principle on which many of Steiner's theorems and constructions are founded: the stereographic projection of the plane onto the sphere.[13] Section 4 of this work contains the following problem (§ 29, X, p. 167): "Draw a circle that intersects at equal angles four arbitrary circles of given size and position" The new methods were applied to the solution of Apollonius' problem (§31, 11,p. 175):"Find a circle tangent to three arbitrary circles of given size and position." Another problem (§ 31,II, p. 175):"Find a circle tangent to three arbitrtary circles of given size and position." Another problem (§ 31, III, p. 182)reads: "Find a circle that intersects three arbitrary circles of given size and position at the angles $\alpha_1, \alpha_2, \alpha_3$." Analogous problems for spheres are given in chapter 2, where the theorems and problems are presented systematically according to the number of spheres involved (from two to eight), with size and position again given–for example (p. 306): "Draw a sphere that intersects five arbitrary spheres of given size and position at one and the same angle" and (p. 333) "Find a sphere that is tangent to a sphere M_1 of given size and position and that cuts at one and the same angle three pairs of spheres, of given size and position, M_2 and M_3, M_4 and M_5, M_6 and M_7, each pair taken singly."

At Berlin, Steiner became friendly with Abel,

Crelle, and Jacobi: and together they introduced a fresh, new current into mathematics. Their efforts were considerably aided by Crelle's founding of the *Journal für die reine und angewandte Mathematik*, to which Steiner contributed sixty-two articles. In the first volume (1826) he published his great work "Einige geometrische Betrachtungen."[14] It contains a selection from the *Allgemeine Theorie* and the first published systematic development of the theory of the power of a point with respect to a circle and of the points of similitude of circles: in his account Steiner mentions Pappus, Viète, and Poncelet. As the first application of these concepts Steiner states, without proof, his solution to Malfatti's problem (§ IV, no. 14). In a given triangle *ABC* draw three circles a, b, and c that are tangent to each other and such that each is

tangent to two sides of the triangle (Figure 1). Steiner then remarks that this is a special case of the next problem (no. 15): "Given three arbitrary circles, M_1, M_2, M_3 of specified size and position, to find three other circles m_1, m_2, m_3 tangent to each other and such that each is tangent to two of the given circles, and that each of the given circles M *i* is tangent to two of the circles m_k that are to be found" (Figure 2).

Steiner did not prove his solution. Examination of his posthumous papers shows that he knew of the principle of inversion and that he used it in finding and proving the above and other theorems.[15]

It was likewise by means of an inversion that Steiner found and proved his famous theorem on series of circles (§ IV, no. 22; see Figure 3):

Two circles *n, N* of assigned size and position, lying one within the other, are given. If, for a definite series of circles M, M_1, ..., M_x, each of which is tangent to *n* and *N* unequally and that are tangent to each other in order, the interval between *n* and *N* is *commensurable*, that is, if the series consists of $x + 1$ members forming a sequence of *u* circuits such that the last circle M_x is tangent to the first one *M*: then this interval is commensurable for any series of circles $m, m_1, ..., m_x$; and the latter series also consists of $x + 1$ members forming *u* circuit, as in the first series.

In this same work (§ VI: see Figure 4), he proves a theorem of Pappus, in the following form:

Given two circles M_1, M_2, of assigned size and position that are tangent to each other in *B*. If one draws two arbitrary circles m_1, m_2 that are tangent to each other externally in *b* and each of which is tangent to the two given circles, and if one drops the perpendiculars m_1P_1, m_2P_2 from the centers m_1, m_2 on the axis M_1M_2 of the given circles and divides these perpendiculars by the radii r_1, r_2 of the circles m_1, m_2:

then the quotient corresponding to the circle, $m_2$is greater by 2 than that corresponding to the former; that is, . Or, as Pappus expressed it: the perpendicular m_1 p_1 plus the diameter of the corresponding circle m_1 is to that diameter as the perpendicular m_2 P_2 is to the diameter of the corresponding circle m_2–that is, .

Steiner furnishes a proof of this proposition, which consists, essentially, of the following steps:

1. The straight line $AB \perp BM_1$ is the line of equal powers with respect to the circles M_1 and M_2.

2. *AB* passes through the exterior center of similitude *A* of the circles m_1 and m_2.

3. $Am_2: Am_1 = r_2:r_1 = BP_2:BP_1$ and $AB=Ab$.

4. The points *B*, *C*, *b D* lie on one straight line.

5. The assertion follows from similarity considerations. In a later paper Steiner applied this "ancient" theorem of Pappus to the sphere.[16]

Also in the first volume of Crelle's *Journal*, Steiner published an expanded version of considerations that stemmed from the period that he was in Yverdon: "Einige Gesetze über die Teilung der Ebene und des Raumes."[17] In it he expressly stated that his ideas were inspired by Pestalozzi's ideas. The simplest result in this paper was presented in the following form:

A plane is divided into two parts by a straight line lying within it; by a second straight line that intersects the first, the number of parts of the plane is increased by 2; by a third straight line that intersects the two first lines at two points, the number is increased by 3; and so forth. That is, each successive straight line increases the number of parts by the number of parts into which it was divided by the preceding straight lines. Therefore, a plane is divided by *n* arbitrary straight lines into at most 2+2+3...++(*n*-1)+ parts.

He then subdivided space by means of planes and spherical surfaces.

In the following years Crelle's *Journal* and Gergonne's *Annales de mathématiques* published many of Steiner's papers, most of which were either problems to be solved or theorems to be proved.[18] In this way Steiner exerted an exceptionally stimulating influence on geometric research that was strengthened by the publication of this first book, *Systematische Etwicklung* (1832).[19] It was originally supposed to consist of five sections, but only the first appeared. Some of the remaining sections were published in the *Vorlesungen*[20]

Steiner believed that the fundamental concepts of plane geometry are the range of points considered as the totality of points *a, b, c*,...of a straight line and the pencil of lines *a, b, c*,...through a point *B* (Figure 5). Since the latter are the intersection points of a, b, c, . . . with straight line, an unambiguous relationship is established between the pencil of lines and the range, a relationship that he called projectivity. In volume II, § 2, of the *Vorlesungen über synthetische Geometrie* (1867), Steiner expressed this property through the statement that the two constructs are of the same *cardinality*, an expression that Georg Cantor adopted and generalized[21]

In the first chapter of this part of the *Vorlesungen*, Steiner discusses the elements of projectivity, emphasizing the duality between point and straight line. In particular he proves the harmonic properties of the complete quadrangle and of the complete quadrilateral. In the second chapter he treats the simple elements of solid geometry. At the center of the epochal work stands the theory of conic sections in the third book. Here Steiner proved his fundamental theorem: The intersection points of corresponding lines of two projective pencils of lines from a conic section. In its metric formulation this theorem was essentially known to Jan de Witt and Newton.[22] Steiner, however, was the first to recognize that it was a theorem of projective geometry, and he made it the cornerstone of the projective treatment of the theory of conic sections.

Steiner knew of the significance of his discovery.

The above investigation of projective figures, by placing them in oblique positions deliberately avoided closer research into the laws that govern the projective rays for two straight lines A, A_1 [Figure 6]. We shall now proceed to this examination. It leads, as will be seen, to the most interesting and fruitful properties of curves of the second order, the so–called conic sections. From these almost all other properties of the conics can be developed in a single, comprehensive framework and in a surprisingly simple and clear manner. This examination shows the necessary emergence of the conic sections from the elementary geometric figures; indeed, it shows, at the same time, a very remarkable double production of these [sections] by means of projective figures. . . . When one considers with what ingenuity past and present mathematicians have investigated the conic sections, and the almost countless number of properties that remained hidden for so long, one is struck that, as will be seen, almost all the known properties (and many new ones) flow from their projective generation as from a spring; and this generation also reveals the inner nature of the conic sections to us. For even if properties are known that are similar to those named here, the latter have never, in my opinion, been explicitly stated; in no case, however, has anyone until now recognized the importance that they derive from our development of them, where they are raised to the level of fundamental theorems.[23]

In proposition no. 37 of this volume, Steiner stated and proved his fundamental theorem for the circle (Figure 6): "Any two tangents A, A_1 are projective with respect to the corresponding pairs of points in which they are cut by the other tangents;and the point of intersection , e_1, of the tangents corresponds to the points ;$_1$, e, where they touch the circle." He also gave the dual of this theorem (Figure 7): "Any two points , $_1$ of a circle are the centers of two projective pencils of lines, the corresponding lines of which intersect in the remaining points of the circle; and the reciprocal tangents d_1, e at the points ,$_1$ correspond to the

lines d, e_1." Applying these theorems to the second degree cone, Steiner obtained the following result: "Any two tangents of a conic section are projective with respect to the pair of points in which they are intersected by the remaining tangents; and conversely."

Steiner emphasized here that "these new theorems on the second–degree cone and its sections are more important for the investigation of these figures than all the theorems previously known about them, for they are, in the strict sense, the true fundamental theorems."

From these fundamental theorems, Steiner derived consequences ranging from the known theorems on conic sections to the Braikenridge–Maclaurin theorem. Propositions 49–53 deal with the production of projective figures in space. An important group of propositions (54–58) contains previously known "composite theorems and problems" that Steiner was the first to derive in a uniform manner from one basic principle. An example is the following problem taken from Möbius: "Given an arbitrary tetrahedron, draw another the vertices of which lie in the faces of the first and the faces of which pass through the vertices of the

first, two vertices of the second tetrahedron being given." Proposition 59, labeled "general observation," contains the "skew projection," a quadratic relationship in space, sometimes called the "Steiner relationship," which had been noted by Poncelet.[24]

The eighty-five "Problems and Theorems" that Steiner appended in a supplement proved especially stimulating to a generation of geometers. They are discussed in the dissertation of Ahmed Karam,[25] who found that, as of 1939, only three problems remained unsolved: no. 70, "What are the properties of a group of similar quadratic surfaces that pass through four or five points of space?": no. 77, "Does a convex polyhedron always have a topological equivalent that can be either circumscribed about or inscribed in a sphere?"; and no. 76, "If polyhedra are distinguished solely according to their boundary surfaces, there exist only one with four faces, two with five faces, and seven with six faces. How many different bodies are possible with 7, 8, . . . , *n* faces?"

The last problem was posed by Steiner's teacher at Heidelberg, Ferdinand Schweins.[26] It was partially solved by Otto Hermes in 1903, and further elements of it have been solved by P. J. Frederico.[27]

Of Steiner's work Jacobi stated:

Starting from a few spatial properties Steiner attempted, by means of a simple schema, to attain a comprehensive view of the multitude of geometric theorems that had been rent asunder. He sought to assign each its special position in relation to the others, to bring order to chaos, to interlock all the parts according to nature, and to assemble them into well-defined groups. In discovering the organism [*Organismus*] through which the most varied phenomena of space are linked, he not only furthered the development of a geometric synthesis; he also provided a model of a complete method and execution for all other branches of mathematics.[28]

Only a year after the appearance of *Systematische Entwicklung*, Steiner published his second book: *Die geometrischen Konstruktionen ausgeführt mittelst der geraden Linie und eines festen Kreises, als Lehrgegenstand auf höheren Unterrichtsanstalten und zur praktischen Benützung* (1833).[29] He took as his point of departure Mascheroni's remark that all constructions made with straightedge and compass can be carried out using compass alone. As a counterpart to this statement, Steiner proved that all such constructions can also be carried out with the straightedge and one fixed circle. To this end he devoted the first chapter to rectilinear figures and especially to teh harmonic properties of the complete quadrilateral. In the third chapter he proved his assertion in a way that enabled him to solve eight fundamental problems, to which all others can be reduced. For example (no. 1): "Draw the parallel to a straight line through a point" and (no. 8) "Find the intersection point of two circles." In the intervening chapter 2, he considered centers of similitude and the radical axis of a pencil of circles. This work, which enjoyed great success, contained an appendix of twenty-one problems that were partly taken from *Systematische Entwicklung*. The first one, for example, was "Given two arbitrary triangles, find a third that is simultaneously circumscribed about the first and inscribed in the second."

At this point we shall present a survey of Steiner's further research, published in volume II of the *Gesammelte Werke* (the page numbers in parentheses refer to that volume). A fuller description of its contents can be found in Louis Kollros' article on Steiner, cited in the bibliography.

Steiner pursued the investigation of conic sections and surfaces in some dozen further publications. Sometimes he merely presented problems and theorems without solutions or proofs. In part the material follows from the general projective approach to geometry; but some of it contains new ideas, as the examination of the extreme-value problems: "Determine an ellipse of greatest surface that is inscribed in a given quadrangle" (333 f.) and "Among all the quadrangles inscribed in an ellipse, that having the gereatest perimeter is the one the vertices of which lie in the tangent points of the sides of a rectangle circumscribed about the ellipse. There are infinitely many such quadrangles. . . . All have the same perimeter, which is equal to twice the diagonal of the rectangle. All these quadrangles of greatest perimeter are parallelograms; and they are, simultaneously, circumscribed about another ellipse the axes of which fall on the corresponding axes of the given ellipse and that is confocal with the latter. . . . Among all the quadrangles circumscribed about a given ellipse, the one with least perimeter is that in which the normals at the tangent points of its sides form a rhombus" (411–412).

In a paper on new methods of determining second-order curves, Steiner considered pairs of such curves and demonstrated propositions of the following type: "If two arbitrary conic sections are inscribed in a complete quadrilateral, then the eight points in which they are tangent to he sides lie on another conic section" (477 f.). To the theory of second-degree surfaces, he contributed the geometric proof of Poisson's theorem: The attraction of a homogeneous elliptical sheet falls on a point *P* in the axis of the cone that has *P* as vertex and is tangent to the ellipsoid.

Steiner dealt on several occasions with center-of-gravity problems. One of the simplest follows.

If, in a given circle *ADBE*, one takes an arc *AB*, of which one end point, *A*, is fixed, and lets it increase steadily from zero, then its center of gravity *C* will describe a curved line ACM. What properties will this barycentric curve possess? . . . The same question can be phrased generally, if instead of the circle an arbitrary curve is given. . . . Questions like those of the above problem occur if one considers the center of gravity of the segment (instead of the arc) *ABC*. Other questions of the same sort arise regarding the center of gravity of a variable sector *AMB*, if *M* is an arbitrary fixed pole and one arm of the sector is fixed, while the other, *MB*, turns about the pole *M* [p. 30; see Figure 8].

Steiner developed a general theory of the center of gravity of mass points in "Von dem krümmungsschwerpunkt ebener Kurven" (97–159). It led to the pedal curve of a given curve and its area, and was followed by the important memoir "Parallele Flächen" (171–176), which generalized a theorem proved in the preceding paper for plane curves: Let *A* and *B* be two parallel polyhedra (surfaces) separated by a distance *h*. Then it is true for *A* and *B* that $B = A + hk + h^2e$; and for the volume *I* between *A* and *B* that $I = hA + h^2k/2 + h^3e/3$, where *k* is "the sum of the edge curvature" and *e* "the sum of the vertex curvature."

Steiner's great two-part paper "Ueber Maximum und Minimum bei den Figuren in der Ebene, auf der Kugelfläche und im Raume überhaupt" (177–308, with 36 figures) was written in Paris during the winter of 1840–1841. It shows the tremendous achievements of which he was still capable–given the necessary time and freedom from distractions. His basic theorem states: "Among all plane figures of the same perimeter, the circle has the greatest area (and conversely)." He gives five ways of demonstrating it, in all of which he assumes the existence of the extremum. All five, moreover, are based on the inequalities of the triangle and of polygons. The first proof proceeds indirectly: Assume that among all figures of equal perimeter the convex figure *EFGH* possesses the greatest area and that it is not a circle (Figure 9).

Let *A* and *B* be two points that bisect the perimeter. Then the surfaces *AEFB* and *AHGB* are equal. For if one of them were smaller, then the other could be substituted for it, whereby the perimeter would remain equal and the surface would be increased. These two surfaces should be considered to have the same form; for if they were different, the mirror image of *AB* could be substituted for one, whereby the perimeter and area would remain the same. According to "Fundamental Theorem II" on triangles, for the extremal figure the angles at *C* and *D* must be right angles. Since this consideration holds for every point *A* of the perimeter, the figure sought is a circle. This is the first occasion on which Steiner employed his principle of symmetrization.

In the fifth proof the basic theorem takes the form that among all plane figures of the same area, the circle has the least perimeter. The principle of the proof is again symmertrization with respect to the axis *X*.

Steiner next effects the transformation of the pentagon *ABCDE* (Figure 10) into the pentagon of equal area *abcde* in such a way that each line segment $B_1B = b_1b$; and $b\ b_1$ is bisected by *X*. As a result, the perimeter decreases. Steiner then turns

to the extremal properties of prisms, pyramids, and the sphere (269–308).

In 1853, while investigating the double tangents of a fourth-degree curve, Steiner encountered a combinational problem (435–438): What number *N* of elements has the property that the elements can be ordered into triplets (*t*-tuples) in such a way that each two (each *t*-1) appear in one and only one combination? For the Steiner triple system, *N* must have the form 6*n* + 1 or 6*n* + 3, and there exist triples, quadruples, and so on. For example, for *N* = 7 there is only one triple system: 123, 145, 167, 246, 257, 347, 356. For *N* = 13 there are two different triple systems. Steiner was unaware of the work on this topic done by Thomas Kirkman (1847).[30]

In a short paper of fundamental importance, published in 1848 and entitled "Allgemeine Eigenschaften der algebraischen Kurven" (493–500), Steiner first defined and examined the various polar curves of a point with respect to a given curve. He then introduced the "Steiner curves" and discussed tangents at points of inflection, double tangents, cusps, and double points. In particular he indicated the resulting relationships for the twenty-eight double tangents of the fourth-degree curve. Luigi Cremona proved the results and coninued Steiner's work in his "Introduzione ad una teoria geometrica delle curve piane."[31]

The desire to find, with the methods of pure geometry, the proofs of the extremely important theorems stated by the celebrated Steiner in his brief memoir "Allgemeine Eigenschaften der algebraischen Kurven" led me to undertake several studies, a sample of which I present here, even though it is incomplete.

In 1851 Steiner wrote "Über solche algebraische Kurven, die einen Mittelpunkt haben . . ." (501–596), a version of which he published in Crelle's *Journal*. In the *Gesammelte Werke* this paper is followed by "Problems and Theorems" (597–601). Steiner's results include the following example: "Through seven given points in the plane there pass, in general, only nine third-degree curves possessing a midpoint." Next follows a discussion of the twenty-eight double tangents of the fourth-degree curve (603–615). In January 1856 Steiner delivered a lecture at the Berlin Academy, "Ueber die Flächen dritten Grades" (649–659), in which he offered four methods of producing these cubic surfaces. The first states: "The nine straight lines in which the surfaces of two arbitrarily given trihedra intersect each other determine, together with one given point, a cubic surface." Aware that Cayley already knew the twenty-seven straight lines of this surface, Steiner introduced the concept of the "nuclear surface" and investigated its properties (656).

Steiner's correspondence with Ludwig Schläfli reveals that the latter discovered his "Doppelsechs" in the course of research on this topic undertaken at Steiner's request.[32] Again, it was Cremona who proved Steiner's theorems in his "Mémoire de géométrie pure sur les surfaces dutroisième ordre" (1866).[33] Cremona began his memoir by declaring that "This work . . . contins the demonstration of all the theorems stated by this great geometer [Steiner] in his memoir *Ueber die Flächen dritten Grades*."

A second treatment of Steiner's theorems appeared in Rudolf Sturm's *Synthetische Untersuchungen über Flächen dritter Ordnung* (Leipzig 1867). In the preface Sturm wrote: "Steiner's paper contains a wealth of theorems on cubic surfaces, although, as had become customary with this celebrated geometer, without any proofs and with only few hints of how they might be arrived at." The works of both Cremona and Sturm were submitted in 1866 as entries in the first competition held by the Berlin Academy for the Steiner Prize, which was divided between them.

During his stay at Rome in 1844, Steiner investigated a fourth-order surface of the third class (721–724, 741–742), but it became known only much later through a communication from Karl Weierstrass (1863).[34] The surface, since called the Roman surface or Steiner surface, has the charateristic property that each of its tangent planes cuts it in a pair of conics. On the surfaces there lie three double straight lines that intersect in a triple point of the surface. The surface was the subject of many studies by later mathematicians.

NOTES

1. See F. Bützberger, "Biographie Jakob Steiners".

2. J. Lange, "Jacob Steiners Lebensjahre in Berlin 1821–1863. Nach seinen Personalakten dargestellt"; E. Jahnke, "Schreiben Jacobis . . . ," in *Archiv der Mathematik und Physik*, 3rd ser., **4** (1903), 278.

3.*Jacob Steiner's Gesammelte Werke*, K. Weierstrass, ed., I, 175–176, and II, 18.

4.*Ibid*., II, 97–159.

5. E. Lampe, "Jakob Steiner": J. H. Graf, "Beiträge zur Biographie Jakob Steiners."

6. K.-R. Biermann, "Jakob Steiner."

7.*Ibid*.

8. J. Lange, *op. cit*.

9.*Gesammelte Werke*, I, 233–234.

10. Steiner, *Allgemeine Theorie über das Berühren und Schneiden der Kreise und der Kugeln*; B. Jegher, "Von Kreisen, die einerlei Kugelfläche liegen. Jakob Steiners Untersuchungen über das Schneiden und Berühren von Kegelkreisen. . . ."

11.*Gesammelte Werke*, I, 17–76.

12. Steiner, *Allgemeine Theorie*.

13.*Ibid*., xiv-xvi.

14.*Gesammelte Werke*, I, 17–76.

15. See F. Bützburger, "Jakob Steiners Nachlass aus den Jahren 1823–1826," § 11, "Die Erfindung der Inversion"; A. Emch, "The Discovery of Inversion"; and Mautz, *op. cit*.

16.*Gesammelte Werke*, I, 133.

17.*Ibid*., 77–94.

18.*Ibid*., 121–228.

19.*Ibid*., 229–460.

20. The MS material is in Steiner, *Allgemeine Theorie*, and in Jegher, *op, cit*.

21. See G. Cantor *Gesammelte Abhandlungen* (Berlin, 1932), 151.

22. See W. L. Schaaf, "Mathematicians and Mathematics on Postage Stamps," in *Journal of Recreational Mathematics*, I (1968), 208; and I. Newton, *Principia mathematica*, 2nd ed. (Cambridge, 1713), Bk. I, 72; and *Universal Arithmetick* (London, 1728), probs. 57, 95.

23.*Vorlesungen*, II, no. 35.

24. V. Poncelet, *Traité des propriétés projectives des figures* (Paris–Metz, 1822), sec. III, ch. 2.

25. A. Karam, *Sur les 85 problèmes de la "dépendance systématique" de Steiner*. See also L. Kollros, "Jakob steiner," p. 10.

26. F. Schweins, *Skizze eines Systems der Geometrie* (Heidelberg, 1810), 14–15.

27. See R. Sturm, "Zusammenstellung von Arbeiten, welche sich mit Steinerschen Aufgaben beschäftigen," in *Bibliotheca mathematica*, 3rd ser., **4** (1903), 160–184; P. J. Frederico, "Enumeration of Polyhedra," in *Journal of Combinatorial Theory*, **7** (1969), 155–161.

28. See F. Bützberger, "Biographie Jakob Steiners," 109; and K.-R. Biermann, *op, cit*., 38.

29.*Gesammelte Weke*, I, 469–522.

30. T. Kirkman, in *Cambridge and Dublin Mathematical Journal*. **2** (1847), 191–204.

31.*Opere matematiche di Luigi Cremona*, I (Milan, 1914), 313–466.

32. See J. H. Graf, *Der, Briefwechsel Steiner-Schläfli*.

33.*Opere matematiche di Luigi Cremona*, III (Milan, 1917), 1–121.

34. In *Monatsberichte der Deutschen Akademie der Wissenschaften zu Berlin* (1863), 339, repr. in *Mathematische werke von Karl Weierstrass*, III (Berlin, 1902), 179–182.

BIBLIOGRAPHY

I. Original Works. Many of Steiner's writings are in *Jacob Steiner's Gesammelte Werke*, K. Weierstrass, ed., 2 vols. (Berlin, 1881–1882). The major ones include *Jacob Steiners Vorlesungen über synthetische Geometrie: I, Die Theorie der Kegelschnitte in elementarer Darstellung*, C. F. Geiser, ed. (Leipzig, 1867; 3rd ed., 1887), and II, *Die Theorie der Kegelschnitte gestützt auf projektive Eigenschaften*, H. Schröter, ed. (Leipzig, 1867; 3rd ed., 1898); *Allgemeine Theorie über das Berühren und Schneiden der Kreise und der Kugeln*, R. Fueter and F. Gonseth, eds. (Zurich–Leipzig, 1931); and Barbara Jegher, "Von Kreisen, die in einerlei Kugelfläche liegen. Jakob Steiners Untersuchungen über das Schneiden und Berühren von Kugelkreisen . . . ," in *Mitteilungen der Naturforschenden Gesellschaft in Bern*, n.s. **24** (1967), 1–20.

Two letters from Steiner to Rudolf Wolf, dated 25 July 1841 and 5 Aug. 1848, are in the autograph collection of the Schweizerische Naturforschende Gesellschaft, at the Bern Burgerbibliothek (nos. 110 and 588 under MSS Hist. Helv. XIV, 150).

The following works by Steiner appeared in Ostwald's Klassiker der Exakten Wissenschaften: *Die geometrischen Konstruktionen, ausgeführt mittels der geraden Linie und eines festen Kreises* (1833), no. 60 (Leipzig, 1895), which contains a short biography of Steiner by the editor, A. J. von Oettingen, pp. 81–84; *Systematische Entwicklung der Abhägigkeit geometrischer Gestalten voneinander* (1832), nos. 82–83 (Leipzig, 1896); and *Einige geometrische Betrachtungen* (1826), no. 123, R. Sturm, ed. (Leipzig, 1901).

II. Secondary Literature. Undated MS material is F. Bützberger, "Kleine Biographie über Jakob Steiner," Bibliothek der Schweizerischen Naturforschenden Gesellschaft, in the Bern Stadt und Universitätsbibliothek, MSS Hist. Helv. XXIb, 347; "Biographie Jakob Steiners," in the same collaction MSS Hist. Helv. XXIb, 348; and "Jakob Steiners Nachlass aus den Jahren 1823–1826," Bibliothek der Eidgenössischen Technischen Hochschule, Zurich, Hs. 92, pp. 30–223.

On Steiner's youth and years in Yverdon, see especially F Bützberger, "Zum 100. Geburtstage Jakob Steiners," in *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, **27** (1896), 161 ff.; on the years in Berlin see Felix Eberty, *Jugenderinnerungeneiner alten Berliners* (Berlin, 1878; repr. 1925), 238–243; and Julius Lange, "Jacob Steiners Lebensjahre in Berlin 1821–1863. Nach seinen Personalakten dargestellt," in *Wissenschaftliche Beilage zum Jahresbericht der Friedrichs-Werderschen Oberrealschule zu Berlin. Ostern 1899*, Program no. 116 (Berlin, 1899). See also Three short obituary notices: C. F. Geiser, "Nekrolog. J. Steiner," in *Die Schweiz: Illustrierte Zeitschrift für Literatur und Kunst* (Nov. 1863), 350–355; Otto Hesse, "Jakob Steiner," in *journal für die reine und angewandte Mathematik*, **62** 199–200; and Bernhard Wyss, "Nekrolog J. Steiner," in *Bund* (Bern) (9 Apr. 1863).

The first detailed biography of Steiner was written by his grandnephew: Carl Friedrich Geiser. *Zur Erinnerung an Jakob Steiner* (Schaffhausen, 1874). Steiner's correspondence with Schläfli was edited by Schläfli's student J. H. Graf: *Der Briefwechsel Steiner-Schläfli* (Bern, 1896); see also the following three works by Graf: *Der Mathematiker Steiner von Utzensdorf* (Bern,1897); "Die Exhuminierung Jakob Steiners und die Einweihung des Grabdenkmals Ludwig Schläflis . . . am 18. März 1896," in *Mitteilungen der Naturforschenden Gesellschaft in Bern*(1897), 8–24; and "Beiträge zur Biographie Jakob Steiners," *ibid*. (1905). Another of Schläfli's students, F. Bützberger, examined Steiner's posthumous MSS and reported on them in "Zum 100. Geburtstage Jakob Steiners" (see above) and in his long MS "Jakob Steiners Nachlass aus den Jahren 1823–1826" (see above).

Recent accounts of Steiner's life and work are Louis Kollros, "Jakob Steiner," supp.2 of *Elemente der Mathematik* (1947), 1–24; and J-P. Sydler, "Apercus sur la vie et sur l'oeuvre de Jakob Steiner," in *Enseignement mathématique*, 2nd ser., **11** (1965), 240–257. Valuable corrections of erros in earlier accounts are given by Kurt-R. Biermann, "Jakob Steiner, in *Nova acta Leopoldina*, n.s. **27** , no,167 (1963), 31–47.

For further information see the following: F. Bützberger, "Jakob Steiner bei Pestalozzi in Yverdon," in *Schweizerische pädagogische Zeitschrift*. **6** (1896), 19–30; and *Bizentrische Polygone, Steinersche Kreis-und Kugelreihen und die Erfindung der Inversion* (Leipzig, 1913): Moritz Cantor, "Jalob Steiner," in *Allgemeine deutsche Biographie*, XXXV (1893), 700–703; A Ecmh, "Unpublished Steiner Manuscripts," in *American Mathematical Monthly*, **36** (1929), 273–275; and "The Discovery of Inversion," in *Bulletin of the American Mathematical Society*, **20** (1913–1914), 412–415, and **21** (1914–1915), 206; R. Fueter, *Grosse schweizer Forscher* (zurich, 1939), 202–203; C. Hanicht, "Die Steinerschen Kreisreihen" (Ph.D.diss., Bern, 1904); A Karam, *Sur, les 85 problèmes de la "dépendence systématique" de Steiner* (Ph.D. diss., Eidgenössische Technische Hochschule, Zurich, 1939): E. Kötter, "Die Entwicklung der synthetischen Geometrie. Dritter Teil: Von Steiner bis auf Stadut," in *Jahresbericht der Deutschen Mathematikervereiningung***5** no. 2(1898), 252 ff.; Emil Lampe, "Jakob Steiner," in *Bibliotheca mathematica*, 3rd ser., I (1900), 129–141; Otto Mautz "Ebene Inversionsgeometrie," in *Wissenschafiliche Beilage zum Bericht ¨ber das Gymnasium Schuljahr* 1908–1909 (Basel, 1909): R. Sturm, "Zusammenstellung von Arbeiten, welche sich mit Steinerschen Aufgaben beschäftigen" in *Bibliotheca mathematica*, 3rd ser., **4** (1903), 160–184; and a series of articles by Rudolf Wolf in *Vierteljahrschrift der Naturforschenden Gesellschaft in Z¨rich;***9** (1864), 145ff.; **13** (1868), 110 ff.; **19** (1874). 325 ff.; **25** (1880), 215 ff.; and **35** (1890), 428 ff.

Johann Jakob Burckhardt