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(*b*. Zwolle, Netherlands, 29 December 1856; *d*. Toulouse, France, 31 December 1894)

mathematics.

To the majority of mathematicians, Stieltjes' name is remembered in association with the Stieltjes integral,

a generalization of the ordinary Riemann integral with wide applications in physics. Yet in his own day, he was renowned as a versatile mathematician whose publications include papers in almost every area of analysis. He is the father of the analytic theory of continued fractions, and his integral was developed as a tool for its study.

Stieltjes, the son of a distinguished Dutch civil engineer, received his principal schooling at the École Polytechnique in Delft. He left the École in 1877 to take up a post at the observatory in Leiden, where he served six years. Evidently he kept up his mathematical studies, since he left Leiden to accept a chair in mathematics at the University of Groningen. Honors came to Stieltjes early. In 1884 the University of Leiden awarded him an honorary doctorate, and in 1885 he was elected to membership in the Royal Academy of Sciences of Amsterdam. Disappointed at Groningen, Stieltjes went to live in Paris, where he received his doctorate of science in 1886. In the same year he was appointed to the faculty at the University of Toulouse, where he remained until his death eight years later. Although not elected to the Academy of Sciences in Paris, Stieltjes was considered for membership in 1892 and won its Ormoy Prize in 1893 for his work on continued fractions.

Stieltjes' published work encompasses almost all of analysis of his time. He made contributions to the theory of ordinary and partial differential equations, studied gamma functions and elliptical functions, and worked in interpolation theory. His thesis was on asymptotic series. A special and increasing interest was the evaluation of particular integrals such as

or

and series of the general form which arise in a natural way from such integrals. These series also occur in the study of continued fractions, and this may have led Stieltjes to analytic continued fraction theory.

His first paper on continued fractions, published in 1884, proves the convergence of

in the slit complex z-plane excluding the interval (–1,1), with use of the series in decreasing powers of z. This convergence, which is locally uniform, was established by transforming the fraction into a definite integral.

Yet Stieltjes' monument is his last memoir, "Recherches sur les fractions continues," written just before he died, and published in two parts (*Annales de la Faculté des sciences de l'Université de Toulouse pour les sciences mathématiques et physiques*, **1** , ser. 1 [1894], 1–122; **1** , ser. 9 [1895], 5–47), the second posthumously. In it he polished and refined all of his previous work on the subject, and here is the first appearance of his integral. The memoir is a beautiful piece of mathematical writing–clear, self-contained, almost lyric in its style.

In this paper the fraction

is considered. The a_i's are assumed to be known real positive quantities, and z is a complex variable. Fraction (1) will be said to converge, or otherwise, according to the convergence, or otherwise, according to the convergence or not of the sequence of "approximates" $P_n(z)/Q_n(z)$. Each approximate is the rational function formed by considering only the first n terms of (1) and simplifying the resulting compound fraction. Thus, $Q_{2n}(z)$ and $P_{2n+1}(z)$ are polynomials of degree n in z, while P_{2n} is of degree n - 1. and Q_{2n+1} is of degree n + 1.

Stieltjes began by studying the roots of the polynomials $P_n(z)Q_n(z)$, which are all real. He proved a whole series of theorems concerning the interlacing of their roots: for example, the roots of $Q_{2n}(z)$ separate the roots of $Q_{2n-2}(z)$. This was then used to prove that the roots of $P_n(z)$ and $Q_n(z)$ ($n = 1, 2, \ldots$) are all nonpositive and distinct. Thus, the approximates have the following partial fraction decomposition:

where in (2) $\{x_1, x_2, \ldots, x_n\}$ are the (positive) roots of $Q_{2n}(-z)$ and in (3) $\{0, y_1, y_2, \ldots, y_n\}$ are the roots of $Q_{2n+1}(-z)$.

Next Stieltjes was able to show that

where the $\{c_k : k = 1, 2, \ldots, n\}$ depend only upon the original fraction (1) and not upon n. Formula (4) led to the definition of the development of (1) in decreasing powers of z:

The c_k are all real and positive, and

Either the sequences of ratios is unbounded, in which case (5) diverges for all z, or (6) is bounded, in which case there is a $\lambda > 0$ with the property that (5) converges for all z satisfying $|z| > \lambda$. Stieltjes then proved that, in the latter case, if the (necessarily positive) roots of $Q_n(-z)$ are ordered according to size, and if the largest is, say, $x_{n,k}$, then .

oscillation of (1). Here Stieltjes showed that for all z with positive real part,

and

Furthermore, for z real ($=x$), $F(x)$ and $F_1(x)$ are real, and $F_1(x) \geq F(x)$: Equality in the right half plane, including the positive real axis, was proved to hold if and only if the series

formed from the terms of fraction (1) is divergent. (Recall that the a_k are all positive.) Also, since the convergence of (7) and (8) is locally uniform, the functions $F(z)$ and $F_1(z)$ are analytic in the right half plane. Thus, to sum up, Stieltjes had shown that the continued fraction (1) was convergent when and only when the series (9) diverged; otherwise the fraction oscillated. The remaining problem was to extend this result to the z in the left half plane, except for certain points on the negative real axis.

To this end, Stieltjes showed that the limits

all exist, that p, q, p_1, q_1 are all analytic, and that

$p_1(z)q(z) - p(z)q_1(z) \equiv 1.$

Then, again he supposed that the roots $\{x_{n,1}, x_{n,2}, \ldots, x_{n,n}\}$ of $Q_{2n}(-z)$ were ordered according to increasing size for each n. Stieltjes then proved that that $\{\lambda_k; k = 1, 2, \ldots\}$ are all distinct real and positive, and that the λ_k are the only zeros of $q(z)$. Similar results hold for $q_1(z)$, $p(z)$, and $p_1(z)$. Next, μ_k is defined by

and it was proved that where the M_k are from expression (2). Furthermore,

is meromorphic in the plane, $S(z) = F(z)$ (see [8]), and finally that for each i

where the c_i's are from (5).

In precisely the same way, an infinite set of pairs of positive real numbers similar to $\{(\mu_k, \lambda_k)\}$ above was associated with the sequence $\{P_{2n+1}(z)/Q_{2n+1}(z)\}$. In particular $F_1(z)$ was shown to be meromorphic, and if ; see [3]), and if then

and

In this way he established the analyticity of $F_1(z)$ and $F(z)$ in the slit plane.

Observe that the above systems (12) (or [13]) can be considered as the equations of the moments of all orders of a system of masses μ_k (or v_k) placed at a distance λ_k (or θ_k) from the origin, and that in either case the i^{th} moment is c_i. Of course, if Σa_k is divergent, $\nu v_k = \mu_k$ and $\lambda_k = \theta_k$ for all k, since $F(z)$ and $F_1(z)$ are the same function. But if Σa_k is convergent, the equalities do not hold, even though the c_i are the same in each case.

The further study of the nature of $F_1(z)$ and $F(z)$ in more detail led Stieltjes to the "moment problem": that is, to find a distribution of mass (an infinite set of ordered pairs of positive numbers) whose moments of all orders are known. If this problem can be solved, then $F_1(z)$ and $F(z)$ will be known, since the c_i's can be calculated from the a_k's of fraction (1). However, it is immediately evident that if Σa_k is convergent, there can be no unique solution, as there are at least two. But Stieltjes was able to show (later on) that if Σa_k diverges, there is a unique solution.

It was to solve the moment problem that Stieltjes introduced his integral. First he considered an increasing real-valued function ϕ defined on the positive real axis, and gave a lengthy discussion of onesided limits. For example, he showed that ϕ is continuous at x, if and only if $\phi^+(x) = \phi^-(x)$. Only bounded functions with countably many discontinuities on the positive axis were considered. Next he supposed that ϕ was a step function, with $\phi(0) = 0$. Then a finite mass condensed at each point of discontinuity can be given by $\phi^+(x) - \phi^-(x)$, and $\phi(b) - \phi(a)$ is the total mass between a and b; in particular $\phi(x)$ is the total mass between x and the origin. Also, changing the value of ϕ at a point of discontinuity does not change the associated mass distribution there.

Stieltjes then defined the integral

to be the limit, as max $(x_{i+1} - x_i) \to 0$ of

where $a = x_0 < x_i < \ldots < x_n = b$ and $x_{i-1} \le \zeta_i \le x_i$. Stieltjes then established the formula for integration by parts

defined the improper integral in the usual way, and established many properties of the integral.

Next he considered the function ϕ_n defined from the even-order approximates by

$\phi_n(0) = 0 \le u < x_1$

Where the M_i and the x_k are defined as in (2), and where a_1 is the first term of (1). After a lengthy discussion of "lim inf:" and "lim sup" (the ideas were new then), he defined, for each u:$\psi(u)$ = lim sup $\phi_n(u)$, $\chi(u)$ = lim inf $\phi_n(u)$, and $\Phi(u) =$, $(\psi(u) + \chi(u))$.Φ was shown to have the property that

and also that the distribution of mass represented by Φ solved the moment problem, since.

A function Φ_1 (u) with similar properties was constructed from the odd-order approximates. He also undertook the study of the inverse problem; that is, given an increasing function $\phi(u)$, with $\phi(0) = 0$, then, by setting

a fraction like (1) can be determined with the property that

Stieltjes' paper, of which only a portion has been summarized here, is a mathematical milestone. The work represents the first general treatment of continued fractions as part of complex analytic function theory; previously, only special cases had been considered. Moreover, it is clearly in the historical line that led to Hilbert spaces and their generalizations. In addition, Stieltjes gave a sort of respectability to discontinuous functions and, together with some earlier work, to divergent series. All together these were astonishing accomplishments for a man who died just two days after his thirty-eighth birthday.

BIBLIOGRAPHY

All Stieltjes' published papers, with some letters, notes, and incomplete works found after his death, are in *Oeuvres complétes de Thomas Jan Stieltjes* (Groningen, 1914–1918). An annotated bibliography of his published works appears with his obituary in *Annales de la Facultè des sciences de l'Université de Toulouse pour les sciences mathématiques et physiques*, **1** , ser. 1 (1895), 1–64.

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