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(*b*. Budapest, Hungar, 28 December 1903; *d*. Washington, D.C., 8 February 1957), *Mathematics, mathematical physics*.

Von Neumann, the eldest of three sons of Max von Neumann, a well-to-do Jewish banker, was privately educated until he entered the Gymnasium in 1914. His unusual mathematical abilities soon came to the attention of his teachers, who pointed out to his father that teaching him conventional school mathematics would be a waste of time; he was therefore tutored in mathematics under the guidance of university professors, and by the age of nineteen he was already recognized as a professional mathematician and had published his first paper. Von Neumann was *Privatdozent* at Berlin from 1927 to 1929 and at Hamburg in 1929-1930, then went to Princeton University for three years; in 1933 he was invited to join the newly opened Institute for Advanced Study, of which he was the youngest permanent member at that time. At the outbreak of World War II, von Neumann was called upon to participate in various scientific projects related to the war effort; in particular, from 1943 he was a consultant on the construction of the atomic bomb at Los Alamos. After the war he retained his membership in numerous government boards and committees, and in 1954 he became a member of the Atomic Energy Commission. His health began to fail in 1955, and he died of cancer two years later.

Von Neumann may have been the last representative of a once-flourishing and numerous group, the great mathematicians who were equally at home in pure and applied mathematics and who throughout their careers manintained a steady production in both directions. Pure and applied mathematics have now become so vast and complex that mastering both seems beyond human capabilities. In von Neumann's generation his ability to absorb and digest an enormous amount of extremely diverse material in a short time was exceptional; and in a profession where quick minds are somewhat commonplace, his amazing rapidity was proverbial. There is hardly a single important part of the mathematics of the 1930's with which he had not at least a passing acquaintance, and the same is probably true of theoretical physics.

Despite his encyclopedic background, con Neumann's work in pure mathematics had a definitely smaller range than that of Poincaré or Hilbert, or even of H. Weyl. His genius lay in analysis and combinatorics, the latter being understood in a very wide sense, including an uncommon ability to organize and axiomatize complex situations that a priori do not seem amenable to mathematical treatment, as in quantum mechanics and the theory of games. As an analyst von Neumann does not be long to the classical school represented by the French and English mathematicians of the early 1900's but, rather, to the tradition of Hilbert, Weyl, and F. Riesz, in which analysis, while being as "hard" as any classical theory, is based on extensive foundations of linear algebra and general topology; however, he never did significant work in number theory, algebraic topology, algebraic geometry, or differential gaeometry. It is only in comparison with the greatest mathematical geniuses of history that von Neumann's scope in pure mathematics may appear somewhat restricted; it was far beyond the range of most of his contemporaries, and his extraordinary work in applied mathematics, in which he certainly equals Gauss, Cauchy, or Poincaré, more than compensates for its limitations.

Pure Mathematics . Von Neumann's work in pure mathematics was accomplished between 1925 and 1940, which might be called his *Sturm und Drange* period, when he seemed to be advancing at a breathless speed on all fronts of logic and analysis at once, not to speak of mathematical physics. This work, omitting a few minor papers, can be classified under five main topics.

Logic and Set Theory. Von Neumann's interest in set theory arose very early; in his second paper (1923) he gave an elegant new definition of ordinal numbers, and in the third (1925) he introduced an axiomatic system for set theory quite different from the one proposed by Zermelo and Fraenkel (it was later adopted by Gödel in his research on the continuum hypothesis). In the late 1920's von Neumann also participated in the Hilbert program of metamathematics and published a few papers on proofs of noncontradiction for parts of arithmetic, before Gödel shattered the hopes for a better result.

Measure Theory. Although it was not in the center of von Neumann's preoccupation, he made several valuable contributions to measure theory. His knowldege of group theory enabled him to "explain" the Hausdorff-Banach-Tarski "paradox," in which two balls of different radii in $\mathbf{R}^n$ ($n \geq 3$) are decomposed into a finite number of (nonmeasurable) subsets that are pairwise congruent (such decompositions cannot exist for $n = 1$ or $n = 2$); he showed that $n = 1$ or $n = 2$ is impossible because the orthogonal group in three or more variables contains free non-Abelian groups, where as it does not for $n \leq 2$.

Another highly ingenious paper established the existence of an algebra of bounded measurable functions on the real line that forms a complete system of representatives of the classes of almost everywhere-equal measurable bounded functions (each class contains one, and only one, function of the algebra). This theorem, later generalized to arbitrary measure spaces by Dorothy Maharam, holds the key to the "disintegration" process of measures (corresponding to the classical notion of “conditional probability”). It is a curious coincidence that in “Operator Methods in Classical Mechanics” von Neumann was the first to prove, by a completely different method, the existence of such disintegrations for fairly general types of measures.

On the borderline between this group of papers and the next lies von Neumann’s basic work on Haar’s measure, which he proved to be unique up to a constant factor; the first proof was valid only for compact groups and used his direct definition of the “mean” of a continuous function over such a group. The extension of that idea to more general groups was the starting point of his subsequent papers, some written in collaboration with Solomon Bochner, on almost-periodic functions ongroups.

Lie Groups. One of the highlights of von Neumann’s career was his 1933 paper solving Hilbert’s “fifth problem” for compact groups, proving that such a group admits a Lie group structure once it is locally homeomorphic with Euclidean space. He had discovered the basic idea behind that paper six years earlier: the fact that closed subgroups of the general linear group are in fact Lie groups. The method of proof of that result was shown a little later by E. Cartan to apply as well to closed subgroups of arbitrary Lie groups.

Spectral Theory of Operators in Hilbert Space. This topic is by far the dominant theme in Von Neumann’s work. For twenty years he was the undisputed master in this area, which contains what is now considered his most profound and most original creation, the theory of rings of operators. The first papers (1927) in which Hilbert space theory appears are those on the foundations of quantum mechanics (see below). These investigations later led von Neumann to a systematic study of unbounded hermitian operators, which previously had been considered only in a few special cases by Weyl and T. Carleman. His papers on unbounded hermitian operators have not been improved upon since their publication, yet within a few years he realized that the traditional idea of representing an operator by an infinite matrix was totally inadequate, and discovered the topological devices that were to replace it: the use of the graph of an unbounded operator and the extension to such an operator of the classical “Cayley transform,” which reduced the structure of a self-adjoint operator to that of a unitary operator (known since Hilbert). At the same time this work led him to discover the defects of a general, densely defined hermitian operator, which later were seen to correspond to the “boundary conditions” for operators stemming from differential and partial differential equations.

The same group of papers includes another famous result from von Neumann’s early years, his proof in 1932 of the ergodic theorem in its “L^2 formulation” given by B.O. Koopman a few months earlier. With G. D. Birkhoff’s almost simultaneous proof of the sharper “almost every where” formulation of the theorem, von Neumann’s results were to form the starting point of all subsequent developments in ergodic theory.

Rings of Operators. Most of von Neumann’s results on unbounded operators in Hilbert space were independently discovered a little later by M.H.Stone. But von Neumann’s ideas on rings of operators broke entirely new ground. He was well acquainted with the noncommutative algebra beautifully developed by Emmy Noether and E. Artin in the 1920’s and he realized how these concepts simplified and illuminated the theory of matrices. This probably provided the motivation for extending such concepts to algebras consisting of (bounded) operators in a given separable Hilbert space, to which he gave the vague name “rings of operators” and which are now known as “von Neumann algebras.” He introduced their theory in the same year as his first paper on unbouned operators, and from the beginning he had the insight to select the two essential features that would allow him further progress: the algebra msut be self-adjoint (that is, for any operator in the algebra, its adjoint must also belong to the algebra) and closed under the strong topology of operators and not merely in the finer topology of the norm.

Von Neumann’s first result was the “double commutant theorem,” which states that the von Neumann algebra generated by a self-adjoint family $\mathcal{F}$ of operators is the commutant of the commutant of $\mathcal{F}$, a generalization of a similar result obtained by I.Schur for semisimple algebras of finite dimension that was to become one of the main tools in his later work. After elucidating the relatively easy study of commutative algebras, von Neumann embarked in 1936, with the partial collaboration of F.J. Murray, on the general study of the noncommutative case. The six major papers in which they developed that theory between 1936 and 1940 certainly rank among the masterpieces of analysis in the twentieth century. They immediately realized that among the von Neumann algebras, the “factors” (those with the ceanter reduced to the scalars) held the key to the structure of the general von Neumann algebras; indeed, in his last major paper on the subject (published in 1949 but dating from around 1940), von Neumann showed how a process of “direct integration” (the analogue of the “direct sum” of the finite dimensional theory) explicitly gave all von Neumann algebras from factors as “building blocks,”

The evidence from classical study of non commutative algebras seemed to lead to the conjecture that all factors would be isomorphic to the algbra $\mathcal{B}$(H) of all bounded operators in a Hilbert space *H* (of infant or separable dimension). Murray and von Neumann therefore startled the mathematical world when they showed that the situation was far more complicated. As in the classical theory. their main tool consisted of hte self-adjoint idempotents in the algebra, which are simply orthogonal projections on closed sunbspaces of the Hilbert space; the novelty was that, in contrast with the classical case (or the case $\mathcal{B}$[H]), minimal idempotents may fail to exist in the algebra, which implies that all idempotents are orthogonal projections on infinite dimensional subspaces. Nevertheless, they may be *compared,* the projection on a subspace E being considered as “smaller” then one on an subspace F when te algebra contains a partial isometry V sending E onto a subspace of F. This is only a “preorder,” : but when one considers the corresponding order relation (between equivalence classes), it turns out that in a factor this is a total order relation that may be described by a “dimension function” that attaches to each equivalence class of projections a real number ≥ 0 or $+\infty$. Murray and von Neumann showed that after proper normalization the range of the dimension could be one of five possibilities: $\{1, 2, \ldots, n\}$ (type I_n, the classical algebras of matrices), $\{1, 2, \ldots, +\infty\}$ (type I_∞, corresponding to the algebraas $\mathcal{B}$[H]), the whole interval [0, 1] in the real line (type II_1), the whole interval[0,+ ∞] in the extended real line (type II $_\infty$;), and the two-element set {0, + ∞} (type III).

It may be said that the algebraic structure of a factor imposes on the set oF corresponding subspaces of *H* (images of *H* by the projections belonging to the factor) an order structure similar to that of the subspaces of a usual projective space, but with completely new possibilities regarding the "dimension" attached to these sunsapces. Intrigued by this geometric interpretation of his results, von Neumann developed it in a series of papers on "continuous geometries" and their algebraic satellites, the "regular rings" (which are to continuous geometries as rings of matrices are to vector spaces). This classification, which required great technical skill in the handling of the spectral theory of operators, immediately led to the question of existence for the new "factors." Murray and von Neumann devoted many of their papers to this question; and they were able to exhibit facvtors of types II_1, II_∞, and III by using ingenious constructions from ergodic theory (at a time when the subject of action of groups on measure spaces was still oin its infancy) and algebras generated by convolution operator. They went even further and initiated the study of isomorphisms between factors, succeeding, in particular, in obtaining two nonisomorphic factors of type II_1; only very recently has it been proved that there are uncountably many isomorphism classes for factors of types II_1 and III.

Applied Mathematics. *Mathematical Physics*. Von Neumann's most famous work in theoretical physics is his axiomatization of quantum mechanics. When he began work in that field in 1927, the methods used by its founders were hard to formulate in precise mathematical terms; "operator" on "functions" were handled without much consideration of their domain or definition to their topological properties: and it was blithely assumed that such "operators," when self-adjoint, could always be "diagonalized" (as in th finite dimensional case), at the expense of introducing "Dirac functions" as "eigenvectors." Von Neumann showed that mathematical rigor could be restored by taking as basic axioms the assumptions that the states of a physical system were poinds of a Hilbert space and that the measurable quantities were Hermitian (generally unbounded) operators densely efined in that space. This formalism. the practical use of which became available after von Neumann had developed the spectral theory of unbounded Hermitian operators (1929), has survived subsequent developments of quantum mechanics and is still the basisi of non relativistic quantum theory; with the introduction of the theory of distributions, it has even become possible to interpret its results in a way similar to Dirac's original intuition.

After 1927 von Neumann also devoted much effort to more specific problems of quantum mechanics, such as the problem of measurement and the foundation of quantum statistics and quantum thermodynamics, proving in particular an ergidic throem for quantum systems. All this work was developed and expanded in *Mathematische Grundlagen der Quantenmechanik* (1932), in which he also discussed the much-debated question of "causality" versus "indeterminacy" and concluded that no introduction of "hidden parameters" could keep the basic structure of quantum theory and restore "causality."

Quantum mechanics was not the only area of theoretical physics in which von Neumann was active. With Subrahmanyan Chandrasekhar he published two papers on the statistics of the fluctuating gravitational field generated by randomly distributed stars. After he started work on the Manhattan project, leading to atomic weapons, he became interested in the theory of shock waves and wrote many reports on their theoretical and computational aspects.

Numerical Analysis and Computers. Von Neuman's uncommon grasp of applied mathematics, treated as a whole without divorcing theory from experimental realization, was nowhere more apparent than in his work on computers. He became interested in numerical computations in connection with the need for quick estimates and approximate results that developed with the technology used for the war effort—particularly the complex problems of hydrodynamics—and the completely new problems presented by the harnessing of nuclear energy, for which no ready-made theoretical solutions were available. Dissatisfied with the computing machines available immediately after the war, he was led to examine from its foundations the optimal method that such machines should follow, and he introduced new procedures in their logical organization, the "codes" by which a fixed system of wiring could solve a great variety of problems. Von Neumann devised various methods of programming a computer, particularly for finding eigenvalues and inverses of matrices, extreme of functions of several variables, and production of random numbers. Although he never lost sight of the theoretical questions involved (as can be seen in his remarkably original papers with Herman Goldstine, on the limitation of the errors in the numerical inversion of a matrix of large order), he also wanted to have a direct acquaintance with the engineering problems that had to be faced, and supervised the construction of a computer at the Institute for Advanced Study; many fundamental devices in the present machines bear the imprint of his ideas.

In the last years of his life, von Neumann broadened his views to the general theory of automata, in a kind of synthesis of his early interest in logic and his later work on computers. With his characteristic boldness and scope of vision, he did not hesitate to attack two of the most complex questions in the field: how to design reliable machines using unreliable components, and the construction of self-reproducing machines. As usual he brought remarkably new ideas in the approach to solutions of these problems and must be considered one of the founders of a flourishing new mathematical discipline.

Theory of Games. The role as founder is even more obvious for the theory of games, which von Neumann, in a 1926 paper, conjured—so to speak—out of nowhere. To give a quantitative mathematical model for games of chance such as poker or bridge might have seemed a priori impossible, since such games involve free choices by the players at each move, constantly reacting on each other. Yet von Neumann did precisely that, by introducing the general concept of "strategy" (qualitatively considered a few years earlier by E. Borel) and by constructing a model that made this concept amenable to mathematical analysis. That this model was well adapted to the problem was shown conclusively by von Neumann in the same paper, with the proof of the famous minimax theorem: for a game with two players in a normalized form, it asserts the existence of a unique numerical value, representing a gain for one player and a loss for the other, such that each can achieve at least this favorable expectation from his own point of view by using a "strategy" of his own choosing; such strategies for the two players are termed optimal strategies, and the unique numerical value, the minimax value of the game.

This was the starting point for far-reaching generalizations, including applications to economics, a topic in which von Neumann became interested as early as 1937 and that he developed in his major treatise written with O. Morgenstern. *Theory of Games and Economic Behavior* (1944). These theories have developed into a full-fledged mathematical discipline, attracting many researchers and branching into several types of applications to the social sciences.

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