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(*b*. Paris, France, 6 May 1906;

d. Princeton, New Jersey, 6 August 1998), *number theory, algebra, analysis, geometry*.

Weil was an arithmetician in the broadest possible sense. His work on Diophantine equations drew on all the fields of pure mathematics and developed methods so deep and elegant as to influence each of those fields in turn. He was a founder of the Bourbaki group and its strongest mathematician, the most prominent mathematician at the University of Chicago when it was arguably the world's leading mathematics department, and a member of the Institute for Advanced Study. He decisively shaped the style and direction of all post–World War II mathematics.

Childhood. The Alsatian Jewish medical doctor Bernard Weil and Russian-Austrian Jewish Salomea (Selma) Rein-herz Weil were very comfortably established in Paris when their first child, André, was born, and three years later their daughter Simone, who became known as a philosopher. The children, raised especially by their mother, were precocious and accomplished. The parents had seen bitter anti-Semitism in eastern Europe and in the Dreyfus affair in France. They raised their children so thoroughly assimilated that André was around thirteen before he learned that Jewish descent could matter in any way. At age eight, as a gift to their father, André taught his five-year-old sister to read the newspaper aloud to the family. By age twelve he worked on university-level mathematics, played the violin, taught himself to read Homer and Plato in Greek, and to read Sanskrit. The family often conversed in English or German.

At age fourteen, three years below the minimum age, he took the state baccalaureate exam by special permission and got the highest scores in the nation. He began preparing for the exam to enter the École Normale Supérieur (ENS), which generally takes two years and not rarely more. He took one. During that year with advice from Jacques Hadamard he began to study analysis and differential geometry.

He entered the ENS with the highest exam scores in the nation. He felt he became a mathematician in Hadamard's nearby seminar at the Collège de France, where he presented at least once: on domains of convergence of power series in several complex variables. He took courses with Henri-Léon Lebesgue and Charles-Émile Picard, which did not prevent his also studying Sanskrit at the Sorbonne and reading the *Bhagavad Gita* in the original, which he would carry with him for the rest of his life both for its poetic beauty and its philosophy.

Beginning a Career. During his time at ENS he lived at home as did many students with family in Paris. The family was extremely close, and brother and sister were devoted to each other. He graduated from the ENS at nineteen, too young for military service, and so had time for what he later regarded as his gift for traveling. A summer with his family in the French Alps left him with notebooks full of Diophantine equations plus a plan to always write so as to draw the reader beyond the manifest content toward yet more distant perspectives.

A scholarship from the Sorbonne took him to Rome for six months of mathematics and study of Italian painting up through the modernists. He heard Francesco Severi on algebraic surfaces and encountered Solomon Lefschetz. He read a paper containing a theorem and a conjecture by Louis Mordell (1922) without guessing how important they would soon be to him. Support from the Rockefeller Foundation let him spend much of 1927 in Göttingen, Germany. He describes encounters with Richard Courant, Emmy Noether, and others in his autobiography (1991). Over this time and the next year he crystallized a thesis topic based on the Mordell paper.

Weil was steeped in the long-prestigious subject, from Niels Abel and Carl Jacobi at the beginning of the nineteenth century to Karl Weierstrass and Jules-Henri Poincaré at the end, of integrals of multiple-valued complex functions or in modern terms integration on Riemann surfaces. His teachers Hadamard and Picard were personally involved in it. An *elliptic curve*, or genus one Riemann surface C, is topologically a torus and algebraically is defined by a nice cubic polynomial $P(X, Y)$ in two variables. Integration along paths on C produces a natural Abelian group structure on the points of C. Any points p, q of C have a kind of geometrical sum $p+q$ and this addition law is associative, commutative, and has a zero point and additive inverses. This group structure efficiently organizes the theory of integration on C. Higher degree polynomials $P(X, Y)$ define Riemann surfaces of higher genus, which topologically are surfaces with more than one torus-handle.

The number of handles g is called the *genus* of the surface. Jacobi already knew in effect that integration on a genus g surface C is organized by a group structure on a space $J(C)$ of complex dimension g, called the *Jacobian* of C. A genus one Riemann surface is its own Jacobian. A higher genus Riemann surface C maps as a complex *1*- dimensional subspace into its complex g-dimensional Jacobian $C \to J(C)$.

Poincaré drew arithmetic conclusions from the trivial observation that if a cubic polynomial $P(X, Y)$ has rational coefficients then any geometrical sum $p+q$ of rational points on its curve C is again rational. He conjectured that the group of rational points is finitely generated: a finite number of rational solutions to the cubic $P(X, Y)$ suffices to generate all the rational solutions by the addition law. Mordell proved this and gave his own conjecture: in genus higher than one a Riemann surface C has at most finitely many rational points. Weil set out to generalize Mordell's proof, prove Mordell's conjecture, and introduce systematically useful tools to do it. He succeeded at the first goal and the last.

In place of rational points Weil proved the theorem for points in any *algebraic number field* k, that is: Fix any finite list of irrational numbers, and take solutions using those irrationals along with the rational numbers. And he proved it for the Jacobian of any Riemann surface: For any Riemann surface of any genus, defined over any algebraic number field k, the group of k-valued points of the Jacobian is finitely generated. Weil's clear organization of the proof made the more general conclusion natural. His proof presaged the arithmetic theory of *heights*, where rational numbers are ordered by increasing complexity so that there are only finitely many below any fixed level of complexity. This allows inductive proofs on the complexity of rational points. And he made elegant use of Galois groups, presaging *Galois cohomology* bringing methods of algebraic topology into arithmetic. Typical of Weil's work, it is a tough, elegant argument and inspired much further progress.

Hadamard encouraged Weil to prove the Mordell conjecture in his thesis so as not to leave the work half done. Weil's result on Jacobians suggested a strategy: The rational points (or k-valued points) of $J(C)$ are finitely generated and so are sparse in $J(C)$. For genus g greater than one, the surface C forms a 1-dimensional subspace of its higher dimensional Jacobian $J(C)$, thus also sparse. Two sparse subsets should meet only rarely—and the right details might show these meet only finitely many times. A proof would have to be much more sophisticated and no one has yet made it work. The theorem was proved fifty-five years later by Gerd Faltings (1983), by quite other means descended from the *Weil conjectures*, described below. Weil liked to say he had done well to reject Hadamard's advice and submit his dissertation as it was.

Weil lived with his family as he wrote the dissertation and indeed during his following year of military service. Because of his age he missed the military training usual at the ENS. So he was placed in the infantry rather than the traditional artillery, and officials secured him an easy station in Paris. He got leave time to correct the printer's proofs of his dissertation.

Having made clear his desire to see India, Weil was offered a job at Aligarh Muslim University near Delhi. He agreed to teach French civilization but then the university could not create a position for it. He would have to teach mathematics. He did so from 1930 to 1932. Back in France he was highly esteemed by top mathematicians, but arithmetic was an odd specialty there at the time. Few could read his dissertation. He found a good position at the University of Strasbourg and held it until 1939.

During these years he worked in analysis, especially integration on topological groups, and on algebraic and arithmetic topics derived from his dissertation. His most widely used innovation was in point set topology, namely the idea of *uniform spaces*. Such a space has no metric giving a distance between points, yet it makes sense to talk of different sequences "converging at the same rate" to different points. In particular there is a well-defined notion of uniform convergence of a series of functions from one uniform space to another.

Bourbaki. France could claim to lead nineteenth-century mathematics. Germany had a decisive lead by the 1930s in part because, unlike France, Germany had a policy of protecting promising academics through World War I. Weil happily visited Germany but was ambitious for his own country. Many young mathematicians in France were unhappy with their outdated curriculum. And classmates from the ENS by design looked to each other as an elite destined precisely to assure French greatness in all things. In 1934 Weil assembled a handful of his friends, all admitted to the ENS between 1922 and 1926, to write a definitive new analysis textbook. He met with Henri Cartan, Claude Chevalley, Jean Delsarte, Jean Dieudonné, and René de Possel at the now-vanished Café Capoulade near the ENS on Monday 10 December 1934. The textbook project expanded into a series of books covering the basics of all pure mathematics, none of which would make any references except to earlier books in the series. The group published under the name Nicolas Bourbaki and kept enough secrecy that for decades many mathematicians were unsure just who this Bourbaki was.

These mathematicians had closely similar backgrounds, tastes, and goals. The work was intensely collaborative and cannot be divided into parts attributable to each separate member. But nothing came out of Bourbaki against Weil's wishes. Indeed nothing came out at all for several years. Weil spent much of 1937 at the Institute for Advanced Study in Princeton, New Jersey. He returned to France via New Orleans and Mexico. Soon after that the book series was named the *Elements of Mathematics*. The French title *Éléments de mathématique* expresses the unity of mathematics by using a made-up singular noun *math-ématique* rather than the standard plural form *mathématiques*.

The first volume appeared in 1939. It was a preliminary treatment of set theory and the basic idea of *structure*. By the 1950s Bourbaki produced books on algebra, topology, functions of one real variable, topological vector spaces, and integration—very close to Weil's range of research topics. Other volumes came later and the *Elements of Mathematics* were never actually completed, but the series had a huge influence on worldwide standards for rigor and style of argument. Bourbaki became the standard reference fixing the definitions of modern terminology in most fields of mathematics. Perhaps the main influence was to reorganize all of pure mathematics around recent abstract techniques instead of traditional subject areas. Weil was the leading member of Bourbaki until, following a rule that he had introduced at the beginning, he retired from the group at age fifty.

World War II and America. During the early Bourbaki years Weil met Évelyne (Eveline) de Possel, then wife of René de Possel, who divorced Possel and in October of 1937 married Weil. They would have two children, daughters, Sylvie born 12 September 1942 and Nicolette born 6 December 1946, after they left Europe for the United States.

As World War II approached, Weil thought of the philosophy of the *Bhagavad Gita* and of the loss France suffered in World War I by not protecting her scientists. He rejected the general pacifism of his sister—as she would also during the war—but resolved that if war came he had a duty to keep himself out of it by going to the United States. When it came he was in Finland with that plan in mind. He stayed there until he was arrested as a suspicious foreigner. The story that he was nearly shot as a spy seems to be exaggerated (Pekonen, 1992). He was shipped to jails in Sweden, England, and finally France, where he was arrested for failing to report for military duty. Convicted in May 1940, he asked to be sent to the front rather than jail, and this was granted. The front collapsed before he reached it. In January 1941 he, Eveline, and his parents left for the United States.

The Rockefeller Foundation helped him get teaching work briefly at Haverford College. There he began influential work in geometry with all of his hallmarks: a classical problem with many easily visualized cases is elegantly solved and generalized by using the latest reputedly abstract tools. Karl Friedrich Gauss showed that the sum of the angles of a triangle on a hyperbolic (constant negative curvature) plane is less than 180°, while the angle sum on an elliptic plane (constant positive curvature) is greater than 180°. Much more, on either kind of plane, the difference from 180° is directly proportional to the area of the triangle. The Gauss-Bonnet theorem generalized this to any region P surrounded by a curve C on a surface S of variable curvature, where P itself may have some complicated topology. The integral of the surface curvature over P replaces the area of the triangle, while the integral of the geodesic curvature along C replaces the angle sum. The sum of the two integrals equals 2p times the *Euler number* of P which measures how many “holes” and “handles” the region P has. In particular the integral of the curvature over an entire surface S is always 2p times some integer uniquely determined by the topology of S. Carl Allendoerfer at Haverford generalized this latter result to n- dimensional manifolds M embedded in some Euclidean space R^m. Allendoerfer and Weil (1943) together generalized the whole Gauss-Bonnet theorem to n-dimensional regions P in Riemannian manifolds and mistakenly thought they had eliminated the need for a Euclidean space.

The next year Shiing-Shen Chern, visiting the Institute for Advanced Study in Princeton from China, sharply simplified the Allendoerfer-Weil proof and did eliminate the Euclidean embeddings. He used a very pretty geometric construction with a *fibre bundle* over the manifold M, that is a map of manifolds $B—>M$ which in Chern’s case bundles together one n-1 dimensional sphere for each point x of M and depicts each as the sphere of unit tangent vectors at x. Fibre bundle techniques were new and growing quickly at Princeton. A series of private letters by Weil used all of the tools of integration on manifolds, and topological groups, and *cohomology* to streamline Chern’s construction. Typical of his best work, Weil showed how a quick and natural treatment of that construction led seamlessly to a vast generalization. The *Chern-Weil homomorphism* gives an analogue of Gauss-Bonnet for any fibre bundle with a *connection*, that is with an abstract analogue of differentiation along tangent vectors. The abstract concept has geometric uses far remote from the original motivation. The result and the means used to prove it became cornerstones of the theory of *characteristic classes* on fibre bundles.

Weil next worked at Lehigh University in Pennsylvania. Depressed by the heavy teaching load and uninterested students, in 1944 he resolved to quit and do anything else. The structural anthropologist Claude Lévi-Strauss got him a position at the Universidade de São Paulo in Brazil, a center for algebraic geometry. With a visit to Paris in 1945 he stayed at São Paulo until 1947, when he was appointed professor at the University of Chicago. He had a leave in Paris for 1957–1958 and then became a professor at the Institute for Advanced Study starting in 1958 and retiring in 1976. He traveled back to India in 1967, and made several trips to Japan.

Simone joined Charles De Gaulle’s Free French movement in England. She had adopted a passionate Christian asceticism and self-denial, she requested to be sent on hopelessly dangerous missions in France, and ate less and less, purportedly to express solidarity with the suffering in France. Stricken with tuberculosis, she refused food and medical care. She died 24 August 1943. André was devastated by the loss and by her role in it. He helped produce her posthumous publications and never got over the pain though he lived another fifty-five years. In his own words he lived less than that. His life “or at least what of it deserves the name” ended when his wife died 24 May 1986 (1991, p. 11).

The Weil Conjectures. Nineteenth-century number theorists already saw deep analogies between the ordinary integers Z on one hand and polynomials in one variable with complex coefficients $C(z)$ on the other. The square root of 2 is an *algebraic number* because it satisfies an integer polynomial equation namely X^2-2=0 so that X=2.The complex square root function is an *algebraic function* since it satisfies a similar equation X^2-z=0 so that X= z where z is not a number but a variable over the complex numbers. Theorems on algebraic numbers often had analogues for complex algebraic functions although the proofs might be quite different. Sometimes the proof was easier for algebraic numbers and sometimes for algebraic functions. No general, routine way of turning proofs for one case into proofs for the other is known as of 2007.

Weil wrote to his sister about this, saying, “we would be badly blocked” if we could find no good link between the cases, but “God beats the devil” because there is a promising intermediary: replace the complex numbers by any *finite field* (Letter of 26 March 1940, in Weil, 1979, vol. 1, p. 252). One simple finite field is the integers modulo 5, which is a five-element set {0,1,2,3,4} with addition and multiplication defined by casting out 5s. So 1+2=3 as usual, but 3+4=2 modulo 5. This is obviously like the ordinary integers, but also like the complex numbers in the key respect that each integer has an inverse modulo 5:1×1 = 2×3 = 4×4 = 1 modulo 5.

Weil proved an analogue to no small theorem for this case, but to the deepest most sought-after theorem in number theory: the Riemann hypothesis. In 1940–1941 he proved a Riemann hypothesis for curves over any finite field.

These curves have genus in the same algebraic sense as Riemann surfaces although they are not continuous curves or surfaces in any way, so their genus has no evident topological meaning. The "coordinates" of points on such a curve lie in finite fields rather than in the continuous complex plane. Yet Weil saw that great results would fall out if he could generalize topological ideas related to the genus of continuous curves.

During Weil's time in Göttingen he heard of Heinz Hopf's work on the *Lefschetz fixed point theorem* using topology to count the solutions to suitable equations without actually solving them. Weil made pioneering use of the theorem in 1935 in an elegant new proof of a known theorem on Lie groups. By the late 1930s the relevant topological tools were embodied in *cohomology* theory. He found an often-useful method of showing when different cohomology theories will agree on specified cases. But the known cohomology theories all relied on the continuity of the real numbers. They had no contact with finite fields or in other words with *finite characteristic*. Yet Weil conjectured that an analogue in finite characteristic could give stunningly simple proofs of some powerful arithmetic claims.

The conjectures became famous as the *Weil conjectures*. They are concise, harmonious, penetrating, and surprising. The known special cases were already impressive. They were too beautiful not to be true. Yet it was nearly inconceivable that they could even be stated precisely. Weil himself would not affirm that such a cohomology theory could exist. Perhaps he did not believe it could. Jean-Pierre Serre did, and convinced Alexander Grothendieck, and took part with him and Pierre Deligne in creating it: "This truly revolutionary idea thrilled the mathematicians of the time, as I can testify at first hand; it has been the origin of a major part of the progress in algebraic geometry since that date. The objective was reached only after about twenty-five years, and then not by Weil himself but (principally) by Grothendieck and Deligne" (Serre, 1999, p. 525).

It is typical of Weil that his sweeping vision of unity among the great branches of mathematics produced specific insights and proved hard theorems in specific branches. In this case the impact began with algebraic geometry and number theory. It required a thorough reconception of topological tools in explicit algebraic terms and these fed back into algebra and topology. It affected the related parts of complex analysis and differential geometry. It had eventual repercussions for more or less all of mathematics. These conjectures were Weil's single most influential and ultimately most productive contribution. No doubt he hoped and intended to prove them as well, but the conjectures as such were a work of transcendent genius, and he knew it.

Weil also contributed to the so-called Taniyama-Shimura-Weil conjecture that all elliptic curves are modular. But this might better have been called a question than a conjecture. None of the three felt strongly that it was true. Weil never claimed he originated it. He worked on it and encouraged work on it. The key advance on it came when Andrew Wiles, using tools descended from the Weil conjectures proper, proved nearly the entire TaniyamaShimura-Weil conjecture as the last step in the proof of Fermat's Last Theorem.

He kept residence in Princeton after retiring in 1976. But he spent each spring in Paris in his parents' apartment overlooking the Jardin du Luxemburg. He spent each summer in the Mayenne where Brittany meets Normandy and the Loire Valley. He wrote history of mathematics in a way that affected current research in number theory, especially his 1976 book on Eisenstein and Kronecker. His wife's death led him to write his autobiography (1991). By age eighty he suffered poor eyesight and failing health and he died of old age while his formidable and sometimes mocking personality left him feeling isolated from colleagues. Despite his attachment to the *Bhagavad Gita* he expected no personal survival after death. He was confident that the work of Bourbaki would endure. In the early twenty-first century that work is less in fashion. Yet it remains a decisive influence on mathematics.

Honors. Weil became an honorary member of the London Mathematical Society in 1959, a Foreign Member of the Royal Society of London in 1966, a member of the U.S. National Academy of Sciences in 1977, and of the French Académie des Sciences in 1982. He received the Wolf Prize in Mathematics for 1979 jointly with Jean Leray, and was presented the Barnard Medal by Columbia University in 1980, the AMS Steele Prize of 1980 for lifetime achievement, and the Kyoto Prize in 1994.

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