Zermelo, Ernst Friedrich Ferdinand | Encyclopedia.com

(*b*. Berlin, Germany, 27 July 1871; *d*. Freiburg im Breisgau, Germany, 21 May 1953)

mathematics.

The son of Ferdinand Rudolf Theodor Zermelo, a college professor, and Maria Augusta Elisabeth Zieger, Zermelo received his secondary education at the Luisenstädtisches Gymnasium in Berlin, where he passed his final examination in 1889. He subsequently studied mathematics, physics, and philosophy at Berlin, Halle, and Freiburg, taking courses taught by Frobenius, Lazarus Fuchs, Planck, Erhard Schmidt, H. A. Schwarz, and Edmund Husserl. In 1894 he received the doctorate at Berlin with the dissertation *Untersuchungen zur Variationsrechnung*. Zermelo went to Göttingen and in 1899 was appointed *Privatdozent* after having submitted the *Habilitationsschrifi* "Hydrodynamische Untersuchungen öber die Wirbelbewe-gungen in einer Kugelfläche." In December 1905, shortly after his sensational proof of the well-ordering theorem (1904). Zermelo was named titular professor at Göttingen. In 1910 he accepted a professorship at Zurich, which poor health forced him to resign in 1916. A year after he had left Göttin-gen, 5000 marks from the interest of the Wolfskehl Fund was awarded him on the initiative of David Hilbert in recognition of his results in set theory (and to enable him to recover his health). After resigning his post at Zurich, Zermelo lived in the Black Forest until 1926, when he was appointed honorary professor at the University of Freiburg im Breisgau. He renounced connection with the university in 1935 because of his disapproval of the Hitler regime. After the war he requested reinstatement, which was granted him in 1946.

Zermelo had a lively interest in physics and a keen sense for the application of mathematics to practical problems. He prepared German editions of Glazebrook's *Light* and Gibbs's *Elementary Principles in Statistical Mechanics;* and after having shown in "Ueber einen Satz der Dynamik" how application of Poincareé's recurrence theorem leads to the nonexistence of irreversible processes in the kinetic theory of gases, he had a penetrating discussion with Boltzmann on the explanation of irreversible processes.

In Zermelo's dissertation, which dealt with the calculus of variations, he extended Weierstrass' method for the extrema of integrals over a class of curves to the case of integrands depending on derivatives of arbitrarily high order, at the same time giving a careful definition of the notion of neighborhood in the space of curves. Throughout his life he was faithful to the calculus of variations, on which he often lectured and to which he contributed a report on its progress written with H. Hahh for the *Encyklopädie der mathematischen Wissenschaften*(1904) and the paper "Über die Navigation..." (1929).

Further examples of his original contributions to practical questions are his method for estimating the strength of participants in tournaments ("Die Berechnung der Turnier-Ergebnisse," 1929), which has been used in chess tournaments, and his investigation of the fracture of a cube of sugar ("Über die Bruchlinien zentrierter Ovale," 1933).

As an assistant at Göttingen, Zermelo lectured during the winter semester of 1900-1901 on set theory, to the development of which he was to theory, to the development of which he was to contribute decisively. He had studied Cantor's work thoroughly, and his conversations with Er–hard Schmidt led to his ingenious proof of the well-ordering theorem, which states that every set can be well-ordered; that is, in every set a relation $a \prec b$, to be read as "*a*comes before *b*," can be introduced, such that (1) for any two elements *a* and *b*, either $a = b$ or b≺a: (2) if for three elements *a,b,c,* we have $a \prec b$ and $b \prec c$, then $a \prec c$: (3) any nonvoid subset has a first element. In a commentary to his own proof, Zermelo pointed out the underlying hypothesis that for any infinite system of sets there always are relations under which every set corresponds to one of its elements. The proof stirred the mathematical world and produced a great deal of criticism-most of it unjustified-which Zermelo answered elegantly in "Neuer Beweis" (1908), where he also gave a second proof of the theorem. His answer to Poincaré's accusation of impredicativity is of some historical interest because he points out certain consequences and peculiarities of the predicative position that have played a role in the development of predicative mathematics.

Also in 1908 Zermelo set up an axiom system for Cantor's set theory that has proved of tremendous importance for the development of mathematics. It consists of seven axioms and uses only two technical terms: set and ∊ , the symbol for the "element of" relation. Zermelo emphasized the descriptive nature of the axioms, starting with a domain *B* of objects and then specifying under what conditions (the axioms) an object is to be called a set. With the exception of the null set introduced in axiom 2, every set a is an object of B for which there is another object *b* of *B* such that $a \in b$.

Axiom 1 (extensionality): $m = n$ if and only if $a \in m$ is equivalent to $a \in n$.

Axiom 2 (elementary sets): There is a null set, having no element at all. Every object *a* of *B* determines a set {a} with *a* as its only element. Any two objects *a, b* of *B* determine a set *{a, b}* with precisely *a* and *b* as elements.

Axiom 3 (separation): If a property ∊ is definite for the elements of a set *m* then there is a subset m_E of *m* consisting of exactly those elements of *m* for which *E* holds.

Axiom 4 (power set): To any set m there is a set *P(m)* that has the subsets of *m* for its elements.

Axiom 5 (union): To any set *m* there is a set ∪ *m* the union of *m*, consisting of the elements of the elements of *m*.

Axiom 6 (axiom of choice): If *m* is a set of disjoint nonvoid sets, then ∪ *m* contains a subset *n* that contains exactly one element from every set of *m*.

Axiom 7 (infinity): There is a set z that has the null set as an element and has the property that if a is an element of z, then {a} is also an element of z.

In order to avoid the paradoxes, particularly Russell's paradox, which would render the system useless, Zermelo restricted set formation by the condition of definiteness of the defining property of a subset. A property *E* definite for set *m* is explained as one for which the basic relations of *B* permit one to decide whether or not *E* holds for any element of *m*. Although this condition seemed to preclude of *m* Although this condition seemed to preclude contradictions in the system, Zermelo explicitly left aside the difficult questions of independence and consistency. This was a wise decision, as one may realize after having seen the solutions of the questions of relative consistency and independence of axiom 6 by Kurt Gödel (1938) and P. J. Cohen (1963), respectively.

Because of its generality the notion of definite property is very elegant. It is rather difficult to apply, however, because it does not yield a general method for proving a proposed property to be definite.

Although nonaxiomatic Cantorian set theory was then flourishing, particularly the branch that developed into point–set topology, there was no progress in axiomatic set theory until 1921, when A. Fraenkel, in his attempts to prove the independence of the axiom of choice, pointed out some defects in Zermelo's system. Fraenkel's objections were threefold. First, the axiom of infinity is too weak; second, the system is by no means categorical; and third, the notion of definite property is too vague to handle in proofs of independence and consistency. These remarks led Fraenkel to add the powerful axiom of replacement, which adds to any set s its image under some function *F,* while the notion of function is introduced by definition. Another way of obtaining a similar result was achieved by T. Skolem, who specified a definite property as one expressible in first-order logic.

After having realized the importance of Fraenkel's and Skolem's remarks, Zermelo set out in "Über den Begriff der Definitheirt in der Axiomatik" (1929) to axiomatize this notion by describing the set of definite properties as the smallest set containing the basic relations if the domain *B* and satisfying certain closure conditions. He admitted that the reason for doing so was methodological: to keep the "pure axiomatic" method, in avoidance of the genetic method and the use of the notion of finite number. Since there is no categoricity, an investigation of the structure of the possible domains *b*-models for axiomatic set theory-makes sense. In "Über Grenzzahlen und Mengenbereiche" (1930) Zermelo investigated the structure of models of an axiom system consisting of his earlier axioms 1, 4, 5, the last part of 2, the unrestricted form of 3, a liberal axiom of replacement and an axiom of well-foundedness (with respect to ∊) stating that every subdomain *T* of domain *B* contains at least one element t_0 that has no element tin *T*.

Zermelo's fragmentary attempt, in "Grundlagen einer allgemeinen Theorie der mathematischen Satzsysteme" (1935), to abolish the limitations of proof theory has not been of great consequence because his conception of a proof as a system of theorems, well-founded with respect to the relation of consequence, seems too general to lead to results of sufficient interest.

BIBLIOGRAPHY

I. Original Works. Zermelo's writings include *Untersachangen zur Variationsrechnung* (Berlin, 1894), his dissertation; "Ueber einen Satz der Dynamik und die mechanische Wärmetheorie," in *Annalen der Physik und Chemie,* n.s. **57** (1896), 485-494; "Ueber mechanische Erklärungen irreversibler Vorgänge. Eine Antwort auf Hrn. Boltzmann's 'Entgegnung,'" *ibid.,***59** (1896), 793–801; *Das Light. Grundriss der Optik fur Studierende und Schüler,* his trans. of R. T. Glazebrook's *Light* (Berlin, 1897); "Ueber die Bewegung eines Punktsystemes bei Bedingungsungleichungen," in *Nachrichten von der K. Gesellschaft der Wissenschaften zu Göttingen,* math-phys. Kl (1899), 306–310; "Über die Anwendung der Wahrscheinlinchkeitsrechnung auf dynamische Systeme," in *Physikalische Zeitschrift,***1** (1899-1900), 317–320; "Ueber die Addition transfiniter Cardinalzahlen," in *Nachrichten von der K. Gesellschaft der Wissenschaften zu Göttingen,* math-phys. Kl. (1901), 34–38; "Hydrodynamische Untersuchungen Über die Wirbelbewegungen in einer Kugelfläche," in *Zeitschrift für Mathematik und Physik,***47** (1902), 201–237, his *Habilitationsschrift;* and "Zur Theorie der kÜrzesten Linien," in *Jahresherichte der Deutschen Mathenatikervereinigung,***11** (1902), 184–187.

Further works are "Über die Herleitung der Differentialgleichung bei Variationsproblemen," in *Mathematische Annalen,***58** (1904), 558-564: "Beweis, dass jede Menge wohlgeordnet werden kann," *ibid.,***59** (1904), 514–516; "Weiterentwickelung der Variationsrechnung in den letzten Jahren," in *Encyklopadie der mathematischen Wissenschaften,***11** pt. 1 (Leipzig, 1904), 626–641, written with H. Hahn; *Elementare Grundlagen der statistischen Mechanik,* his trans. of Gibbs's work (Leipzig, 1905); "Neuer Beweis für die Moglichkeit einer Wohlordnung," in *Mathematische Annalen***65** (1908), 107–128; "Untersuchungen uber die Grundlagen der Mengenlehre I," *ibid.,* 261–281; "Sur les ensembles finis et le principe de l'induction complète," in *Acta mathematica,***32** (1909), 185–193; "Die Einstellung der Grenzkonzentrationen an der Trennungs-flache zweier Lösungsmittel," in *Physikalische Zeitschrift,***10** (1909), 958–961, written with E. H. Riesenfeld; and "Ueber die Grundlagen der Arithmetik," in *Atti del IV Congresso internazionale dei matematici,***II** (Rome, 1909), 8–11.

See also "Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels," in *Proceedings of the Fifth International Congress of Mathematicians,***II** (Cambridge, 1913), 501–504; "Uber ganze transzendente Zahlen," in *Mathematische Annalen,***75** (1914), 434–442; "Ueber das Masz und die Diskrepanz von Punktmengen," in *Journal fur die reine und angewandte Mathematik,***158** (1927), 154–167; "Uber den Begriff der Definitheit in der Axiomatik," in *Fundamenta mathematicae,***14** (1929), 339–344; "Die Berechnung der Turnier-Ergebnisse als ein Maximumproblem der Wahrscheinlichkeitsrechnung," in *Mathematische Zeitschrift,***29** (1929), 436–460; and "Über die Navigation in der Luft als Problem der Variation srechnung," in *Jahresherichte der Deutschen Muthenuutikcrrerinigurrg***39** (1929), 44–48.

Additional works are "Über Grenzzahlen und Mengenbereiche. neue Untersuchungn iiber die Grundlagen der Mengenlehre," in *Fundamenta mathematicae***16** (1930), 29–47; "Über die logische Form der mahmatistchen Theorien," in *Annales de la société polonaise de mathématique,***9** (1930), 187; "Über das Navigations problem bei ruhender oder veränderlicher Windverteilung," in *Zeitschrift für angewandte Mathematik und Mechanik,***11** (1931), (114–124: "Über mathematische Systeme und die Logik des Unendlichen," in *Forschungn und Fortschritte,***8** (1932), 6–7; "Über stufen der quantifikation und die Logik des Unendlichen," in *Jahresberichte der Dutschen Mathmatikervreinigung,***41** (1932), 85–88; "Über die Bruchlinien Zentrierter Ovale. (Wie Zerbricht ein Stück Zucker?)," in *Zeits chrift für angwandte Mathmatik und Mechnaik,***13** (1933), 168–170; "Elementare Betrachtungen Zur Theorie der Primzahlen," in *Nachrichtn von der Gesells chaft der Wisschaften zugottingen,* Fachgruppe 1, **1** (1934), 43–46; and "Grundlagen einer allgemeinen Theorie dermathematischen Satzsysteme. (Erste Mitteilung)," in *Fundamenta mathematicae,***25** (1935), 136–146.

A collection of papers left by Zermelo is in the library of the University of Freiburg im Breisgau. A short description, furnished by H. Gericke, is as follows: a set of copies of articles by Zermelo and other mathematicians, a collection of letters and MSS and sketches of published papers, lecture notes in shorthand, parts of a translation of Homer in German verse, the second part of his Hahilitationsschrift, and a sketch of a patent application –;Kreisel zur Stabilisierung von Fahr–and Motorradernä (gyroscope for stabilizing bicycles and motorcycles)..

II. Secondary Literature. Quite a number of relevant papers on set theory, including three memoirs by zermelo are reprinted in J. van Heijenoort, *From Frege to Giidel* (Cambridg, Mass., 1967), which also contains references to the literure up to 1966. See also P.j. cohen. *Set Theory and the Continuum Hypohesis* (New York, 1966); S. Fefermann, "Systms of Predicative Analysis," in *journal of Symbolic Logic,***29** (1964). 1-30; A. Fraenkel, "Über die Zermelosche Begrundujng der Mengnlhre," in *Jahresberichte der Deautschen Mathmatikervereinigung,***30** (1921), 97–98; "Zu den Grundlagen der Cantor-Zermloschen mengnlechr," in *Mathematische Annalen,***86** (1922), 230–237; "Der Begriff 'definit' und die Unabhängigkeit des auswahl-saxioms," in *Sitzungsberichte der Preussischen Akademie der wissenschaften zu Berlin* (1922), 253–257; and *Foundations of Set Theory* (Amsterdam, 1958), written with Y. Bar-Hillel: H. Gericke, *Beiträge zur Freiburger Wissenchafts-und universitatsgeschichte* VII *Zur Geschichte der Mathematick and der Universität freiburg* i. Br., J. Vincke, ed. (Freibug im Breisgau, 1955). 72–73; K. Gödel, *The Consistency of the continuum Hypothesis* (Princeton, N.J., 1940); G. Kreisel and J.L. Krivine. *Elements of Mathematical Logic* (Amsterdam, 1967); M.Pinl, "Kollegen in einer dunklen Zeit." in *Jahresberichte der Deutsche Mathematikervereinigung***71** (1969), 167–228, esp. 221-222; C. Reid. *Hilbert* (Berlin, 1970); and J. Barkeley Rosser, *Simplified Independence Proofs* (New York, 1969).

Also see T. Skolem, "Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer sätze nebst einem Theoreme über dichte Mengen," in *Skrifter utgitt av videnskapsselskapet i Kristiania* I. Mat-naturvid. kl. (1920), no. 4; "Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre," in *Matematiker kongressen i Helsingors den* 4–7 *Juli* 1922 (Helsinki, 1923), 217–232; "Über einige Grundlagenfragen der Mathematick," in *skrifter utgitt av det Norske videnskaps-akademi i Oslo,* I. Mat.-naturvid. kl. (1929), no. 4; and "Einige Bemerkungn zu der abhandlung von E. Zermelo: 'Über die Definitheit in der Axiomatik,'" in *Fundamenta mathematicae,***15** (1930), 337–341; and L. Zoretti and A. Rosenthal, "Die Punktmengen," in *Encyklpädie der mathematischen Wissenschaften,***II** . pt. 3 (Leipzig, 1923), 855–1030.

B. Van Rootselaar